ON A CLASS OF CAHN–HILLIARD MODELS WITH NONLINEAR DIFFUSION

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Abstract. In the present work, we address a class of Cahn–Hilliard equations characterized by a nonlinear diffusive dynamics and possibly containing an additional sixth order term. This model describes the separation properties of oil-water mixtures when a substance enforcing the mixing of the phases (a surfactant) is added. However, the model is also closely connected with other Cahn–Hilliard-like equations relevant in different types of applications. We first discuss the existence of a weak solution to the sixth order model in the case when the configuration potential of the system has a singular (e.g., logarithmic) character. Then we study the behavior of the solutions in the case when the sixth order term is allowed to tend to 0, proving convergence to solutions of the fourth order system in a special case. The fourth order system is then investigated by a direct approach, and existence of a weak solution is shown under very general conditions by means of a fixed point argument. Finally, additional properties of the solutions, like uniqueness and parabolic regularization, are discussed, both for the sixth order and for the fourth order model, under more restrictive assumptions on the nonlinear diffusion term.

Key words. Cahn–Hilliard equation, nonlinear diffusion, variational formulation, existence theorem

AMS subject classifications. 35K35, 35K55, 35A01, 47H05

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1. Introduction. This paper is devoted to the mathematical analysis of the following class of parabolic systems:

\begin{align}
(1.1) & \quad u_t - \Delta w = 0, \\
(1.2) & \quad w = \delta \Delta^2 u - a(u) \Delta u - \frac{a'(u)}{2} |\nabla u|^2 + f(u) + \varepsilon w \\
\end{align}
on \{(0, T) \times \Omega, \Omega \text{ being a bounded smooth subset of } \mathbb{R}^3 \text{ and } T > 0 \text{ an assigned final time.} \}

The restriction to the three-dimensional setting is motivated by physical applications. Similar, or better, results are expected to hold in space dimensions 1 and 2. The system is coupled with the initial and boundary conditions

\begin{align}
(1.3) & \quad u|_{t=0} = u_0 \quad \text{in } \Omega, \\
(1.4) & \quad \partial_n u = \partial_n w = \delta \partial_n \Delta u = 0 \quad \text{on } \partial \Omega \quad \text{for } t \in (0, T) \\
\end{align}

and represents a variant of the Cahn–Hilliard model for phase separation in binary materials. The function \( f \) stands for the derivative of a singular potential \( F \) of a double obstacle type. Namely, \( F \) is assumed to be \( +\infty \) outside a bounded interval (assumed equal to \([-1, 1]\) for simplicity), where the extrema correspond to the pure states. A physically significant example is given by the so-called Flory–Huggins logarithmic
potential

\[ F(r) = (1 - r) \log(1 - r) + (1 + r) \log(1 + r) - \frac{\lambda}{2} r^2, \quad \lambda \geq 0. \]

As in this example, we will assume \( F \) to be at least \( \lambda \)-convex, i.e., convex up to a quadratic perturbation. In this way, we can also allow for singular potentials having more than two minima in the interval \([-1, 1]\) (as happens in the case of the oil-water-surfactant models described below, where the third minimum appears in relation to the so-called microemulsion phase).

We assume the coefficients \( \delta, \varepsilon \) to be \( \geq 0 \), with the case \( \delta > 0 \) giving rise to a sixth order model and the case \( \varepsilon > 0 \) related to possible viscosity effects that are likely to appear in several models of Cahn–Hilliard type (see, e.g., [31]). The investigation of the limits as \( \delta \) or \( \varepsilon \) tend to zero provides validation of these models as the approximates of the limit fourth order model.

The main novelty of system (1.1)–(1.2) is related to the presence of the nonlinear function \( a \) in (1.2), which is supposed smooth, bounded, and strongly positive (i.e., everywhere larger than some constant \( a > 0 \)). Mathematically, the latter is an unavoidable assumption as we are mainly interested in the behavior of the problem when \( \delta \) is allowed to tend to 0 and in the properties of the (fourth order) limit system \( \delta = 0 \). On the other hand, at least in the physical context of the sixth order model, it would also be meaningful to admit \( a \) to take negative values, as may happen in the “microemulsion” phase (see [28, 29]). We will not deal with this situation, but we just point out that, as long as \( \delta > 0 \) is fixed, this should create no additional mathematical difficulties since the nonlinear diffusion term is then dominated by the sixth order term.

From the analytical point of view, as a basic observation we can notice that this class of systems has an evident variational structure. Indeed, (formally) testing (1.1) by \( w \), (1.2) by \( u_t \), taking the difference of the obtained relations, integrating with respect to space variables, using the no-flux conditions (1.4), and performing suitable integrations by parts, one readily gets the a priori bound

\[ \frac{d}{dt} E_\delta(u) + \| \nabla u \|_{L^2(\Omega)}^2 + \varepsilon \| u_t \|_{L^2(\Omega)}^2 = 0, \]

which has the form of an energy equality for the energy functional

\[ E_\delta(u) = \int_\Omega \left( \frac{\delta}{2} |\Delta u|^2 + \frac{a(u)}{2} |\nabla u|^2 + F(u) \right), \]

where the interface (gradient) part contains the nonlinear function \( a \). In other words, the system (1.1)–(1.2) arises as the \( (H^1)' \)-gradient flow problem for the functional \( E_\delta \).

While the literature on the fourth order Cahn–Hilliard model with logarithmic free energy is very wide (starting from the pioneering work [22] up to more recent works like, e.g., [1, 27, 39]; see also the recent review [19] and the references therein), it seems that potentials of logarithmic type have never been considered in the case of a nonconstant coefficient \( a \). Similarly, the sixth order Cahn–Hilliard-type equations, which appear as models of various physical phenomena and have recently attracted a notable interest in the mathematical literature (see the discussion below), seem not to have garnered much interest in the case of logarithmic potentials.

The sixth order system (1.1)–(1.2) arises as a model of dynamics of ternary oil-water-surfactant mixtures in which three phases occupying a region \( \Omega \) in \( \mathbb{R}^3 \)—a microemulsion, almost pure oil, and almost pure water—can coexist in equilibrium. The
phenomenological Landau–Ginzburg theory for such mixtures has been proposed in a series of papers by Gompper and coauthors (see, e.g., [28, 29, 30] and other references in [41]). This theory is based on the free energy functional (1.7) with constant \( \delta > 0 \) (in general, however, this coefficient can depend on \( u \); see [46]), and with \( F(u), a(u) \) approximated, respectively, by a sixth and a second order polynomial:

\[
F(u) = (u + 1)^2(u^2 + h_0)(u - 1)^2, \quad a(u) = g_0 + g_2u^2,
\]

where the constant parameters \( h_0, g_0, g_2 \) are adjusted experimentally, \( g_2 > 0 \), and \( h_0, g_0 \) are of arbitrary sign. In this model, \( u \) is the scalar, conserved order parameter representing the local difference between oil and water concentrations; \( u = -1, u = 1, \) and \( u = 0 \) correspond to oil-rich, water-rich, and microemulsion phases, respectively, and the parameter \( h_0 \) measures the deviation from oil-water-microemulsion coexistence.

The associated evolution system (1.1)–(1.2) has the standard Cahn–Hilliard structure. Equation (1.1) expresses the conservation law

\[
\frac{\partial u}{\partial t} + \nabla \cdot j = 0
\]

with the mass flux \( j \) given by

\[
j = -M \nabla w.
\]

Here \( M > 0 \) is the constant mobility (we set \( M = 1 \) for simplicity), and \( w \) is the chemical potential difference between the oil and water phases. The chemical potential is defined by the constitutive equation

\[
w = \frac{\delta E_\delta(u)}{\delta u} + \varepsilon u_t,
\]

where \( \frac{\delta E_\delta(u)}{\delta u} \) is the first variation of the functional \( E_\delta(u) \), and the constant \( \varepsilon \geq 0 \) represents possible viscous effects. For energy (1.7), equation (1.11) yields (1.2). We note also that the boundary conditions \( \partial_n u = \delta \partial_n \Delta u = 0 \) are standardly used in the frame of sixth order Cahn–Hilliard models due to their mathematical simplicity. Moreover, they are related to the variational structure of the problem in terms of the functional (1.7). However, other types of boundary conditions for \( u \) might be considered as well, paying the price of technical complications in the proofs. Concerning, instead, the condition \( \partial_n w = 0 \), in view of (1.10), it simply represents the mass isolation at the boundary of \( \Omega \).

The system (1.1)–(1.4) with functions \( F(u), a(u) \) in the polynomial form (1.8), and with no viscous term \( (\varepsilon = 0) \), has been recently studied in [41]. It has been proved there that for a sufficiently smooth initial datum \( u_0 \) the system admits a unique global solution in the strong sense.

The sixth order Cahn–Hilliard-type equation with the same structure as (1.1)–(1.2), \( \delta > 0 \), polynomial \( F(u) \), and negative constant \( a \) arises also as the so-called phase-field crystal (PFC) atomistic model of crystal growth, developed by Elder and coauthors; see, e.g., [25, 10, 9], and [26] for the overview and up-to-date references. It is also worth mentioning a class of sixth order convective Cahn–Hilliard-type equations with different (nonconservative) structure than (1.1)–(1.2). This type of equation arises in particular as a model of the faceting of a growing crystalline surface, derived by Savina et al. [45] (for a review of other convective fourth and sixth order Cahn–Hilliard models, see [33]). In this model, contrary to (1.1)–(1.2), the order parameter
$u$ is not a conserved quantity due to the presence of a force-like term related to the deposition rate. Such a class of models has been recently studied mathematically in one- and two-dimensional cases by Korzec and coauthors [33, 34, 35].

Finally, let us note that in the case $\delta = 0$, $a(u) = \text{const} > 0$, the functional (1.7) represents the classical Cahn–Hilliard free energy [17, 18]. The original Cahn–Hilliard free energy derivation has been extended by Lass, Johnson, and Shiflet [36] to account for composition dependence of the gradient energy coefficient $a(u)$. For a face-centered cubic crystal the following expressions for $a(u)$ have been derived, depending on the level of approximation of the nearest-neighbor interactions:

\begin{equation}
(1.12) \quad a(u) = a_0 + a_1 u + a_2 u^2,
\end{equation}

where $a_0 > 0$, $a_1, a_2 \in \mathbb{R}$ in the case of four-body interactions, $a_2 = 0$ in the case of three-body interactions, and $a_1 = a_2 = 0$ in the case of pairwise interactions.

Numerical experiments in [36] indicate that these three different approximations (all reflecting the face-centered cubic crystal symmetry) have a substantial effect on the shape of the equilibrium composition profile and the interfacial energy.

A specific free energy with composition-dependent gradient energy coefficient $a(u)$ also arises in the modeling of phase separation in polymers [23]. This energy, known as Flory–Huggins–de Gennes energy, has the form (1.7) with $\delta = 0$, $F(u)$ being the logarithmic potential (1.5), and the singular coefficient

\begin{equation}
(1.13) \quad a(u) = \frac{1}{(1-u)(1+u)}.
\end{equation}

We mention also that various formulations of phase-field models with gradient energy coefficient dependent on the order parameter (and possibly on other fields) appear, e.g., in [2, 14].

Our objective in this paper is threefold. First, we would like to extend the result of [41] both to the viscous problem ($\varepsilon > 0$) and to the case when the configuration potential is singular (e.g., of the form (1.5)). While the first extension is almost straightforward, considering constraint (singular) terms in fourth order equations ((1.2), in the specific case) gives rise to regularity problems since it is not possible, to the best of our knowledge, to estimate all the terms of (1.2) in $L^p$-spaces. For this reason, the nonlinear term $f(u)$ needs to be interpreted in a weaker sense, namely, as a selection of a nonlinear, and possibly multivalued, mapping acting from $V = H^1(\Omega)$ to $V'$.

This involves a monotone operator technique that is developed in subsection 2.1.

As a second step, we investigate the behavior of the solutions to the sixth order system as the parameter $\delta$ is allowed to tend to 0. In particular, we would like to show that, at least up to subsequences, we can obtain in the limit suitably defined solutions to the fourth order system obtained by setting $\delta = 0$ in (1.2). Unfortunately, we are able to prove this fact only under additional conditions. The reason is that the natural estimate required to control second space derivatives of $u$, i.e., testing (1.2) by $-\Delta u$, is compatible with the nonlinear term in $\nabla u$ only under additional assumptions on $a$ (e.g., if $a$ is concave). This nontrivial fact depends on an integration by parts formula devised by Dal Passo, Garcke, and Grün in [21] in the frame of the thin-film equation and whose use is necessary to control the nonlinear gradient term. It is, however, likely that the use of more refined integration by parts techniques may permit one to control the nonlinear gradient term under more general conditions on $a$.

Since we are able to take the limit $\delta \searrow 0$ only in special cases, in a subsequent part of the paper we address the fourth order problem by using a direct approach.
In this way, we can obtain existence of a weak solution under general conditions on $a$ (we notice, however, that uniqueness is no longer guaranteed for $\delta = 0$). The proof of existence is based on an “ad hoc” regularization of the equations by means of a system of phase-field type. This kind of approach has been proved to be effective also in the framework of other types of Cahn–Hilliard equations (see, e.g., [6]). Local existence for the regularized system is then shown by means of the Schauder theorem, and, finally, the regularization is removed by means of suitable a priori estimates and compactness methods. This procedure involves some technicalities since parabolic spaces of Hölder type have to be used for the fixed point argument. Indeed, the use of Sobolev techniques seems not suitable due to the nonlinearity in the highest order term, which prevents compactness of the fixed point map with respect to Sobolev norms. A further difficulty is related to the necessity of estimating the second order space derivatives of $u$ in the presence of the nonlinear term in the gradient. This is obtained by introducing a proper transformed variable and rewriting (1.2) in terms of it. Proceeding in this way, we can get rid of that nonlinearity, but at the same time we can still exploit the good monotonicity properties of $f$. We note here that a different method based on entropy estimates could also be used to estimate $\Delta u$ without making the change of variable, which seems, however, a simpler technique.

