Transmission problems for nonlinear parabolic systems of phase-field type

Giulio Schimperna[1]

November 9, 2006

[1]Dottorato di Ricerca in Matematica, XI Ciclo, Università di Milano

Contents

0.1 Introduction

Phase-field models account for a significant attempt to give a rigorous mathematical presentation to some modern thermodynamical theories involving rather general fusion-solidification processes; for instance, solidification of fluids, phase transitions in metallic alloys, diffusion processes in solutes with variable concentration, austenitic-martensitic structures in steel, formation of glasses are only some phenomena which can be described in this framework. The introduction of the model goes back to the mid 80's and is essentially due to Caginalp [19] and Fix [40], who constructed it as an extension of the weak formulation of the Stefan problem describing the phase transitions in ice-water mixtures (or in similarly-behaving substances). In the latest years, phase-field models have rapidly become very popular among the scientific community, so that they are now extensively studied both from the thermodynamical and the mathematical points of view; new generalizations are also continually provided, which are able to apply to very complex and detailed physical situations.

In our dissertation, anyway, we are not interested in describing a complicated thermodynamical setting; we rather refer to the simpler original situation concerning heat diffusion processes in substances allowed to phase transitions. So, let us briefly present the precise physical situation and, in particular, start by listing the involved variables, which are the temperature θ and the order parameter, or *phase-field* χ. Clearly, it is not necessary to insist on the physical sense of θ, which is normally intended as the relative temperature with respect to the fusion-solidification point $\theta^* = 0$ (i.e. not as the absolute Kelvin temperature); as far as χ is concerned, instead, we now try and explain its sense in the simplest thermodynamical setting, i.e. that of an ice-water mixture; we point out anyway that the "interpretation" of the variable χ can be much less intuitive in the general case (we refer to Chapter 2 for the detailed construction of the phase-field model and for several related comments and generalizations).

So, take a bounded region $\Omega \subset \mathbb{R}^N$ (with $N = 2$ or $N = 3$) and suppose it filled with water and ice, which are subject to the heat diffusion in the course of a fixed time period $[0, T]$. Then, for every point x and time instant t, $\chi(x, t)$ represents the *proportion of phases*; for instance, if ice is present at x at the time t, then it turns out to be $\chi(x, t) = -1$ (pure solid), while $\chi(x, t) = 1$ accounts for pure water; furthermore, also the intermediate states, called *mushy regions* and corresponding to

$-1 < \chi(x,t) < 1$, are allowed. We can now present the precise mathematical form of the diffusion equations constituting what in the course of the dissertation will be referred to as the (parabolic) phase-field model; in the following formulation, anyway, we omit some (nonnegative) "physical" coefficients, since our interest is only that of "giving an idea" of the related mathematical setting: we have

$$\partial_t \theta + \partial_t \chi - \Delta \theta = f \tag{1}$$

$$\mu \partial_t \chi - \nu \Delta \chi + \alpha(\chi) + \gamma(\chi) \ni \theta, \tag{2}$$

where, in order to obtain a well-posed problem under an analytical point of view, the system must clearly be complemented by means of the usual Cauchy and boundary conditions (which should be of the Neumann type as far as χ is concerned). Notice that the first equation is nothing else but the standard heat equation, where, anyway, the time derivative of χ is added to the left hand side. The reason of this choice is explained in detail in Chapter 2; for the present, just notice that the term $\partial_t \chi$ is zero in the pure-solid or pure-liquid states, so that its contribution is only significant "near" the interface between the solid and liquid phases. As (2) is concerned instead, observe that we assume that a diffusion process (both in space and in time) is allowed also for the order parameter χ. On the mathematical side, we also point out that equation (2) can be seen as a *gradient flow problem* (see Subsection 2.1.2) for the following so-called Ginzburg-Landau free energy functional:

$$F(\theta, \chi) = \int_\Omega \Big(\frac{\nu}{2} |\nabla \chi|^2 + \phi(\chi) - \theta \chi \Big)\, dx. \tag{3}$$