Finally, in the last section of the paper, we discuss another property of weak solutions. More precisely, we address the problems of uniqueness (only for the fourth order system, since in the case $\delta > 0$ it is always guaranteed) and of parabolic time regularization of solutions (both for the sixth and for the fourth order systems). We are able to prove such properties only when the energy functional $\mathcal{E}_\delta$ is $\lambda$-convex (so that its gradient is monotone up to a linear perturbation). In terms of the coefficient $a$, this corresponds to asking that $a$ is a convex function and, moreover, $1/a$ is concave (cf. [24] for generalizations and further comments regarding this condition). If these conditions fail, then the gradient of the energy functional exhibits a nonmonotone structure in terms of the space derivatives of the highest order. For this reason, proving an estimate of contractive type (which would be a requirement for uniqueness) appears to be difficult in that case.

As a final result, we will show that, both in the sixth and in the viscous fourth order cases, all weak solutions satisfy the energy equality (1.6), at least in an integrated form (and not just an energy inequality). This property is the starting point for proving existence of the global attractor for the dynamical process associated to system (1.1)–(1.2), an issue that we intend to investigate in a forthcoming paper. Actually, it is not difficult to show that the set of initial data having finite energy constitutes a complete metric space (see, e.g., [43, Lem. 3.8]), which can be used as a phase space for the system. Then, by applying the so-called energy method (cf., e.g., [38, 5]), one can see that the energy equality implies precompactness of trajectories for $t \to \infty$ with respect to the metric of the phase space. In turn, this gives existence of the global attractor with respect to the same metric. On the other hand, the question of whether the energy equality holds in the nonviscous fourth order case seems to be more delicate, and, actually, we could not give a positive answer to it.

It is also worth noticing an important issue concerned with the sharp interface limit of the Cahn–Hilliard equation with a nonlinear gradient energy coefficient $a(u)$. To the best of our knowledge this issue has not been addressed in the literature. Let us mention that, using the method of matched asymptotic expansions, the sharp interface limits of the Cahn–Hilliard equation with constant coefficient $a$ have been investigated by Pego [42] and rigorously by Alikakos, Bates, and Chen [3]. Such a
method has also been successfully applied to a number of phase-field models of phase transition problems; see, e.g., [16, 15]. In view of the various physical applications described above, it would be of interest to apply the matched asymptotic expansions in the case of a nonlinear coefficient \( a(u) \) to investigate what kind of corrections it may introduce to conditions on the sharp interface.

The plan of the paper is as follows. In section 2, we report our notation and hypotheses, together with some general tools that will be used in the proofs. Section 3 contains the analysis of the sixth order model. The limit \( \delta \downarrow 0 \) is analyzed in section 4. Section 5 is devoted to the analysis of the fourth order model. Finally, in section 6 uniqueness and regularization properties of the solutions are discussed, as well as the validity of the energy equality.

**2. Notation and technical tools.** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^3 \) of boundary \( \Gamma \), \( T > 0 \) a given final time, and \( Q := (0, T) \times \Omega \). We let \( H := L^2(\Omega) \), endowed with the standard scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). For \( s > 0 \) and \( p \in [1, \infty] \), we use the notation \( W^{s,p}(\Omega) \) to indicate Sobolev spaces of positive (possibly fractional) order. We also set \( H^s(\Omega) := W^{s,2}(\Omega) \) and let \( V := H^1(\Omega) \). We note by \( \langle \cdot, \cdot \rangle \) the duality between \( V' \) and \( V \) and by \( \| \cdot \|_X \) the norm in the generic Banach space \( X \). We identify \( H \) with \( H' \) in such a way that \( H \) can be seen as a subspace of \( V' \) or, in other words, \( (V, H, V') \) form a Hilbert triplet.

We make the following assumptions on the nonlinear terms in (1.1)–(1.2):

\[
\begin{align*}
(2.1) \quad & a \in C^2_b(\mathbb{R}; \mathbb{R}), \ |a_\alpha r > 0: \ a \leq a(r) \leq \bar{a} \ \forall r \in \mathbb{R}; \\
(2.2) \quad & \exists a_-, a_+ \in [a, \bar{a}]: \ a(r) \equiv a_- \ \forall r \leq -2, \ a(r) \equiv a_+ \ \forall r \geq 2; \\
(2.3) \quad & f \in C^1((-1, 1); \mathbb{R}), \ f(0) = 0, \ \exists \lambda \geq 0: \ f'(r) \geq -\lambda \ \forall r \in (-1, 1); \\
(2.4) \quad & \lim_{|r| \to 1} f(r) = \lim_{|r| \to 1} \frac{f'(r)}{|f'(r)|} = +\infty.
\end{align*}
\]

The latter condition in (2.4) is just a technical hypotheses which is actually verified in all significant cases. We also notice that, due to the choice of a singular potential (mathematically represented here by assumptions (2.3)–(2.4)), any weak solution \( u \) will take its values only in the physical interval \([-1, 1]\). For this reason, the behavior of \( a \) is also significant only in that interval, and we have extended it outside \([-1, 1]\) just for the purpose of properly constructing the approximating problem (see subsection 3.1 below).

Note that our assumptions on \( a \) are not in conflict with (1.8) or (1.12) since these conditions (or, more generally, any condition on the values of \( a(u) \) for large \( u \)) make sense in the different situation of a function \( f \) with polynomial growth (which does not constrain \( u \) in the interval \((-1, 1)\)). It should be pointed out, however, that the assumptions (2.1)–(2.4) do not admit the singular Flory–Huggins–de Gennes free energy model with \( a(u) \) given by (1.13). We expect that the analysis of such a singular model could require different techniques.

In (2.1), \( C^2_b \) denotes the space of functions that are continuous and globally bounded together with their derivatives up to the second order. Concerning \( f \), (2.3) states that it can be written in the form

\[
(2.5) \quad f(r) = f_0(r) - \lambda r,
\]

i.e., as the difference between a (dominating) monotone part \( f_0 \) and a linear pertur-
bation. By (2.3)–(2.4), we can also set, for $\ r\in(-1,1)$,

\[(2.6)\quad F_0(r) := \int_0^r f_0(s) \, ds \quad \text{and} \quad F(r) := F_0(r) - \frac{\lambda}{2} r^2,\]

so that $F' = f$. Notice that $F_0$ may be bounded in $(-1,1)$ (e.g., this occurs in the case of the logarithmic potential (1.5)). If this is the case, we extend it by continuity to $[-1,1]$. Then $F_0$ is set to be $+\infty$ either outside $(-1,1)$ (if it is unbounded in $(-1,1)$) or outside $[-1,1]$ (if it is bounded in $(-1,1)$). This standard procedure permits us to penalize the nonphysical values of the variable $u$ and to intend $f_0$ as the subdifferential of the (extended) convex function $F_0 : \mathbb{R} \to [0, +\infty]$.

That said, we define a number of operators. First, we set

\[(2.7)\quad A : V \to V', \quad \langle Av, z \rangle := \int_\Omega \nabla v \cdot \nabla z \quad \text{for} \quad v, z \in V.\]

Then we define

\[(2.8)\quad W := \{ z \in H^2(\Omega) : \partial_n z = 0 \text{ on } \Gamma \}\]

and recall that (a suitable restriction of) $A$ can be seen as an unbounded linear operator on $H$ having domain $W$. The space $W$ is endowed with the natural $H^2$-norm. We then introduce

\[(2.9)\quad A : W \to H, \quad A(z) := -a(z) \Delta z - \frac{a'(z)}{2} |\nabla z|^2.\]

It is a standard issue to check that, indeed, $A$ takes its values in $H$.

**2.1. Weak subdifferential operators.** To state the weak formulation of the sixth order system, we need to introduce a proper relaxed form of the maximal monotone operator associated to the function $f_0$ and acting in the duality between $V'$ and $V$ (rather than in the scalar product of $H$). Actually, it is well known (see, e.g., [12, Ex. 2.1.3, p. 21]) that $f_0$ can be interpreted as a maximal monotone operator on $H$ by setting, for $v, \xi \in H$,

\[(2.10)\quad \xi = f_0(v) \quad \text{in} \quad H \quad \iff \quad \xi(x) = f_0(v(x)) \quad \text{a.e. in } \Omega.\]

If no danger of confusion occurs, the new operator on $H$ will be still noted by the letter $f_0$. Correspondingly, $f_0$ is the $H$-subdifferential of the convex functional

\[(2.11)\quad \mathcal{F}_0 : H \to [0, +\infty], \quad \mathcal{F}_0(v) := \int_\Omega F_0(v(x)),\]

where the integral might possibly be $+\infty$ (this happens, e.g., when $|v| > 1$ on a set of strictly positive Lebesgue measure).

The weak form of $f_0$ can be introduced by setting

\[(2.12)\quad \xi \in f_{0,w}(v) \iff \langle \xi, z - v \rangle \leq \mathcal{F}_0(z) - \mathcal{F}_0(v) \quad \text{for any} \quad z \in V.\]

Actually, this is nothing else than the definition of the subdifferential of (the restriction to $V$ of) $\mathcal{F}_0$ with respect to the duality pairing between $V'$ and $V$. In general, $f_{0,w}$ can be a *multivalued* operator; namely, $f_{0,w}$ is a *subset* of $V'$ that may contain more
than one element. It is not difficult to prove (see, e.g., [8, Prop. 2.5]) that if \( v \in V \) and \( f_0(v) \in H \), then

\[
\{ f_0(v) \} \subset f_{0,w}(v).
\]

Moreover,

\[
\text{if } v \in V \text{ and } \xi \in f_{0,w}(v) \cap H, \text{ then } \xi = f_0(v) \text{ a.e. in } \Omega.
\]

In general, the inclusion in (2.13) is strict and, for instance, it can happen that \( f_0(v) \not\in H \) (i.e., \( v \) does not belong to the \( H \)-domain of \( f_0 \)), while \( f_{0,w}(v) \) is nonempty. Nevertheless, we still have some “automatic” gain of regularity for any element of \( f_{0,w}(v) \).

**Proposition 2.1.** Let \( v \in V, \xi \in f_{0,w}(v) \). Then \( \xi \) can be seen as an element of the space \( \mathcal{M}(\Omega) = C^0(\Omega)' \) of the bounded real-valued Borel measures on \( \Omega \). More precisely, there exists \( T \in \mathcal{M}(\Omega) \), such that

\[
\langle \xi, z \rangle = \int_{\Omega} z \, dT \quad \text{for any } z \in V \cap C^0(\Omega).
\]

**Proof.** Let \( z \in C^0(\Omega) \cap V \) with \( z \neq 0 \). Then, using definition (2.12), it is easy to see that

\[
\langle \xi, z \rangle = 2 \| z \|_{L^\infty(\Omega)} \left( \frac{\xi}{2 \| z \|_{L^\infty(\Omega)}} \right) \leq 2 \| z \|_{L^\infty(\Omega)} \left( \langle \xi, v \rangle + \lambda_0 \left( \frac{z}{2 \| z \|_{L^\infty(\Omega)}} - \lambda_0(v) \right) \right)
\]

\[
\leq 2 \| z \|_{L^\infty(\Omega)} \left( |\langle \xi, v \rangle| + |\Omega| (\lambda_0(-1/2) + \lambda_0(1/2)) \right).
\]

This actually shows that the linear functional \( z \mapsto \langle \xi, z \rangle \) defined on \( C^0(\Omega) \cap V \) (that is a dense subspace of \( C^0(\Omega) \)); recall that \( \Omega \) is smooth) is continuous with respect to the sup-norm. Thus, by the Riesz representation theorem, it can be represented over \( C^0(\Omega) \) by a measure \( T \in \mathcal{M}(\Omega) \).

Actually, we can give a general definition, saying that a functional \( \xi \in V' \) belongs to the space \( V' \cap \mathcal{M}(\Omega) \), provided that \( \xi \) is continuous with respect to the sup-norm on \( \Omega \). In this case, we can use (2.15) and say that the measure \( T \) represents \( \xi \) on \( \mathcal{M}(\Omega) \). We now recall a result [11, Thm. 3] that will be exploited in what follows.

**Theorem 2.2.** Let \( v \in V, \xi \in f_{0,w}(v) \). Then, denoting by \( \xi_a, \xi_s \) the the Lebesgue decomposition of \( \xi \), with \( \xi_a \) (respectively, \( \xi_s \)) standing for the absolute continuous (respectively, singular) part of \( \xi \), we have

\[
\langle \xi, v \rangle = \int_{\Omega} \xi_a(v) \, dx = \sup \left\{ \int_{\Omega} z \, d\xi_s, \ z \in C^0(\Omega), \ z(\Omega) \subset [-1,1] \right\}.
\]

Actually, in [11] a slightly different result is proved, where \( V \) is replaced by \( H^1_0(\Omega) \) and, correspondingly, \( \mathcal{M}(\Omega) \) is replaced by \( \mathcal{M}(\Omega) \) (i.e., the dual of \( C^0(\Omega) \)). Nevertheless, thanks to the smoothness of \( \Omega \), one can easily realize that the approximation procedure used in the proof of the theorem can be extended to cover the present
situation. The only difference is given by the fact that the singular part $\xi_s$ may be supported also on the boundary.

Let us now recall that, given a pair $X, Y$ of Banach spaces, a sequence of (multi-valued) operators $T_n : X \to 2^Y$ is said to G-converge (strongly) to $T$ iff
\begin{equation}
(2.21) \quad \forall (x, y) \in T, \exists (x_n, y_n) \in T_n \text{ such that } (x_n, y_n) \to (x, y) \text{ strongly in } X \times Y.
\end{equation}

We would like to apply this condition to an approximation of the monotone function $f_0$ that we now construct. Namely, for $\sigma \in (0, 1)$ (intended to go to 0 in the limit), we would like to have a family $\{f_\sigma\}$ of monotone functions such that
\begin{align}
(2.22) & \quad f_\sigma \in C^1(\mathbb{R}), \quad f'_\sigma \in L^\infty(\mathbb{R}), \quad f_\sigma(0) = 0, \\
(2.23) & \quad f_\sigma \to f_0 \text{ uniformly on compact subsets of } (-1, 1).
\end{align}

Moreover, noting
\begin{equation}
(2.24) \quad F_\sigma(r) := \int_0^r f_\sigma(s) \, ds \text{ for } r \in \mathbb{R},
\end{equation}
we ask that
\begin{equation}
(2.25) \quad F_\sigma(r) \geq \lambda r^2 - c
\end{equation}
for some $c \geq 0$ independent of $\sigma$ and for all $r \in \mathbb{R}$, $\sigma \in (0, 1)$, where $\lambda$ is as in (2.3) (note that the analogue of the above property holds for $F$ thanks to the first of (2.4)).

Moreover, we ask the monotonicity condition
\begin{equation}
(2.26) \quad F_{\sigma_1}(r) \leq F_{\sigma_2}(r) \quad \text{if } \sigma_2 \leq \sigma_1 \quad \text{and } \forall r \in \mathbb{R}.
\end{equation}

Finally, on account of the last assumption (2.4), we require that
\begin{equation}
(2.27) \quad \forall m > 0, \exists C_m \geq 0 : \quad f'_\sigma(r) - m|f_\sigma(r)| \geq -C_m \quad \forall r \in [-2, 2],
\end{equation}
with $C_m$ being independent of $\sigma$. Notice that it is sufficient to ask the above property for $r \in [-2, 2]$. The details of the construction of a family $\{f_\sigma\}$ fulfilling (2.22)–(2.27) are standard, and hence we leave them to the reader. For instance, one can first take Yosida regularizations (see, e.g., [12, Chap. 2]) and then mollify in order to get additional smoothness.

Thanks to the monotonicity property (2.26), we can apply [4, Thm. 3.20], which gives that
\begin{align}
(2.28) & \quad f_\sigma \text{ G-converges to } f_0 \text{ in } H \times H, \\
(2.29) & \quad f_\sigma \text{ G-converges to } f_{0,w} \text{ in } V \times V'.
\end{align}

A notable consequence of G-convergence is the following property, whose proof can be obtained by slightly modifying [7, Prop. 1.1, p. 42].

**Lemma 2.3.** Let $X$ be an Hilbert space, and let $B_\sigma, B$ be maximal monotone operators in $X \times X'$ such that
\begin{equation}
(2.30) \quad B_\sigma \text{ G-converges to } B \text{ in } X \times X',
\end{equation}
as $\sigma \searrow 0$. Let also, for any $\sigma > 0$, $v_\sigma \in X, \xi_\sigma \in X'$ such that $\xi_\sigma \in B_\sigma(v_\sigma)$. Finally, let us assume that, for some $v \in X, \xi \in X'$, there holds
\begin{align}
(2.31) & \quad v_\sigma \to v \text{ weakly in } X, \quad \xi_\sigma \to \xi \text{ weakly in } X', \\
(2.32) & \quad \limsup_{\sigma \searrow 0} \langle \xi_\sigma, v_\sigma \rangle_X \leq \langle \xi, v \rangle_X.
\end{align}

Then $\xi \in B(v)$. 

Next, we present an integration-by-parts formula.

**Lemma 2.4.** Let \( u \in W \cap H^{3}(\Omega) \), \( \xi \in V' \) such that \( \xi \in f_{0,w}(u) \). Then we have that

(2.33) \( \langle \xi, Au \rangle \geq 0 \).

**Proof.** Let us first note that the duality above surely makes sense in the assigned regularity setting. Actually, we have that \( Au \in V' \). We then consider the elliptic problem

(2.34) \[ u_{\sigma} \in V, \quad u_{\sigma} + A^{2}u_{\sigma} + f_{\sigma}(u_{\sigma}) = u + A^{2}u + \xi \quad \text{in} \quad V'. \]

Since \( f_{\sigma} \) is monotone and Lipschitz continuous and the above right-hand side lies in \( V' \), it is not difficult to show that the above problem admits a unique solution \( u_{\sigma} \in W \cap H^{3}(\Omega) \).

Moreover, the standard a priori estimates for \( u_{\sigma} \) lead to the following convergence relations, which hold, for some \( v \in V \) and \( \zeta \in V' \), up to the extraction of (nonrelabeled) subsequences (in fact, uniqueness guarantees them for the whole \( \sigma \searrow 0 \)):

(2.35) \[ u_{\sigma} \rightharpoonup v \quad \text{weakly in} \quad H^{3}(\Omega) \] and strongly in \( W \),

(2.36) \[ A^{2}u_{\sigma} \rightharpoonup A^{2}v \quad \text{weakly in} \quad V' \],

(2.37) \[ f_{\sigma}(u_{\sigma}) \rightharpoonup \zeta \quad \text{weakly in} \quad V'. \]

As a byproduct, the limit functions satisfy \( v + A^{2}v + \zeta = u + A^{2}u + \xi \) in \( V' \). Moreover, we deduce from (2.34)

(2.38) \[ (f_{\sigma}(u_{\sigma}), u_{\sigma}) = \langle u + A^{2}u + \xi - u_{\sigma} - A^{2}u_{\sigma}, u_{\sigma} \rangle, \]

whence

(2.39) \[ \lim_{\sigma \to 0} (f_{\sigma}(u_{\sigma}), u_{\sigma}) = \langle u + A^{2}u + \xi - v - A^{2}v, v \rangle = \langle \zeta, v \rangle. \]

Then, on account of (2.35), (2.37), (2.29), and Lemma 2.3 (cf., in particular, relation (2.29)) applied to the sequence \( \{f_{\sigma}(v_{\sigma})\} \), we readily obtain that \( \zeta \in f_{0,w}(v) \). By uniqueness, \( v = u \) and \( \zeta = \xi \).

Let us finally verify the required property. Actually, for \( \sigma > 0 \), thanks to monotonicity of \( f_{\sigma} \) we have

(2.40) \[ 0 \leq (f_{\sigma}(u_{\sigma}), Au_{\sigma}) = \langle u + A^{2}u + \xi - u_{\sigma} - A^{2}u_{\sigma}, Au_{\sigma} \rangle. \]

Taking the supremum limit, we then obtain

(2.41) \[ 0 \leq \limsup_{\sigma \searrow 0} \langle u + A^{2}u + \xi - u_{\sigma} - A^{2}u_{\sigma}, Au_{\sigma} \rangle \]

\[ = \langle u + A^{2}u + \xi - u, Au \rangle - \liminf_{\sigma \searrow 0} \langle A^{2}u_{\sigma}, Au_{\sigma} \rangle. \]

Then, using (2.35) and semicontinuity of norms with respect to weak convergence,

(2.42) \[ -\liminf_{\sigma \searrow 0} \langle A^{2}u_{\sigma}, Au_{\sigma} \rangle = -\liminf_{\sigma \searrow 0} \| \nabla Au_{\sigma} \|^{2} \leq -\| \nabla Au \|^{2} = -\langle A^{2}u, Au \rangle, \]

whence we finally obtain

(2.43) \[ 0 \leq \langle u + A^{2}u + \xi - u - A^{2}u, Au \rangle = \langle \xi, Au \rangle, \]

as desired. \( \square \)
Next, we recall a further integration-by-parts formula that extends the classical result [12, Lem. 3.3, p. 73] (see, e.g., [43, Lem. 4.1] for a proof).

**Lemma 2.5.** Let \( T > 0 \) and let \( J : H \to [0, +\infty] \) be a convex, lower semicontinuous, and proper functional. Let \( u \in H^1(0, T; V') \cap L^2(0, T; V) \), \( \eta \in L^2(0, T; V) \) and let \( \eta(t) \in \partial J(u(t)) \) for a.e. \( t \in (0, T) \), where \( \partial J \) is the \( H \)-subdifferential of \( J \). Moreover, let us suppose the coercivity property

\[ \exists k_1 > 0, k_2 \geq 0 \text{ such that } J(v) \geq k_1 \|v\|^2 - k_2 \quad \forall v \in H. \]

Then the function \( t \mapsto J(u(t)) \) is absolutely continuous in \( [0, T] \) and

\[ \frac{d}{dt} J(u(t)) = \langle u_t(t), \eta(t) \rangle \quad \text{for a.e. } t \in (0, T). \]

In particular, integrating in time, we have

\[ \int_s^t \langle u_t(r), \eta(r) \rangle \, dr = J(u(t)) - J(u(s)) \quad \forall s, t \in [0, T]. \]

We conclude this section by stating an integration-by-parts formula for the operator \( A \).

**Lemma 2.6.** Let \( a \) satisfy (2.1) and let either

\[ v \in H^1(0, T; H) \cap L^2(0, T; W) \cap L^\infty(Q) \]

or

\[ v \in H^1(0, T; V') \cap L^\infty(0, T; W) \cap L^2(0, T; H^3(\Omega)). \]

Then the function

\[ t \mapsto \int_{\Omega} \frac{a(v(t))}{2} |\nabla v(t)|^2 \]

is absolutely continuous over \( [0, T] \). Moreover, for all \( s, t \in [0, T] \) we have that

\[ \int_s^t \langle A(v(r)), v_t(r) \rangle \, dr = \int_{\Omega} \frac{a(v(t))}{2} |\nabla v(t)|^2 - \int_{\Omega} \frac{a(v(s))}{2} |\nabla v(s)|^2, \]

where, in the case (2.48), the scalar product in the integral on the left-hand side has to be replaced with the duality \( \langle v_t(r), A(v(r)) \rangle \).

**Proof.** We first notice that (2.49)–(2.50) surely hold if \( v \) is smoother. Then we can proceed by first regularizing \( v \) and then passing to the limit. Namely, we define \( v_{\sigma} \), a.e. in \( (0, T) \), as the solution of the singular perturbation problem

\[ v_{\sigma} + \sigma A v_{\sigma} = v \quad \text{for } \sigma \in (0, 1). \]

Then, in the case (2.47), we have

\[ v_{\sigma} \in H^1(0, T; W) \cap L^2(0, T; H^4(\Omega)), \]

whereas if (2.48) holds, we get

\[ v_{\sigma} \in H^1(0, T; V) \cap L^\infty(0, T; H^4(\Omega)). \]
Moreover, proceeding as in [20, Appendix] (cf., in particular, Proposition 6.1 therein) and applying the Lebesgue dominated convergence theorem in order to control the dependence on the time variable, we can easily prove that

\[ v_\sigma \to v \ 	ext{strongly in } H^1(0,T; H) \cap L^2(0,T; W) \]  
(2.54) and weakly star in \( L^\infty(0,T) \) (the latter condition following from the maximum principle) if (2.47) holds, or

\[ v_\sigma \to v \ 	ext{strongly in } H^1(0,T; V') \cap L^2(0,T; H^3(\Omega)) \]  
(2.55) and weakly star in \( L^\infty(0,T; W) \) if (2.48) is satisfied instead.

To prove that

\[ A(v_\sigma) \to A(v) \]  
(at least) weakly in \( L^2(0,T; H) \).

In particular, to control the square gradient term in \( A \), we use the Gagliardo–Nirenberg inequality (cf. [40])

\[ \| \nabla z \|_{L^4(\Omega)} \leq C_\Omega \| z \|_{W}^{1/2} \| z \|_{L^\infty(\Omega)}^{1/2} \quad \forall z \in W, \]  
(2.58)  
so that, thanks also to (2.1),

\[ \| a'(v_\sigma) \nabla v_\sigma \|_{L^2(0,T;H)}^2 \leq \| a'(v_\sigma) \|_{L^\infty(Q)} \| \nabla v_\sigma \|_{L^4(Q)}^2 \leq c \| v_\sigma \|_{L^\infty(Q)} (1 + \| A v_\sigma \|_{L^2(0,T;H)}), \]  
(2.59)

and (2.57) follows. Moreover, by (2.54) and the continuous embedding \( H^1(0,T; H) \cap L^2(0,T; W) \subset C^0([0,T]; V) \), we also have that

\[ v_\sigma \to v \ 	ext{strongly in } C^0([0,T]; V). \]  
(2.60)

Combining (2.54), (2.57), and (2.60), we can take the limit \( \sigma \searrow 0 \) in (2.56) and get back (2.50). Then the absolute continuity property of the functional in (2.49) follows from the summability of the integrand on the left-hand side of (2.50).

Finally, let us come to the case (2.48). Then (2.55) and the Aubin–Lions theorem give directly (2.60), so that we can pass to the limit on the right-hand side of (2.56). To take the limit of the left-hand side, on account of the first (2.55), it is sufficient to prove that

\[ A(v_\sigma) \to A(v) \]  
(at least weakly in \( L^2(0,T; V) \).

Since weak convergence surely holds in \( L^2(0,T; H) \), it is then sufficient to prove uniform boundedness in \( L^2(0,T; V) \). With this aim, we compute

\[ \nabla \left( a(v_\sigma) \Delta v_\sigma + \frac{a'(v_\sigma)}{2} |\nabla v_\sigma|^2 \right) \]

\[ = a'(v_\sigma) \nabla v_\sigma \Delta v_\sigma + a(v_\sigma) \nabla \Delta v_\sigma + \frac{a''(v_\sigma)}{2} |\nabla v_\sigma|^2 \nabla v_\sigma + a'(v_\sigma) D^2 v_\sigma \nabla v_\sigma, \]  
(2.62)

and, using (2.55), (2.1), and standard embedding properties of Sobolev spaces, it is a standard procedure to verify that the right-hand side is uniformly bounded in \( L^2(0,T; H) \) (and, consequently, so is the left). This concludes the proof.
3. The sixth order problem. We start by introducing the concept of weak solution to the sixth order problem associated with system (1.1–(1.4).