Indeed, F gives a measure of the internal free energy of the fluid in terms of θ and χ; equation (2), roughly speaking, states that the system tends to relax in a (small) time (whose magnitude order is μ) towards an extreme (with respect to χ) point of F. In (2), we have precisely chosen $\alpha(r) + \gamma(r) = \phi'(r)$; notice anyway that, in general, the function $\phi(r)$ in (3) is not convex; hence, we can expect not only minimum energy configuration, but also unstable states which account for relative minimizers (or maximizers) of ϕ. In most "physical" cases, we have that $\phi(r) \sim (r^2-1)^2$, so that two distinct minima (which, in a more general setting, could also not attain the same minimal value) and a relative maximum for such a potential are present; here, in the expression of (2), we have chosen to denote by $\alpha(r)$ the increasingly monotone part of $\phi'(r)$ ($\sim r^3$ in the above case) and by $\gamma(r)$ the rest (behaving as $-r$); in general, on the mathematical viewpoint, α is assumed to be a *maximal monotone graph* in $\mathbb{R} \times \mathbb{R}$. Moreover, we observe that the coefficient ν is related to the *interfacial energy* of the fluid ($\nu^{1/2}$ gives a measure of the thickness of the mushy regions) and we emphasize that the term $\theta\chi$ actually should also depend on the latent heat of the fluid; finally, the interpretation of χ as the proportion of one phase is now not completely immediate, since we no longer restrict its range to $[-1,1]$ (think indeed to have now pure solid for $\chi \le -1$ and pure liquid for $\chi \ge 1$).

Referring to equations (1–2), in this dissertation we are mainly concerned with their extension to the case when the heat (and phase) exchange takes places between two different substances. So, let us suppose that the set Ω is now subdivided into two subdomains Ω_1 and Ω_2 by a smooth interface Γ. Assume also that two materials, allowed to a phase transition process, are placed in those subregions and allow them to exhibit different physical characteristics; in particular, suppose that both in Ω_1 and in Ω_2 equations analogous to (1–2) are satisfied, but naturally with different physical coefficients (also those which do not explicitely appear in (1–2), indeed). Denoting by θ_i, χ_i the unknowns in the domain Ω_i, $i = 1, 2$, we add the transmission conditions at the interface in the following form

$$\theta_1 = \theta_2 \quad \text{and} \quad \chi_1 = \chi_2 \quad \text{on } \Gamma\times]0,T[, \tag{4}$$

$$\partial_{\mathbf{n}}\theta_1 = \partial_{\mathbf{n}}\theta_2 \quad \text{and} \quad \nu_1\partial_{\mathbf{n}}\chi_1 = \nu_2\partial_{\mathbf{n}}\chi_2 \quad \text{on } \Gamma\times]0,T[, \tag{5}$$

where ν_i replaces the ν of (2), referring now to the domain Ω_i and $\mathbf{n}$ is the normal unit vector on Γ, outer, e.g., with respect to Ω_1. Equations (1–2), together with (4–5), constitute the *phase-field transmission problem*, which will be the main object of this dissertation.

Notice that relations (4–5) account for a diffusion of θ and χ through the interface Γ; while this is a standard hypothesis as far as temperature is concerned, maybe it can be surprising that the same holds for the phase variable; consider anyway that, due to the above mentioned physical interpretation of ν, it seems realistic to assume that no source of interfacial energy is present on the common boundary Γ, and this leads exactly to the flux condition for χ of (5).

The reason of the choice of the transmission problem resides in the further interest which it presents, especially from the mathematical point of view, with respect to the standard (one-substance) situation, and which is particularly due to the discontinuity of coefficients at the common boundary Γ. More specifically, while the management of most of the discontinuous parameters is not difficult, what gives some troubles is the simultaneous presence in the phase-field equation of a diffusion term (the $-\nu\Delta\chi$ of (2)) and a discontinuous (though monotone) nonlinearity, which is due to the *two* graphs α_1 and α_2 corresponding to the α of (2), but depending now on the domain Ω_1 or Ω_2. This kind of discontinuity gives rise to the occurrence of nontrivial surface terms on Γ in the derivation of the a priori estimates; consequently, some further hypotheses on α_1, α_2 seem now to be necessary in order to get an existence result for the transmission system and we believe that a considerable part of the work performed in this thesis is aimed to get over this kind of troubles.

We now outline the plan of the dissertation, which consists of two introductory chapters and of two original ones. In Chapter 1, the mathematical preliminaries are developed and some notation, which will be extensively used in the sequel, is specified. In particular, after recalling several properties of the Sobolev spaces, together with some related trace and compact embedding theorems (also in the infinite-dimensional vectorial case), we especially insist on the theory of maximal monotone operators in Hilbert spaces and on its connexions with the Γ-convergence theory for convex functionals. Naturally, we advance no claim of completeness in this part, referring indeed to the textbooks [1, 5, 14, 36] for more details; we only intend to present in an as much as possible self-contained form some results which are essential for the management of our physical problems; in particular, almost no proofs are provided in this chapter. Among the mathematical instruments which are needed in the second part of the dissertation, instead, we do not discuss here (except for some scattered remarks) the basic theory of the variational formulation of elliptic and parabolic problems. Anyway, all the related machinery which we are going to use in the sequel is rather standard; hence, we can refer for instance to the very classical textbooks of Brezis [15], Nečas [67], and Ladyzhenskaja-Solonnikov-Ural'ceva [50]. If more specifical results in this area are used somewhere, we shall give at that point the precise references.