**Definition 3.1.** Let $\delta > 0$ and $\varepsilon \geq 0$. Let us consider the sixth order problem given by the system

\begin{align*}
(3.1) & \quad u_t + Aw = 0 \quad \text{in } V', \\
(3.2) & \quad w = \delta A^2 u + A(u) + \xi - \lambda u + \varepsilon u_t \quad \text{in } V', \\
(3.3) & \quad \xi \in f_{0,w}(u),
\end{align*}

together with the initial condition

\begin{align*}
(3.4) & \quad u|_{t=0} = u_0 \quad \text{a.e. in } \Omega.
\end{align*}

A (global-in-time) weak solution to the sixth order problem (3.1)–(3.4) is a triplet $(u, w, \xi)$, with

\begin{align*}
(3.5) & \quad u \in H^1(0, T; V') \cap L^\infty(0, T; W) \cap L^2(0, T; H^3(\Omega)), \\
(3.6) & \quad F(u) \in L^\infty(0, T; L^1(\Omega)), \\
(3.7) & \quad \xi \in L^2(0, T; V'), \\
(3.8) & \quad w \in L^2(0, T; V),
\end{align*}

satisfying (3.1)–(3.3) a.e. in $(0, T)$ together with (3.4).

We can then state the main result of this section.

**Theorem 3.2.** Let us assume (2.1)–(2.4). Let $\varepsilon \geq 0$ and $\delta > 0$. Moreover, let us suppose that

\begin{align*}
(3.9) & \quad u_0 \in W, \quad F(u_0) \in L^1(\Omega), \quad (u_0)_\Omega \in (-1, 1),
\end{align*}

where $(u_0)_\Omega$ is the spatial mean of $u_0$. Then the sixth order problem admits one and only one weak solution.

The proof of the theorem will be carried out in several steps, presented as separate subsequences.

**Remark 3.3.** We observe that the last condition in (3.9), which is a common assumption when dealing with Cahn–Hilliard equations with constraints (cf. [32] for more details), does not simply follow from the requirement $F(u_0) \in L^1(\Omega)$. Indeed, $F$ may be bounded over $[-1, 1]$, as happens, for instance, with the logarithmic potential (1.5). In that case, $F(u_0) \in L^1(\Omega)$ simply means $-1 \leq u_0 \leq 1$ almost everywhere and, without the last (3.9), we could have initial data that coincide almost everywhere with either of the pure states $\pm 1$. However, solutions that assume (for example) the value $+1$ in a set of strictly positive measure cannot be considered, at least not in our regularity setting. Indeed, if $|\{u = 1\}| > 0$, then regularity (3.7) (which is crucial for passing to the limit in our approximation scheme) is broken, because $f(r)$ is unbounded for $r \nearrow +1$ and $\xi$ is nothing else than a relaxed version of $f(u)$.

### 3.1. Approximation and local existence.

First of all, we introduce a suitably approximated statement. The monotone function $f_0$ is regularized by taking a family $\{f_\sigma\}, \sigma \in (0, 1)$, defined as in subsection 2.1. Next, we regularize $u_0$ by singular perturbation, similarly as before (cf. (2.51)). Namely, we take $u_{0,\sigma}$ as the solution to the elliptic problem

\begin{align*}
(3.10) & \quad u_{0,\sigma} + \sigma Au_{0,\sigma} = u_0,
\end{align*}
and we clearly have, by Hilbert elliptic regularity results,

\[(3.11)\quad u_{0,\sigma} \in D(A^2) \quad \forall \sigma \in (0,1).\]

Other types of approximations of the initial datum are possible, of course. The choice \((3.10)\), beyond its simplicity, has the advantage that it preserves the mean value.

**Approximate problem.** For \(\sigma \in (0,1)\), we consider the problem

\[(3.12)\quad u_t + A w = 0,\]
\[(3.13)\quad w = \delta A^2 u + A(u) + f_\sigma(u) - \lambda u + (\varepsilon + \sigma) u_t,\]
\[(3.14)\quad u|_{t=0} = u_{0,\sigma} \quad \text{a.e. in } \Omega.\]

We shall now show that it admits at least one local-in-time weak solution. Namely, there holds the following.

**Lemma 3.4.** Let us assume \((2.1)–(2.4)\). Then, for any \(\sigma \in (0,1)\), there exist \(T_0 \in (0,T]\) (possibly depending on \(\sigma\)) and a pair \((u,w)\) with

\[(3.15)\quad u \in H^1(0,T_0;H) \cap L^\infty(0,T_0;W) \cap L^2(0,T_0;D(A^2)),\]
\[(3.16)\quad w \in L^2(0,T_0;W),\]

such that \((3.12)–(3.13)\) hold a.e. in \((0,T_0]\) and the initial condition \((3.14)\) is satisfied.

**Proof.** The theorem will be proved by using the Schauder fixed point theorem. We take

\[(3.17)\quad B_R := \{ v \in L^2(0,T_0;W) \cap L^4(0,T_0;W^{1,4}(\Omega)) : \|v\|_{L^2(0,T_0;W)} + \|v\|_{L^4(0,T_0;W^{1,4}(\Omega))} \leq R \}\]

for \(T_0\) and \(R\) to be chosen below. Then we take \(\overline{\pi} \in B_R\) and consider the problem given by \((3.14)\) and

\[(3.18)\quad u_t + A w = 0 \quad \text{in } H,\]
\[(3.19)\quad w = \delta A^2 u + A(\overline{\pi}) + f_\sigma(u) - \lambda u + (\varepsilon + \sigma) u_t \quad \text{in } H.\]

Then, as \(\overline{\pi} \in B_R\) is fixed, we can notice that

\[(3.20)\quad \|A(\overline{\pi})\|_{L^2(0,T_0;H)} \leq c(\|\overline{\pi}\|_{L^2(0,T_0;W)} + \|\overline{\pi}\|_{L^4(0,T_0;W^{1,4}(\Omega))}) \leq Q(R).\]

Here and below, \(Q\) denotes a computable function, possibly depending on \(\sigma\), defined for any nonnegative value of its argument(s) and increasingly monotone in (each of) its argument(s).

As we substitute into \((3.18)\) the expression for \(w\) given by \((3.19)\) and apply the inverse operator \((\text{Id} + (\varepsilon + \sigma) A)^{-1}\), we obtain a parabolic equation in \(u\) which is linear up to the Lipschitz perturbation \(f_\sigma(u)\). Hence, owing to the regularity \((3.20)\) of the forcing term, to the regularity \((3.11)\) of the initial datum, and to the standard Hilbert theory of linear parabolic equations, there exists a unique pair \((u,w)\) solving the problem given by \((3.18)–(3.19)\) and the initial condition \((3.14)\). Such a pair satisfies the regularity properties \((3.15)–(3.16)\) (as it will also be apparent from the forthcoming a priori estimates). We then note as \(K\) the map such that \(K : \overline{\pi} \mapsto u\). To conclude the proof we will have to show the following three properties:

(i) \(K\) takes its values in \(B_R\);

(ii) \(K\) is continuous with respect to the \(L^2(0,T_0;W)\) and the \(L^4(0,T_0;W^{1,4}(\Omega))\) norms;
(iii) $\mathcal{K}$ is a compact map.
To prove these facts, we perform a couple of a priori estimates. To start, we test (3.18) by $w$ and (3.19) by $u_t$ (energy estimate). This gives
\begin{equation}
\frac{d}{dt} \left( \frac{\delta}{2} \|Au\|^2 + \int_{\Omega} \left( F_\sigma(u) - \frac{\lambda}{2} u^2 \right) \right) + (\varepsilon + \sigma)\|u_t\|^2 + \|\nabla w\|^2
\end{equation}
and, after integration in time, the latter term can be estimated using (3.20). Next, we observe that, thanks to (2.25), we have
\begin{equation}
\frac{\delta}{2} \|Au\|^2 + \int_{\Omega} \left( F_\sigma(u) - \frac{\lambda}{2} u^2 \right) \geq \eta\|u\|_{W}^2 - c
\end{equation}
for some $\eta > 0$, $c \geq 0$ independent of $\sigma$, and for all $u$ in $W$. Thus, (3.21) provides the bounds
\begin{equation}
\|u\|_{L^\infty(0,T_0;W)} + \|u_t\|_{L^2(0,T_0;H)} + \|\nabla w\|_{L^2(0,T_0;H)} \leq Q(R,T_0,\|u_{0,\sigma}\|_W).
\end{equation}
Next, testing (3.19) by $A^2u$ and performing some standard computations (in particular, the terms $(A(\varpi), A^2u)$ and $(f_\sigma(u), A^2u)$ are controlled by using (3.20), Hölder’s and Young’s inequalities, and the Lipschitz continuity of $f_\sigma$), we obtain the further bound
\begin{equation}
\|A^2u\|_{L^2(0,T_0;H)} \leq Q(R,T_0,\|u_{0,\sigma}\|_W).
\end{equation}
Hence, estimates (3.23) and (3.24) and a standard application of the Aubin–Lions lemma permit us to see that the range of $\mathcal{K}$ is relatively compact both in $L^2(0,T_0;W)$ and in $L^4(0,T_0;W^{1,4}(\Omega))$. Thus, (iii) follows.

Concerning (i), we can now simply observe that, by (3.23),
\begin{equation}
\|u\|_{L^2(0,T_0;W)} \leq T_0^{1/2} \|u\|_{L^\infty(0,T_0;W)} \leq T_0^{1/2}Q(R,T_0,\|u_{0,\sigma}\|_W),
\end{equation}
whence the right-hand side can be made smaller than $R$ if $T_0$ is chosen small enough. A similar estimate works also for the $L^4(0,T_0;W^{1,4}(\Omega))$-norm since $W \subset W^{1,4}(\Omega)$ continuously. Thus, also (i) is proved.

Finally, to prove condition (ii), we first observe that if $\{\varpi_n\} \subset B_R$ converges strongly to $\varpi$ in $L^2(0,T_0;W) \cap L^4(0,T_0;W^{1,4}(\Omega))$, then, using proper weak compactness theorems, it is not difficult to prove that
\begin{equation}
A(\varpi_n) \to A(\varpi) \text{ weakly in } L^2(0,T_0;H).
\end{equation}
Consequently, if $u_n$ (respectively, $u$) is the solution to (3.18)–(3.19) corresponding to $\varpi_n$ (respectively, $\varpi$), then estimates (3.23)–(3.24) hold for the sequence $\{u_n\}$ with a function $Q$ independent of $n$. Hence, standard weak compactness arguments together with the Lipschitz continuity of $f_\sigma$ permit us to prove that
\begin{equation}
u_n = \mathcal{K}(\varpi_n) \to u = \mathcal{K}(\varpi) \text{ strongly in } L^2(0,T_0;W) \cap L^4(0,T_0;W^{1,4}(\Omega)),
\end{equation}
i.e., condition (ii). The proof of the lemma is concluded. □
3.2. A priori estimates. In this section we will show that the local solutions constructed in the previous section satisfy uniform estimates with respect to both the approximation parameter \( \sigma \) and the time \( T_0 \). By standard extension methods this will yield a global-in-time solution (i.e., defined over the whole of \((0,T)\)) in the limit. However, to avoid technical complications, we will directly assume that the approximating solutions are already defined over \((0,T)\). Of course, to justify this, we will have to take care that all the constants appearing in the forthcoming estimates be independent of \( T_0 \). To be precise, in what follows we will note by \( c > 0 \) a computable positive constant (whose value can vary on occurrence) independent of all approximation parameters (in particular of \( T_0 \) and \( \sigma \)) and also of the parameters \( \varepsilon \) and \( \delta \).

Energy estimate. First, integrating (3.12) in space and recalling (3.10), we obtain the mass conservation property

\[
(u(t))_\Omega = (u_{0,\sigma})_\Omega = (u_0)_\Omega.
\]

Next, we can test (3.12) by \( w \) and (3.13) by \( u_t \) and take the difference, arriving at

\[
\frac{d}{dt} E_{\sigma,\delta}(u) + \| \nabla w \|^2 + (\varepsilon + \sigma) \| u_t \|^2 = 0,
\]

where the “approximate energy” \( E_{\sigma,\delta}(u) \) is defined as

\[
E_{\sigma,\delta}(u) = \int _\Omega \left( \frac{\delta}{2} |Au|^2 + \frac{a(u)}{2} |\nabla u|^2 + F_\sigma(u) - \frac{\lambda}{2} u^2 \right).
\]

Actually, it is clear that the high regularity of approximate solutions (cf. (3.15)–(3.16)) allows the integration by parts necessary to write (3.29) (at least) almost everywhere in time. Indeed, all single terms in (3.13) lie in \( L^2(0,T;H) \), and the same holds for the test function \( u_t \).

Then we integrate (3.29) in time and notice that, by (2.25),

\[
E_{\sigma,\delta}(u) \geq \eta (\delta \| u \|^2_W + \| u \|^2_W) - c \quad \forall t \in (0,T).
\]

Consequently, (3.29) provides the bounds

\[
\| u \|_{L^\infty(0,T;V)} + \delta^{1/2} \| u \|_{L^\infty(0,T;W)} + (\varepsilon + \sigma)^{1/2} \| u_t \|_{L^2(0,T;H)} \leq c,
\]

\[
\| \nabla w \|_{L^2(0,T;H)} \leq c,
\]

\[
\| F_\sigma(u) \|_{L^\infty(0,T;L^1(\Omega))} \leq c,
\]

where it is worth stressing once more that the above constants \( c \) depend explicitly neither on \( \delta \) nor on \( \varepsilon \).

Second estimate. We test (3.13) by \( u - u_\Omega \), \( u_\Omega \) denoting the (constant-in-time) spatial mean of \( u \). Integrating by parts the term \( A(u) \), we obtain

\[
\delta \| Au \|^2 + \int _\Omega a(u) |\nabla u|^2 + \int _\Omega f_\sigma(u)(u - u_\Omega)
\]

\[
\leq (w + \lambda u - (\varepsilon + \sigma) u_t, u - u_\Omega) - \int _\Omega \frac{a'(u)}{2} |\nabla u|^2 (u - u_\Omega),
\]
and we have to estimate some terms. First of all, we observe that there exists a constant $c$, depending on the (assigned once $u_0$ is fixed) value of $u_{\Omega}$, but independent of $\sigma$, such that

\begin{equation}
\int_{\Omega} f_{\sigma}(u)(u-u_{\Omega}) \geq \frac{1}{2} ||f_{\sigma}(u)||_{L^1(\Omega)} - c.
\end{equation}

To prove this inequality, one basically uses the monotonicity of $f_{\sigma}$ and the fact that $f_{\sigma}(0) = 0$ (cf. [39, Appendix] or [27, third a priori estimate] for the details). Next, by (2.2), the function $r \mapsto a'(r)(r - u_{\Omega})$ is uniformly bounded, whence

\begin{equation}
- \int_{\Omega} \frac{a'(u)}{2} |\nabla u|^2 (u - u_{\Omega}) \leq c ||\nabla u||^2.
\end{equation}

Finally, using that $(w + \lambda u, u - u_{\Omega}) = 0$ since $u_{\Omega} + \lambda u_{\Omega}$ is constant with respect to space variables, and applying the Poincaré–Wirtinger inequality,

\begin{align*}
(w + \lambda u - (\varepsilon + \sigma) u_t, u - u_{\Omega}) &= (w - w_{\Omega} + \lambda(u - u_{\Omega}) - (\varepsilon + \sigma) u_t, u - u_{\Omega}) \\
&\leq c ||\nabla w|| ||\nabla u|| + c ||\nabla u||^2 + c(\varepsilon + \sigma) ||u_t|| ||\nabla u|| \\
&\leq c(||\nabla w|| + (\varepsilon + \sigma) ||u_t|| + 1),
\end{align*}

the latter inequality following from estimate (3.32).

Thus, squaring (3.35), using (3.36)–(3.38), and integrating in time, we arrive after recalling (3.32), (3.33) at

\begin{equation}
\|f_{\sigma}(u)\|_{L^2(0,T;L^1(\Omega))} \leq c.
\end{equation}

Next, integrating (1.13) with respect to space variables (and, in particular, integrating by parts the term $A(u)$), using (3.39), and recalling (3.33), we obtain (still for $c$ independent of $\varepsilon$ and $\delta$)

\begin{equation}
\|w\|_{L^2(0,T;V)} \leq c.
\end{equation}

**Third estimate.** We test (1.13) by $Au$. Using the monotonicity of $f_{\sigma}$ and (2.1), it is not difficult to arrive at

\begin{equation}
\frac{\varepsilon + \sigma}{2} \frac{d}{dt} ||\nabla u||^2 + \delta ||\nabla u||^2 + \frac{\alpha}{2} ||Au||^2 \leq (\nabla w + \lambda \nabla u, \nabla u) + c ||\nabla u||^{4/3}_{L^4(\Omega)}.
\end{equation}

Using the continuous embedding $H^{3/4}(\Omega) \subset L^4(\Omega)$ (so that, in particular, $H^{7/4}(\Omega) \subset W^{1,4}(\Omega)$) together with the interpolation inequality

\begin{equation}
\|v\|_{H^{3/4}(\Omega)} \leq \|v\|^{3/8}_{H^3(\Omega)} ||v||^{5/8}_{H^1(\Omega)} \quad \forall v \in H^{3/4}(\Omega),
\end{equation}

and recalling estimate (3.32), the last term is treated as follows:

\begin{equation}
c ||\nabla u||^{4/3}_{L^4(\Omega)} \leq c ||u||^{3/2}_{H^1(\Omega)} ||u||^{5/2}_{H^1(\Omega)} \leq \frac{\delta}{2} ||\nabla u||^2 + c(\delta).
\end{equation}

Note that the latter constant $c(\delta)$ is expected to explode as $\delta \searrow 0$ but, on the other hand, is independent of $\sigma$. Next, noting that

\begin{equation}
(\nabla w + \lambda \nabla u, \nabla u) \leq c(||\nabla u||^2 + ||\nabla w||^2),
\end{equation}
from (3.41) we readily deduce

\[(3.45)\quad \|u\|_{L^2(0,T;H^3(\Omega))} \leq c(\delta) .\]

A similar (and even simpler) argument permits us to check that it is also

\[(3.46)\quad \|\mathcal{A}(u)\|_{L^2(0,T;H)} \leq c(\delta) .\]

Thus, using (3.32), (3.40), (3.45)–(3.46) and comparing terms in (3.13), we arrive at

\[(3.47)\quad \|f_\sigma(u)\|_{L^2(0,T;W')} \leq c(\delta) .\]

### 3.3. Limit $\sigma \searrow 0$. We now use the machinery introduced in subsection 2.1 to take the limit $\sigma \searrow 0$ in (3.12)–(3.13). For convenience, we then rename as $(u_\sigma, w_\sigma)$ the solution. Then, recalling estimates (3.32)–(3.34), (3.40), and (3.45)–(3.47), and using the Aubin–Lions compactness lemma, we deduce

\[(3.48)\quad u_\sigma \rightarrow u \quad \text{strongly in } C^0([0,T];H^{2,\epsilon}(\Omega)) \cap L^2(0,T;H^{3,\epsilon}(\Omega)),\]

\[(3.49)\quad u_\sigma \rightarrow u \quad \text{weakly in } H^1(0,T;V') \cap L^\infty(0,T;W) \cap L^2(0,T;H^3(\Omega)),\]

\[(3.50)\quad (\epsilon + \sigma)u_{\sigma,t} \rightarrow \epsilon u_t \quad \text{weakly in } L^2(0,T;H),\]

\[(3.51)\quad w_\sigma \rightarrow w \quad \text{weakly in } L^2(0,T;V),\]

\[(3.52)\quad f_\sigma(u_\sigma) \rightarrow \xi \quad \text{weakly in } L^2(0,T;V').\]

for suitable limit functions $u, w, \xi$, where $\epsilon > 0$ is arbitrarily small. It is readily checked that the above relations ((3.48) in particular) are strong enough to guarantee that

\[(3.53)\quad \mathcal{A}(u_\sigma) \rightarrow \mathcal{A}(u) \quad \text{strongly in } L^2(0,T;H).\]

This allows us to take the limit $\sigma \searrow 0$ in (3.12)–(3.14) (rewritten for $u_\sigma, w_\sigma$) and get

\[(3.54)\quad u_t + Aw = 0 \quad \text{in } V',\]

\[(3.55)\quad w = \delta A^2 u + \mathcal{A}(u) + \xi - \lambda u + \epsilon u_t \quad \text{in } V',\]

\[(3.56)\quad u|_{t=0} = u_0 \quad \text{a.e. in } \Omega.\]

To identify $\xi$, we observe that, thanks to (3.48), (3.52), and Lemma 2.3 applied with the choices of $X = V, X' = V', B_\sigma = f_\sigma, B = f_{0,w}, v_\sigma = u_\sigma, v = u,$ and $\xi_\sigma = f_\sigma(u_\sigma)$, it follows that

\[(3.57)\quad \xi \in f_{0,w}(u).\]

Namely, $\xi$ is identified with respect to the weak (duality) expression of the function $f_0$. This concludes the proof of Theorem 3.2 for what concerns existence.

### 3.4. Uniqueness. To conclude the proof of Theorem 3.2, it remains to prove uniqueness. To this purpose, we write both (3.1) and (3.2) for a couple of solutions $(u_1, w_1, \xi_1), (u_2, w_2, \xi_2)$ and take the difference. This gives

\[(3.58)\quad u_t + Aw = 0 \quad \text{in } V',\]

\[w = \delta A^2 u - a(u_1)\Delta u - (a(u_1) - a(u_2))\Delta w_2 - \frac{a'(u_1)}{2}(|\nabla u_1|^2 - |\nabla u_2|^2)\]

\[- \frac{a'(u_1) - a'(u_2)}{2}|\nabla u_2|^2 + \xi_1 - \xi_2 - \lambda u + \epsilon u_t \quad \text{in } V'.\]
where we have set \((u, w, \xi) := (u_1, w_1, \xi_1) - (u_2, w_2, \xi_2)\). Then we test (3.58) by \(A^{-1}u\) and (3.59) by \(u\) and take the difference. Notice that, indeed, \(u\) has zero mean value by (3.28). Thus, the operator \(A^{-1}\) makes sense since \(A\) is bijective from \(W_0\) to \(H_0\), the subscript 0 indicating the zero-mean condition.

A straightforward computation involving use of standard embedding properties of Sobolev spaces then gives

\[
\begin{align*}
\left(-a(u_1)\Delta - a(u_1)\Delta + a(u_2)\Delta - a(u_2)\Delta\frac{1}{2}(|\nabla u_1|^2 - |\nabla u_2|^2) - a(u_1) - a(u_2)\right)^{\frac{1}{2}} u
\right)
\leq Q(\|u_1\|_{L^\infty(\Omega ; W)}, \|u_2\|_{L^\infty(\Omega ; W)}) \|u\|_W \|u\|
\end{align*}
\]

and we notice that the norms inside the function \(Q\) are controlled thanks to (3.5). Thus, also on account of the monotonicity of \(f_0, w\), we arrive at

\[
\begin{align*}
\frac{d}{dt} \frac{1}{2} \|u\|_V^2 + \frac{\varepsilon}{2} \|u\|_V^2 + \delta \|Au\|_V^2 \leq c \|u\|_W \|u\| + \lambda \|u\|^2
\end{align*}
\]

where, to deduce the last two inequalities, we used the interpolation inequality \(\|u\| \leq \|u\|_V^2/\|u\|_V^{1/3}\) (note that \((V, H, V')\) form a Hilbert triplet; cf., e.g., [13, Chap. 5]), together with Young’s inequality and the fact that the function \(\|\cdot\|_V + \|A\cdot\|\) is an equivalent norm on \(W\). Thus, the thesis of Theorem 3.2 follows by applying Gronwall’s lemma to (3.61).

4. From the sixth order to the fourth order model. In this section, we analyze the behavior of solutions to the sixth order problem as \(\delta\) tends to 0. To start with, we specify the concept of weak solution in the fourth order case.

Definition 4.1. Let \(\delta = 0\) and \(\varepsilon \geq 0\). Let us consider the fourth order problem given by the system

\[
\begin{align*}
\text{(4.1)} & \quad u_t + Aw = 0 \quad \text{in} \ V', \\
\text{(4.2)} & \quad w = A(u) + f(u) + \varepsilon u_t \quad \text{in} \ H,
\end{align*}
\]

together with the initial condition (3.4). A (global-in-time) weak solution to the fourth order problem (4.1)–(4.2), (3.4) is a pair \((u, w)\), with

\[
\begin{align*}
\text{(4.3)} & \quad u \in H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad \varepsilon u \in H^1(0, T; H), \\
\text{(4.4)} & \quad F(u) \in L^\infty(0, T; L^1(\Omega)), \\
\text{(4.5)} & \quad f_0(u) \in L^2(0, T; H), \\
\text{(4.6)} & \quad w \in L^2(0, T; V),
\end{align*}
\]

satisfying (4.1)–(4.2) a.e. in \((0, T)\) together with (3.4).

Theorem 4.2. Let us assume (2.1)–(2.4) together with

\[
\text{(4.7)} \quad a \text{ is concave on } [-1, 1].
\]

Also let \(\varepsilon \geq 0\) and let, for all \(\delta \in (0, 1)\), \(u_{0, \delta}\) be an initial datum satisfying (3.9). Moreover, let us suppose

\[
\begin{align*}
\text{(4.8)} & \quad u_{0, \delta} \to u_0 \text{ strongly in } V, \quad \mathcal{E}_\delta(u_{0, \delta}) \to \mathcal{E}_0(u_0), \text{ where } (u_0)_\Omega \in (-1, 1).
\end{align*}
\]
Let, for any \( \delta \in (0, 1) \), \((u_\delta, w_\delta, \xi_\delta)\) be a weak solution to the sixth order system in the sense of Definition 3.1. Then we have that, up to a (nonrelabeled) subsequence of \( \delta \searrow 0 \),

\[
\begin{align*}
&u_\delta \rightarrow u \text{ weakly star in } H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^2(0, T; W), \\
&\varepsilon w_\delta \rightarrow \varepsilon u \text{ weakly in } H^1(0, T; H), \\
&w_\delta \rightarrow w \text{ weakly in } L^2(0, T; V), \\
&\delta u_\delta \rightarrow 0 \text{ strongly in } L^2(0, T; H^3(\Omega)), \\
&\xi_\delta \rightarrow f_0(u) \text{ weakly in } L^2(0, T; V'),
\end{align*}
\]

and \((u, w)\) is a weak solution to the fourth order problem.

Proof. The first part of the proof consists in repeating the “energy estimate” and the “second estimate” of the previous section. In fact, we could avoid this procedure since we already noted that the constants appearing in those estimates were independent of \( \delta \). However, we choose to perform once more the estimates working directly on the sixth order problem (rather than on its approximation) for various reasons. First, this will show that the estimates do not depend on the chosen regularization scheme. Second, the procedure has an independent interest since we will see that the use of “weak” subdifferential operators still permits us to rely on suitable integration-by-parts formulas and on monotonicity methods. Of course, many passages, which were trivial in the “strong” setting, now need a precise justification. Finally, in this way we are able to prove, as a byproduct, that any solution to the sixth order system satisfies an energy equality (and not just an inequality). Actually, this property may be useful for addressing the long-time behavior of the system.

Energy estimate. As before, we would like to test (3.1) by \( w_\delta \) and (3.2) by \( u_{\delta,t} \) and take the difference. To justify this procedure, we start observing that \( w_\delta \in L^2(0, T; V) \) by (3.8). Actually, since (3.1) is in fact a relation in \( L^2(0, T; V') \), the use of \( w_\delta \) as a test function makes sense. The problem, instead, arises when working on (3.2) and, to justify the estimate, we can just consider the (more difficult) case \( \varepsilon = 0 \).

Then it is easy to check that the assumptions of Lemma 2.6 are satisfied. In particular, we have (2.48) thanks to (3.5). Hence, (2.50) gives

\[
\langle u_{\delta,t}, A(u_\delta) \rangle = \frac{1}{2} \frac{d}{dt} \int_\Omega a(u_\delta)|\nabla u_\delta|^2 \text{ a.e. in } (0, T).
\]

Thus, it remains to show that

\[
\langle u_{\delta,t}, \delta A^2 u_\delta + \xi_\delta \rangle = \frac{d}{dt} \int_\Omega \left( \frac{\delta}{2} |Au_\delta|^2 + F(u_\delta) \right) \text{ a.e. in } (0, T).
\]

To prove this, we observe that

\[
\delta A^2 u_\delta + \xi_\delta \in L^2(0, T; V') \quad \forall \delta \in (0, 1).
\]

Actually, we already noted above that \( w_\delta, A(u_\delta) \) lie in \( L^2(0, T; V) \). Since \( u_\delta \in L^2(0, T; V) \) by (3.5) and we assumed \( \varepsilon = 0 \), (4.16) simply follows by comparing terms in (3.2). Thus, the duality on the left-hand side of (4.15) makes sense. Moreover, as we set

\[
\mathcal{J}_\delta(v) := \int_\Omega \left( \frac{\delta}{2} |Av|^2 + F(v) \right),
\]
then a direct computation permits us to check that
\[
\delta A^2 u_\delta + \xi_\delta \in \partial J_\delta(u_\delta) \quad \text{a.e. in } (0,T).
\]
Indeed, by the definition of the $H$-subdifferential, this corresponds to the relation
\[
\langle \delta A^2 u_\delta + \xi_\delta, v - u_\delta \rangle \leq J_\delta(v) - J_\delta(u_\delta) \quad \forall v \in H,
\]
and it is sufficient to check it for $v \in V$ since for $v \in V \setminus H$ the right-hand side is $+\infty$ and consequently the relation is trivial. However, for $v \in V$, (4.19) follows by the definition of the relaxed operator $J_{f_0,\omega}$. Thanks to (4.18), (4.15) is then a direct consequence of inequality (2.45) of Lemma 2.5.

Thus, the above procedure permits us to see that (any) weak solution $(u_\delta, w_\delta, \xi_\delta)$ to the sixth order problem satisfies the energy equality
\[
\frac{d}{dt} E_\delta(u(t)) + \|\nabla w(t)\|^2 + \varepsilon \|u_t(t)\|^2 = 0
\]
for almost all $t \in [0, T]$. As a consequence, we get back the first two convergence relations in (4.9) as well as (4.10). Moreover, we have
\[
\|\nabla u_\delta\|_{L^2(0,T;H)} \leq c.
\]

**Second estimate.** Next, to get (4.11) and (4.13), we essentially need to repeat the “second estimate” of the previous section. Indeed, we see that $u_\delta - (u_\delta)_\Omega$ is an admissible test function in (3.1). However, we now have to obtain an estimate of $\xi_\delta$ from the duality product
\[
\langle \xi_\delta, u_\delta - (u_\delta)_\Omega \rangle.
\]
Actually, if $\xi_\delta = \xi_{\delta,a} + \xi_{\delta,s}$ is the Lebesgue decomposition of the measure $\xi_\delta$ given in Theorem 2.2, then, noting that for all $t \in [0, T]$ we have $u_\delta(t) \in W \subset C^0(\Omega)$, we can write
\[
\langle \xi_\delta(t), u_\delta(t) - (u_\delta)_\Omega \rangle = \int_\Omega \xi_{\delta,a}(t)(u_\delta(t) - (u_\delta)_\Omega) \, dx + \int_\Omega (u_\delta(t) - (u_\delta)_\Omega) \, d\xi_{\delta,s}(t).
\]
Next, we notice that, as a direct consequence of assumption (4.8),
\[
\exists \mu \in (0, 1) : \quad -1 + \mu \leq (u_{0,\delta})_\Omega \leq 1 - \mu, \quad \forall \delta \in (0, 1),
\]
where $\mu$ is independent of $\delta$. In other words, the spatial means $(u_{0,\delta})_\Omega$ are uniformly separated from $\pm 1$. Then, recalling (2.19) and proceeding as in (3.36), we have
\[
\int_\Omega \xi_{\delta,a}(t)(u_\delta(t) - (u_\delta)_\Omega) \, dx \geq \frac{1}{2} \|\xi_{\delta,a}(t)\|_{L^1(\Omega)} - c,
\]
where $c$ does not depend on $\delta$.

On the other hand, let us denote $d\xi_{\delta,s} = \phi_{\delta,s} \, d|\xi_{\delta,s}|$ to be the polar decomposition of $\xi_{\delta,s}$, where $|\xi_{\delta,s}|$ is the total variation of $\xi_{\delta,s}$ (cf., e.g., [44, Chap. 6]). Then, introducing the bounded linear functional $S_\delta : C^0(\Omega) \to \mathbb{R}$ given by
\[
S_\delta(z) := \int_\Omega z \, d\xi_{\delta,s},
\]
using, e.g., [44, Thm. 6.19], and recalling (2.20), we can estimate the norm of \( S \) as follows:

\[
|\xi_{\delta,s}(\Omega)| = \int_{\Omega} d|\xi_{\delta,s}| = \|S\|_{\mathcal{M}(\Omega)} = \sup \left\{ \int_{\Omega} z \, d\xi_{\delta,s}, \ z \in C^0(\Omega), \ z(\Omega) \subset [-1,1] \right\}
\]

(4.27)

where we also used that \( u_{\delta} \in C^0(\Omega) \). Comparing terms, it then follows that

(4.28)

\[
u_{\delta} = \phi_{\delta,s}, \quad |\xi_{\delta,s}| \text{-a.e. in } \Omega.
\]

Then, since is clear that

(4.29)

\[
u_{\delta} = \pm 1 \quad \Rightarrow \quad \frac{\nu_{\delta} - (\nu_{\delta})_{\Omega}}{|\nu_{\delta} - (\nu_{\delta})_{\Omega}|} = \pm 1,
\]

coming back to (4.27) we deduce

(4.30)

\[
\int_{\Omega} d|\xi_{\delta,s}| = \int_{\Omega} \phi_{\delta,s} \frac{\nu_{\delta} - (\nu_{\delta})_{\Omega}}{|\nu_{\delta} - (\nu_{\delta})_{\Omega}|} \, d|\xi_{\delta,s}| \leq c \int_{\Omega} \phi_{\delta,s} (\nu_{\delta} - (\nu_{\delta})_{\Omega}) \, d|\xi_{\delta,s}|
\]

Here we used again in an essential way the uniform separation property (4.24).

Collecting (4.23)–(4.30), we then have

(4.31)

\[
\langle \xi_{\delta}, u_{\delta} - (u_{\delta})_{\Omega} \rangle \geq \frac{1}{2} \|\xi_{\delta,a}(t)\|_{L^1(\Omega)} + \eta \int_{\Omega} d|\xi_{\delta,s}| - c
\]

for some \( c \geq 0, \eta > 0 \) independent of \( \delta \). On the other hand, mimicking (3.35)–(3.38), we obtain

(4.32)

\[
\delta \|Au_{\delta}\|^2 + g^2 \|\nabla u_{\delta}\|^2 + \langle \xi_{\delta}, u_{\delta} - (u_{\delta})_{\Omega} \rangle \leq c \left( \|\nabla w_{\delta}\| + \epsilon \|u_{\delta,t}\| + 1 \right),
\]

whence squaring, integrating in time, and using (4.10) and (4.21), we obtain that the function

(4.33)

\[ t \mapsto \|\xi_{\delta,a}(t)\|_{L^1(\Omega)} + \int_{\Omega} d|\xi_{\delta,s}(t)| \] is bounded in \( L^2(0,T) \), independently of \( \delta \).

Now integrating (3.2) in space, we deduce

(4.34)

\[
\int_{\Omega} w_{\delta} = \frac{1}{2} \int_{\Omega} a'(u_{\delta})|\nabla u_{\delta}|^2 + \int_{\Omega} \xi_{\delta} - \lambda(u_{\delta})_{\Omega},
\]

whence

(4.35)

\[
\left| \int_{\Omega} w_{\delta} \right| \leq c \left( \|\nabla u_{\delta}\|^2 + \|\xi_{\delta,a}(t)\|_{L^1(\Omega)} + \int_{\Omega} d|\xi_{\delta,s}(t)| + 1 \right).
\]

Thus, squaring, integrating in time, and recalling (4.21) and (4.33), we finally obtain (4.11).
**Key estimate.** To take the limit $\delta > 0$, we have to provide a bound on $A(u_\delta)$ independent of $\delta$. This will be obtained by means of the following integration-by-parts formula due to Dal Passo, Garcke, and Grün [21, Lem. 2.3].

**Lemma 4.3.** Let $h \in W^{2,\infty}(\mathbb{R})$ and $z \in W$. Then

$$
\int_{\Omega} h'(z) |\nabla z|^2 \, dz = -\frac{1}{3} \int_{\Omega} h''(z) |\nabla z|^4
$$

$$
+ \frac{2}{3} \int_{\Omega} h(z) (|D^2 z|^2 - |\Delta z|^2) + \frac{2}{3} \int_{\Gamma} h(z) II(\nabla z),
$$

where $II(\cdot)$ denotes the second fundamental form of $\Gamma$.

We then test (3.2) by $A u_\delta$ in the duality between $V'$ and $V$. This gives the relation

$$
\frac{\varepsilon}{2} \int_{\Omega} |\nabla w_\delta|^2 + \delta |\nabla A u_\delta|^2 + (A(u_\delta), A u_\delta) + (\xi_\delta, A u_\delta) = (\nabla w_\delta, \nabla u_\delta) + \lambda |\nabla u_\delta|^2
$$

and some terms have to be estimated. First, we note that

$$
(A(u_\delta), A u_\delta) = \left( a(u_\delta) \Delta u_\delta + \frac{a'(u_\delta)}{2} |\nabla u_\delta|^2, \Delta u_\delta \right).
$$

Thus, using Lemma 4.3 with the choice of $h(\cdot) = a'(\cdot)/2$, we obtain

$$
(A(u_\delta), A u_\delta) = \int_{\Omega} a(u_\delta) |\Delta u_\delta|^2 + \frac{1}{3} \int_{\Omega} a(u_\delta) (|D^2 u_\delta|^2 - |\Delta u_\delta|^2)
$$

$$
- \frac{1}{6} \int_{\Omega} a''(u_\delta) |\nabla u_\delta|^4 + \frac{1}{3} \int_{\Gamma} a(u_\delta) II(\nabla u_\delta).
$$

Let us now point out that, with $\Gamma$ smooth, we can estimate

$$
\frac{1}{3} \int_{\Gamma} a(u_\delta) II(\nabla u_\delta) \leq c \|\nabla u_\delta\|_{L^2(\Gamma)}^2 \leq \omega \|A u_\delta\|_{V'}^2 + c_\omega \|u_\delta\|_{W}^2
$$

for small $\omega > 0$ to be chosen below, the last inequality following from the continuity of the trace operator (applied to $\nabla u$) from $H^s(\Omega)$ into $L^2(\Gamma)$ for $s \in (1/2, 1)$ and the compactness of the embedding $W \subset H^{1+s}(\Omega)$ for $s$ in the same range.

Thus, using the concavity assumption (4.7) on $a$ and the fact that $|u_\delta| \leq 1$ a.e. in $(0, T) \times \Omega$, we get

$$
(A(u_\delta), A u_\delta) \geq \eta \|A u_\delta\|^2 - c
$$

for proper strictly positive constants $\eta$ and $c$, both independent of $\delta$. Next, we observe that, by (3.57) and Lemma 2.4, we obtain $\langle \xi, A u_\delta \rangle \geq 0$. Finally, we have

$$
-(\nabla w_\delta, \nabla u_\delta) \leq c \|\nabla w_\delta\| \|\nabla u_\delta\|,
$$

and the right-hand side is readily estimated thanks to (4.10) and (4.11).

Thus, on account of (4.41), integrating (4.37) in time, we readily obtain the last of (4.9) as well as (4.12). Moreover, since $-1 \leq \delta \leq 1$ a.e., we have for free

$$
\|u_\delta\|_{L^\infty((0, T) \times \Omega)} \leq 1.
$$

Thus, using the Gagliardo–Nirenberg inequality (2.58), we have also

$$
u_\delta \to u \quad \text{weakly in } L^4((0, T); W^{1,4}(\Omega)).$$
This readily entails

\[(4.45) \quad \mathcal{A}(u_\delta) \to \mathcal{A}(u) \text{ weakly in } L^2(0,T;H). \]

Thus, a comparison of terms in (3.2) also gives

\[(4.46) \quad \xi_\delta \to \xi \text{ weakly in } L^2(0,T;V'). \]

Then we can take the limit \( \delta \searrow 0 \) in (3.1) and get (4.1). On the other hand, if we take the limit of (3.2), we obtain

\[(4.47) \quad w = \mathcal{A}(u) + \xi - \lambda u + \varepsilon u_t \]

and we have to identify \( \xi \). Actually, (4.46), the strong convergence \( u_\delta \to u \) in \( L^2(0,T;V) \) (following from (4.9) and the Aubin–Lions lemma), and Lemma 2.3 permit us to show that

\[(4.48) \quad \xi \in f_{0,w}(u) \text{ a.e. in } (0,T). \]

On the other hand, a comparison argument in (4.47) permits us to see that \( \xi \in L^2(0,T;H) \), whence, thanks to (2.14), we obtain that \( \xi(t) = f_0(u(t)) \in H \) for a.e. \( t \in (0,T) \). This concludes the proof of Theorem 4.2. \( \square \)

5. Analysis of the fourth order problem. In this section, we will prove existence of a weak solution to problem (1.1)–(1.4) in the fourth order case \( \delta = 0 \) by means of a direct approach not relying on the sixth order approximation. This will allow us to consider a general function \( a \) (without the concavity assumption (4.7)). More precisely, we have the following.

**Theorem 5.1.** Let assumptions (2.1)–(2.4) hold, let \( \varepsilon \geq 0 \), and let

\[(5.1) \quad u_0 \in V, \quad F(u_0) \in L^1(\Omega), \quad (u_0)_{\Omega} \in (-1,1). \]

Then there exists at least one weak solution to the fourth order problem, in the sense of Definition 4.1.

The rest of the section is devoted to the proof of the above result, which is divided into several steps.

**Phase-field approximation.** For \( \sigma \in (0,1) \), we consider the system

\[
(5.2) \quad u_t + \sigma w_t + Aw = 0, \\
(5.3) \quad w = \mathcal{A}(u) + f_\sigma(u) - \lambda u + (\varepsilon + \sigma)u_t.
\]

This will be endowed with the initial conditions

\[(5.4) \quad u|_{t=0} = u_{0,\sigma}, \quad w|_{t=0} = 0. \]

Similarly as before (compare with (3.10)), we have set

\[(5.5) \quad u_{0,\sigma} + \sigma A^2 u_{0,\sigma} = u_0 \]

and, by standard elliptic regularity, we have that

\[(5.6) \quad u_{0,\sigma} \in H^5(\Omega) \subset C^{3+\alpha}(\overline{\Omega}) \quad \text{for } \alpha \in (0,1/2), \quad \partial_\nu u_{0,\sigma} = \partial_\nu Au_{0,\sigma} = 0 \text{ on } \Gamma. \]

Moreover, of course, \( u_{0,\sigma} \to u_0 \) in a suitable sense as \( \sigma \searrow 0 \).
Fixed point argument. We now prove existence of a local solution to the phase-field approximation by a further Schauder fixed point argument. Namely, we introduce the system

\begin{align}
(5.7) & \quad u_t + \sigma w_t + Aw = 0, \\
(5.8) & \quad \pi = -a(\pi)\Delta u - \frac{a'(\pi)}{2}|\nabla \pi|^2 + f_\sigma(\pi) - \lambda \pi + (\varepsilon + \sigma)u_t, \quad \partial_n u = 0 \quad \text{on } \Gamma,
\end{align}

which we still endow with the condition (5.4). Here, \( f_\sigma \) is chosen as in (2.22).

Next, we set

\begin{align}
(5.9) & \quad \mathcal{U} := \{ u \in C^{0,1+\alpha}([0,T_0] \times \overline{\Omega}) : u|_{t=0} = u_{0,\sigma}, \| u \|_{C^{0,1+\alpha}} \leq 2R \},
\end{align}

where \( R := \max\{ 1, \| u_{0,\sigma} \|_{C^{1+\alpha}(\overline{\Omega})} \} \) and \( T_0 \) will be chosen at the end of the argument.

It is clear that \( R \) depends in fact on \( \sigma \) (so that the same will happen for \( T_0 \)). This dependence, however, is not emphasized here. For the definition of the parabolic Hölder spaces used in this proof, we refer the reader to [37, Chap. 5], whose notation we have adopted. Moreover, in what follows in place of \( \text{H¨older spaces used in this proof, we refer the reader to } [37, \text{Chap. 5}], \) whose notation we still endow with the condition (5.4). Here, \( \Pi \) is chosen at the end of the argument.

Then, choosing \( (\overline{\pi}, \overline{w}) \) in \( \mathcal{U} \times \mathcal{W} \) and inserting it in (5.8), we observe that, by the Lipschitz regularity of \( a \) (cf. (2.1)) and standard multiplication properties of Hölder spaces, there exists a computable monotone function \( Q \), also depending on \( \sigma \), but independent of the time \( T_0 \), such that

\begin{align}
(5.11) & \quad \| a(\pi) \|_{C^{0,\alpha}} + \| a'(\pi)|\nabla \pi|^2 \|_{C^{0,\alpha}} + \| f_\sigma(\pi) \|_{C^{0,\alpha}} \leq Q(R).
\end{align}

Thanks to [37, Thm. 5.1.21], then there exists one and only one solution \( u \) to (5.8) with the first initial condition (5.4). This solution satisfies

\begin{align}
(5.12) & \quad \| u \|_{C^{1,2+\alpha}} \leq Q(R).
\end{align}

Substituting \( u_t \) in (5.7) and applying the same theorem of [37] to this equation with the second initial condition (5.4), we then obtain one and only one solution \( w \), with

\begin{align}
(5.13) & \quad \| w \|_{C^{1,2+\alpha}} \leq Q(R).
\end{align}

We then denote by \( \mathcal{T} \) the map such that \( \mathcal{T} : (\overline{\pi}, \overline{w}) \mapsto (u, w) \). As before, we need to show that

(i) \( \mathcal{T} \) takes its values in \( \mathcal{U} \times \mathcal{W} \);

(ii) \( \mathcal{T} \) is continuous with respect to the \( C^{0,1+\alpha} \times C^{0,\alpha} \)-norm of \( \mathcal{U} \times \mathcal{W} \);

(iii) \( \mathcal{T} \) is a compact map.

First of all let us prove (i). We just refer to the component \( u \), the argument for \( w \) being analogous and in fact simpler. We start observing that if \( u \in \Pi_1(\mathcal{T}(\mathcal{U} \times \mathcal{W})) \), \( \Pi_1 \) denoting the projection on the first component, then

\begin{align}
(5.14) & \quad \| u(t) \|_{C^{0,\alpha}(\overline{\Omega})} \leq \| u_0 \|_{C^{0,\alpha}(\overline{\Omega})} + \int_0^t \| u_t(s) \|_{C^{0,\alpha}(\overline{\Omega})} \, ds \leq R + T_0 Q(R) \quad \forall t \in [0,T_0],
\end{align}

which is smaller than \( 2R \) if \( T_0 \) is chosen suitably.
Next, using the continuous embedding (cf. [37, Lem. 5.1.1])

\[ C^{1,2+\alpha} \subset C^{1/2}([0,T_0]; C^{1+\alpha}(\Omega)) \cap C^{\alpha/2}([0,T_0]; C^2(\Omega)), \]

we obtain that, analogously,

\[ \| \nabla u(t) \|_{C^{\alpha}(\Omega)} \leq \| \nabla u_0 \|_{C^{\alpha}(\Omega)} + T_0^{1/2} \| u \|_{C^{1/2}([0,T_0]; C^{1+\alpha}(\Omega))} \leq R + T_0^{1/2} Q(R). \]

Hence, passing to the supremum for \( t \in [0,T_0] \), we see that the norm of \( u \) in \( C^{0,1+\alpha} \) can be made smaller than \( 2R \) if \( T_0 \) is small enough. Thus, (i) is proved.

Let us now come to (iii). As before, we just deal with the component \( u \). Namely, on account of (5.12), we have to show that the space \( C^{1,2+\alpha} \) is compactly embedded into \( C^{0,1+\alpha} \). Actually, by (5.15) and using standard compact inclusion properties of Hölder spaces, this relation is proved easily. Hence, we have (iii).

Finally, we have to prove (ii). This property is, however, straightforward. Actually, taking \( (\varpi_n, \overline{\varpi}_n) \rightarrow (\varpi, \overline{\varpi}) \) in \( \mathcal{U} \times \mathcal{W} \), we have that the corresponding solutions \( (u_n, w_n) = \mathcal{T}(\varpi_n, \overline{\varpi}_n) \) are bounded in the sense of (5.12)–(5.13) uniformly in \( n \). Consequently, a standard weak compactness argument, together with the uniqueness property for the initial value problems associated to (5.7) and to (5.8), permit us to see that the whole sequence \( (u_n, w_n) \) converges to a unique limit point \((u, w)\) solving (5.7)–(5.8) with respect to the limit data \( (\varpi, \overline{\varpi}) \). Moreover, by the compactness property proved in (iii), this convergence holds with respect to the original topology of \( \mathcal{U} \times \mathcal{W} \). This proves that \((u, w) = \mathcal{T}(\varpi, \overline{\varpi})\), i.e., (ii) holds.

**A priori estimates.** For any \( \sigma > 0 \), we have obtained a local (i.e., with a final time \( T_0 \) depending on \( \sigma \)) solution to (5.2)–(5.3) with the initial conditions (5.4). To emphasize the \( \sigma \)-dependence, we will note it by \((u_\sigma, w_\sigma)\) in what follows. To let \( \sigma \searrow 0 \), we now devise some a priori estimates uniform both with respect to \( \sigma \) and with respect to \( T_0 \). As before, this will give a global solution in the limit and, to avoid technicalities, we can directly work on the time interval \([0,T]\). Notice that the high regularity of \((u_\sigma, w_\sigma)\) gives sense to all the calculations performed below (in particular, to all the integrations by parts). That said, we repeat the “energy estimate” exactly as in the previous sections. This now gives

\[ \| u_\sigma \|_{L^\infty(0,T; V)} + \| F_\sigma(u_\sigma) \|_{L^\infty(0,T; L^1(\Omega))} \leq c, \]

\[ (\sigma + \varepsilon)^{1/2} \| u_{\sigma,t} \|_{L^2(0,T; H)} \leq c, \]

\[ \sigma^{1/2} \| w_\sigma \|_{L^\infty(0,T; H)} + \| \nabla w_\sigma \|_{L^2(0,T; H)} \leq c. \]

Next, working as in the “second estimate” of subsection 3.2, we obtain the analogue of (3.39) and (3.40).

To estimate \( f_\sigma(u_\sigma) \) in \( H \), we now test (5.3) by \( f_\sigma(u_\sigma) \), to get

\[ \frac{\varepsilon + \sigma}{2} \frac{d}{dt} \int_{\Omega} F_\sigma(u_\sigma) + \int_{\Omega} \left( a(u_\sigma) f_\sigma'(u_\sigma) + \frac{a'(u_\sigma)}{2} f_\sigma(u_\sigma) \right) |\nabla u_\sigma|^2 + \| f_\sigma(u_\sigma) \|^2 \]

\[ = \left( w_\sigma + \lambda u_\sigma, f_\sigma(u_\sigma) \right), \]

and it is a standard matter to estimate the right-hand side by using the last term on the left-hand side, Hölder’s and Young’s inequalities, and properties (5.17) and (3.40). Now, we notice that, thanks to (2.27),

\[ a(r) f_\sigma'(r) + \frac{a'(r)}{2} f_\sigma(r) \geq \mathcal{A} f_\sigma'(r) - c |f_\sigma(r)| \geq \frac{\mathcal{A}}{2} f_\sigma'(r) - c \quad \forall r \in [-2,2], \]
with the last \( c \) being independent of \( \sigma \). On the other hand, for \( r \not\in [-2, 2] \) we have that \( a'(r) = 0 \) by (2.2). Hence, also thanks to (5.17), the second term on the left-hand side of (5.20) can be controlled. We then arrive at

\[
\|f_\sigma(u_\sigma)\|_{L^2(0,T; H)} \leq c.
\]

The key point is represented by the next estimate, which is used to control the second space derivatives of \( u \). To do this, we first have to operate a change of variable. Namely, we set

\[
\phi(s) := \int_0^s a^{1/2}(r) \, dr, \quad z_\sigma := \phi(u_\sigma)
\]

and notice that, by (2.1)–(2.2), \( \phi \) is monotone and Lipschitz together with its inverse. Then, by (5.17),

\[
\|z_\sigma\|_{L^\infty(0,T; V)} \leq c,
\]

and it is straightforward to realize that (5.3) can be rewritten as

\[
w_\sigma = -\phi'(u_\sigma)\Delta z_\sigma + f_\sigma \circ \phi^{-1}(z_\sigma) - \lambda u_\sigma + (\varepsilon + \sigma) u_{\sigma,t}, \quad \partial_n u_\sigma = 0 \text{ on } \Gamma.
\]

By the Hölder continuity of \( u_\sigma \) up to its second space derivatives and the Lipschitz continuity of \( a \) and \( a' \) (cf. (2.1)–(2.2)), \( -\Delta z_\sigma \) is also Hölder continuous in space. Thus, we can use it as a test function in (5.25). Using the monotonicity of \( f_\sigma \) and \( \phi^{-1} \), and recalling (5.22), we then easily obtain

\[
\|z_\sigma\|_{L^2(0,T; W)} \leq c.
\]

**Passage to the limit.** As a consequence of (5.17)–(5.19), (3.39)–(3.40), and (5.22), we have

\[
u_\sigma \to u \text{ weakly star in } H^1(0,T; V') \cap L^\infty(0,T; V),
\]

\[
(\sigma + \varepsilon) u_{\sigma,t} \to \varepsilon u_t \text{ weakly in } L^2(0,T; H),
\]

\[
f_\sigma(u_\sigma) \to \overline{f} \text{ weakly in } L^2(0,T; H),
\]

\[
w_\sigma \to w \text{ weakly in } L^2(0,T; V),
\]

\[
u_{\sigma,t} + \sigma w_{\sigma,t} \to u_t \text{ weakly in } L^2(0,T; V')
\]

for suitable limit functions \( u, w, \) and \( \overline{f} \). Here and below, all convergence relations have to be intended to hold up to (nonrelabeled) subsequences of \( \sigma \searrow 0 \). Now, by the Aubin–Lions lemma, we have

\[
u_\sigma \to u \text{ strongly in } C^0([0,T]; H) \text{ and a.e. in } Q.
\]

Then (5.29) and a standard monotonicity argument (cf. [7, Prop. 1.1]) imply that \( \overline{f} = f(u) \) a.e. in \( Q \). Furthermore, by (2.1)–(2.2) and the generalized Lebesgue theorem, we have

\[
a(u_\sigma) \to a(u), \quad a'(u_\sigma) \to a'(u) \text{ strongly in } L^q(Q) \forall q \in [1, +\infty).
\]

Analogously, recalling (5.24), \( z_\sigma = \phi(u_\sigma) \to \phi(u) =: z \text{ strongly in } L^q(Q) \) for all \( q \in [1, 6] \). Actually, the latter relation holds also weakly in \( L^2(0,T; W) \) thanks to the bound (5.26). Moreover, by (5.24), (5.26), and interpolation, we obtain

\[
\|\nabla z_\sigma\|_{L^{10/3}(Q)} \leq c,
\]
whence, clearly, we also obtain

$$\|\nabla u_\sigma\|_{L^{10/3}(Q)} \leq c.$$  

As a consequence, being

$$-\Delta u_\sigma = -\frac{1}{a^{1/2}(u_\sigma)} \Delta z_\sigma + \frac{a'(u_\sigma)}{2a(u_\sigma)} |\nabla u_\sigma|^2,$$

we also have that

$$\Delta u_\sigma \to \Delta u \text{ weakly in } L^{5/3}(Q).$$

Combining this with (5.27) and using the generalized Aubin–Lions lemma (cf., e.g., [47]), we then arrive at

$$u_\sigma \to u \text{ strongly in } L^{5/3}(0,T;W^{2-\varepsilon, 5/3}(\Omega)) \cap C^0([0,T];H^{1-\varepsilon}(\Omega)) \quad \forall \varepsilon > 0,$$

whence, by standard interpolation and embedding properties of Sobolev spaces, we obtain

$$\nabla u_\sigma \to \nabla u \text{ strongly in } L^q(Q) \text{ for some } q > 2.$$  

Consequently, recalling (5.33),

$$(a'(u_\sigma)|\nabla u_\sigma|^2 \to a'(u)|\nabla u|^2, \text{ say, weakly in } L^1(Q).$$

This is sufficient to take the limit $\sigma \searrow 0$ in (5.3) and get back (4.2). To conclude the proof, it only remains to show the regularity (4.3) for what concerns the second space derivatives of $u$. Actually, by (5.26) and the Gagliardo–Nirenberg inequality (2.58),

$$z \in L^2(0,T;W) \cap L^\infty(Q) \subset L^4(0,T;W^{1,4}(\Omega)).$$

Thus, we have also $u \in L^4(0,T;W^{1,4}(\Omega))$ and, consequently, a comparison of terms in (4.2) permits us to see that $\Delta u \in L^2(0,T;H)$, whence (4.3) follows from elliptic regularity. The proof of Theorem 5.1 is concluded.

6. Further properties of weak solutions.

6.1. Uniqueness for the fourth order problem. We will now prove that if the interfacial (i.e., gradient) part of the free energy $E_\delta$ satisfies a convexity condition (in the viscous case $\varepsilon > 0$) or, respectively, a strict convexity condition (in the nonviscous case $\varepsilon = 0$), then the solution is unique also in the fourth order case. Actually, the stronger assumption (corresponding to $\kappa > 0$ in the statement below) required in the nonviscous case is needed for the purpose of controlling the nonmonotone part of $f(u)$, while in the viscous case we can use the term $\varepsilon u_t$ for that aim.

It is worth noting that, also from a merely thermodynamical point of view, the convexity condition is a rather natural requirement. Indeed, it corresponds to asking the second differential of $E_\delta$ to be positive definite, to ensure that the stationary solutions are dynamically stable (cf., e.g., [48] for more details).

Theorem 6.1. Let the assumptions of Theorem 5.1 hold and assume that, in addition,

$$a''(r) \geq 0, \quad \left(\frac{1}{a'}\right)''(r) \leq -\kappa \quad \forall r \in [-1, 1],$$
where $\kappa > 0$ if $\varepsilon = 0$ and $\kappa \geq 0$ if $\varepsilon > 0$. Then the fourth order problem admits a unique weak solution.

Proof. Let us denote by $J$ the gradient part of the energy, i.e.,

\[
J : V \to [0, +\infty), \quad J(u) := \int_{\Omega} \frac{a(u)}{2} |\nabla u|^2.
\]

Then we clearly have

\[
\langle J'(u), v \rangle = \int_{\Omega} \left( a(u) \nabla u \cdot \nabla v + \frac{a'(u)}{2} |\nabla u|^2 v \right),
\]

and we can correspondingly compute the second derivative of $J$ as

\[
\langle J''(u)v, z \rangle = \int_{\Omega} \left( a''(u) |\nabla u|^2 vz + a'(u)v |\nabla v|^2 + a(u) |\nabla v|^2 z \right).
\]

To be more precise, we have that $J'(u) \in V'$ and $J''(u) \in L(V, V')$ at least for $u \in W$ (this may instead not be true if we only have $u \in V$, due to the quadratic terms in the gradient). This is, however, the case for the fourth order system since for any weak solution we have that $u(t) \in W$ at least for a.e. $t \in (0, T)$.

From (6.4), we then have in particular

\[
\langle J''(u)v, v \rangle = \int_{\Omega} \left( a''(u) |\nabla u|^2 v^2 + 2a'(u)v |\nabla u|^2 v + a(u) |\nabla v|^2 \right)
\]

\[
\geq \int_{\Omega} \left( a(u) - \frac{2a'(u)^2}{a''(u)} \right) |\nabla v|^2,
\]

whence the functional $J$ is convex, at least when restricted to functions $u$ such that

\[
u \in W, \quad u(\Omega) \subset [-1, 1],
\]

provided that $a$ satisfies

\[
a(r)a''(r) - 2a'(r)^2 \geq 0 \quad \forall r \in [-1, 1].
\]

Noting that

\[
\left( \frac{1}{a} \right)'' = \frac{2(a')^2 - aa''}{a^3},
\]

we have that $J$ is (strictly) convex if $1/a$ is (strictly) concave; i.e., (6.1) holds (cf. also [24, sec. 3] for related results). Note that, in deducing the last inequality in (6.5), we worked as if it were $a'' > 0$. However, if $a''(r) = 0$ for some $r$, then also $a'(r)$ has to be 0 due to (6.7). So, this means that in the set $\{u = r\}$ the first two summands on the right-hand side of the first line of (6.5) identically vanish.

That said, let us write both (3.1) and (3.2) for a couple of solutions $(u_1, w_1)$, $(u_2, w_2)$ and take the difference. Setting $(u, w) := (u_1, w_1) - (u_2, w_2)$, we obtain

\[
u_t + Au = 0,
\]

\[
w = J'(u_1) - J'(u_2) + f(u_1) - f(u_2) + \varepsilon u_t.
\]
Then we can test (6.9) by $A^{-1}u$ and (6.10) by $u$ and take the difference. Indeed, $u = u_1 - u_2$ has zero-mean value by (3.28). We obtain

$$\frac{1}{2}\frac{d}{dt} \left( ||u||_{V'}^2 + \varepsilon ||u||^2 \right) + \langle J'(u_1) - J'(u_2), u \rangle + (f(u_1) - f(u_2), u) = 0,$$

and, using the convexity of $J$ coming from (6.1) and the $\lambda$-monotonicity of $f$ (see (2.3)), we have, for some function $\xi$ belonging to $W$ a.e. in time and taking its values in $[-1, 1]$,

$$\frac{1}{2}\frac{d}{dt} \left( ||u||_{V'}^2 + \varepsilon ||u||^2 \right) + \lambda ||u||^2 \leq \frac{1}{2}\frac{d}{dt} \left( ||u||_{V'}^2 + \varepsilon ||u||^2 \right) + \langle J''(\xi)u, u \rangle \leq \lambda ||u||^2.$$

Thus, in the case $\varepsilon > 0$ (where it may be $\kappa = 0$), we can just use Gronwall’s lemma. Instead, if $\varepsilon = 0$ (so that we assumed $\kappa > 0$), by the Poincaré–Wirtinger inequality we have

$$\lambda ||u||^2 \leq \frac{\kappa}{2} ||\nabla u||^2 + c||u||_{V'}^2,$$

and the thesis follows again by applying Gronwall’s lemma to (6.12). \[\square\]

6.2. Additional regularity. We prove here parabolic regularization properties of the solutions to the fourth order system holding in the case of a convex energy functional. An analogous result would hold also for the sixth order system under general conditions on $a$ since the bi-Laplacian in that case dominates the lower order terms (we omit the details).

**Theorem 6.2.** Let the assumptions of Theorem 6.1 hold. Then the solution satisfies the additional regularity property

$$\|u\|_{L^\infty(\tau, T; W)} + \|u\|_{L^\infty(\tau, T; W^{1,4}(\Omega))} \leq Q(\tau^{-1}) \quad \forall \tau > 0,$$

where $Q$ is a computable monotone function whose expression depends on the data of the problem and, in particular, on $u_0$.

**Proof.** The proof is based on a further a priori estimate, which has unfortunately a formal character in the present regularity setting. To justify it, one should proceed by regularization. For instance, a natural choice would be that of refining the fixed point argument leading to existence of a weak solution (cf. section 5) by showing (e.g., using a bootstrap regularity argument) that, at least locally in time, the solution lies in higher order H"older spaces. We leave the details to the reader.

That said, we test (3.1) by $w_t$ and subtract the result from the time derivative of (3.2) tested by $u_t$. We obtain

$$\frac{1}{2}\frac{d}{dt} ||\nabla u||^2 + \varepsilon \frac{d}{dt} ||u_t||^2 + \langle J''(u)u_t, u_t \rangle + \int_\Omega f'(u)u_t^2 \leq 0.$$

Then, by convexity of $J$,

$$\langle J''(u)u_t, u_t \rangle \geq \kappa ||u_t||^2_{L^2(0, T; V)}.$$

On the other hand, the $\lambda$-monotonicity of $f$ gives

$$\int_\Omega f'(u)u_t^2 \geq -\lambda ||u_t||^2_{L^2(0, T; H)}.$$
and, if \( \varepsilon = 0 \) (so that \( \kappa > 0 \)), we have as before
\[
-\lambda \|u_t\|_{L^2(0,T;H)}^2 \geq -\frac{\kappa}{2} \|u_t\|_{L^2(0,T;V)}^2 - c \|u_t\|_{L^2(0,T;V)}^2.
\]

Thus, recalling the first of (3.5) and applying the uniform Gronwall lemma (cf. [49, Lem. I.1.1]), it is not difficult to infer
\[
\|\nabla w\|_{L^\infty(\tau,T;H)} + \varepsilon^{1/2} \|u_t\|_{L^\infty(\tau,T;H)} + \kappa \|u_t\|_{L^2(\tau,T;V)} \leq Q(\tau^{-1}) \quad \forall \tau > 0.
\]

Next, testing (3.2) by \( u - u_\Omega \) and proceeding as in the “second estimate” of subsection 3.2, but taking now the essential supremum as time varies in \([\tau,T]\), we arrive at
\[
\|w\|_{L^\infty(\tau,T;V)} + \|f(u)\|_{L^\infty(\tau,T;L^1(\Omega))} \leq Q(\tau^{-1}) \quad \forall \tau > 0.
\]

Thus, thanks to (6.20), we can test (4.2) by \(-\Delta z\), with \( z = \phi(u) \) (cf. (5.23)). Proceeding similarly with section 5 (but taking now the supremum over \([\tau,T]\) rather than integrating in time), we easily get (6.14), which concludes the proof.

### 6.3. Energy equality.

As noted in section 4, any weak solution to the sixth order system satisfies the energy equality (4.20). We will now see that the same property holds also in the viscous fourth order case (i.e., if \( \delta = 0 \) and \( \varepsilon > 0 \)). More precisely, we can prove the following.

**Proposition 6.3.** Let the assumptions of Theorem 5.1 hold and let \( \varepsilon > 0 \). Then any weak solution to the fourth order system satisfies the integrated energy equality
\[
\mathcal{E}_0(u(t)) = \mathcal{E}_0(u_0) - \int_0^t \left( \|\nabla w(s)\|^2 - \varepsilon \|u_t(s)\|^2 \right) ds \quad \forall t \in [0,T].
\]

**Proof.** As before, we proceed by testing (4.1) by \( w \) and (4.2) by \( u_t \) and taking the difference. As \( w_t \in L^2(0,T;H) \) and \( f_0(u) \in L^2(0,T;H) \) (cf. (4.3) and (4.5)), then the integration by parts
\[
(f(u), u_t) = \frac{d}{dt} \int_\Omega F(u) \quad \text{a.e. in } (0,T)
\]
is straightforward (it follows directly from [12, Lem. 3.3, p. 73]).

Moreover, in view of (4.3), assumption (2.47) of Lemma 2.6 is satisfied. Hence, by (2.50), we deduce that
\[
\int_0^t (A(u(s)), u_t(s)) \, ds = \int_\Omega \frac{a(u(t))}{2} |\nabla u(t)|^2 - \int_\Omega \frac{a(u_0)}{2} |\nabla u_0|^2.
\]

Combining (6.22) and (6.23), we immediately get the assertion.

It is worth noting that the energy equality obtained above has a key relevance in the investigation of the long-time behavior of the system. In particular, given \( m \in (-1,1) \) (the spatial mean of the initial datum, which is a conserved quantity due to (3.28)), we can define the phase space
\[
\mathcal{X}_{\delta,m} := \{ u \in V : \delta u \in W, F(u) \in L^1(\Omega), u_\Omega = m \}
\]
and view the system (both for \( \delta > 0 \) and for \( \delta = 0 \)) as a (generalized) dynamical process in \( \mathcal{X}_{\delta,m} \). Then (6.21) (or its sixth order analogue) stands at the basis of the so-called energy method (cf. [5, 38]) for proving existence of the global attractor with respect to the “strong” topology of the phase space.
This issue will be analyzed in a forthcoming work.

Remark 6.4. Whether the equality (6.21) still holds in the nonviscous case \( \varepsilon = 0 \) seems to be a nontrivial question. The answer would be positive in case one could prove the integration-by-parts formula

\[
\int_0^T \langle u_t, A(u) + f(u) \rangle = \int_\Omega \left( \frac{a(u(t))}{2} |\nabla u(t)|^2 + F(u(t)) \right) - \int_\Omega \left( \frac{a(u_0)}{2} |\nabla u(t)|^2 + F(u_0) \right)
\]

under the conditions

\[
(0, T) \cap L^2(0, T; W) \cap L^\infty(Q), \quad A(u) + f(u) \in L^2(0, T; V),
\]

which are satisfied by our solution (in particular, the latter (6.26) follows by a comparison of terms in (4.2), where now \( \varepsilon = 0 \)). Actually, if (6.26) holds, then both sides of (6.25) make sense. However, devising an approximation argument suitable for proving (6.25) could be a rather delicate problem.

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REFERENCES