In Chapter 2, we present in some detail the physical derivation of equations (1–2), together with some related comments and we also discuss several extensions of the physical models of phase-field and of Stefan, and, finally, some known results about the precise mathematical formulations of these problems and of related ones. Naturally, doing all this in detail would be rather heavy and go indeed far beyond the purposes of a Ph.D. dissertation; so, most of the finest thermodynamical features of the model are only briefly mentioned; furthermore, we chose to report only a number of mathematical results, which, in our opinion, are hystorically important and which also might serve as elements of comparison with respect to our subsequent approach.

We point out that the subsections devoted to the resolution of the phase-field model are not directly based on any specific paper or textbook; anyway, the mathematical procedure, which consists of a rather standard Faedo-Galerkin approximation – a priori estimates – compactness argument, is relatively simple and cannot be considered original under any aspect. We decided to present it in some detail for two precise reasons: first, it can be compared with the more abstract approach of Chapter 3, which gives results which are in good accordance with it. Second, it is simpler to prove some estimates concerning the transmission problem in this framework than in the more general abstract one; in fact, even if this procedure is actually referred to a problem related to a sole substance, there is no great difficulty in extending it to the transmission case, except for the management of the above mentioned surface terms on Γ (which is performed indeed in Chapter 3).

In any case, the objectives of this chapter are only those of outlining, in a sufficiently compact way, which are the physical features of the model we decided to study, and what there is of mathematically interesting to do about it; we hope to have been able, in very few pages, to give a reasonable idea of these points; for the sake of obtaining further details, some basic references are quoted in the course of the exposition.

The original research part of the dissertation starts from Chapter 3. Here, our objective is twofold: first, we describe, in Section 3.1, an abstract version of a generalization of equations (1–2) and we solve it by a backward finite differences approximation scheme; then, we discuss some applications to important physical cases. In particular, as a first example, we give an alternative proof of the "known" results of Chapter 2, by deducing them from the new, and more general, setting; then the abstract approach is adapted to the case of transmission problems. In particular, the question of the discontinuity of the graphs α_1, α_2 at the common boundary is deeply analyzed and solved at least under suitable compatibility or coerciveness conditions.

We particularly emphasize that the abstract method provides a solution to a problem which is stated in a weaker setting with respect to that of Chapter 2. Actually, this argument guarantees the existence of solutions under much less restrictive hypotheses on coefficients; however, it is sometimes difficult to give an interpretation of these solutions in a physical sense, that corresponds to a stronger mathematical setting. The required procedure exploits, in the case of a single substance, the machinery of the paper [10]; in the transmission case, instead, a mathematical analysis of some questions related with monotone operators of subdifferential type is needed. This study is performed in detail in Section 3.2 and the related conclusions are also compared with the standard theory of monotone operators outlined among the Preliminaries. That section concludes with the proof of the existence and uniqueness results concerning the transmission problem and with some further remarks about its thermodynamical assumptions.

Finally, the chapter ends with a last application of the abstract results for the phase-field system, which still goes in the direction of a transmission problem, which, anyway, is now related to a conduction dynamics of a *concentrated capacity* type. Indeed, this kind of model is related to a diffusion process where the thermal and phase conductivities in the domain Ω_2 are assumed to be very large at least in a privileged direction (which is the normal direction to the common boundary Γ). Then, it is possible to prove [76] that the transmission problem is well approximated by a statement where the phase-field system in Ω_1 is complemented with an analogous system on Γ, which has as unknowns the traces of the variables of Ω_1 and also presents a further source term, deriving from the original trasmission conditions (5). An existence and uniqueness theorem is proved for the concentrated capacity problem; indeed, due to the generality of the abstract approach, the procedure results to be rather similar to the standard transmission case, with the only remarkable difference concerning the choice of the functional spaces. So, the method of resolution of this problem is just briefly outlined; we refer for the full computations to the paper [77], where the same analysis has been performed by means of a more direct approach.

In the last Chapter 4, we eventually examine a number of applications of the abstract results concerning the phase field systems, again referring in particular to the case of the transmission problem. First of all, in Section 4.1, we perform a detailed asymptotic study of this situation under a blow out (or simply a variation) of some coefficients of the phase-field equation (2) in the sole domain Ω_2 (the contribution of Ω_1, instead, is essentially kept fixed). This analysis is considerably more difficult with respect to that of the problem arising from the simultaneous variation of coefficients in both domains; indeed, as we have seen for the transmission problem, the more different the equations are in Ω_1 and Ω_2, the more difficult their resolution becomes. In particular, we are able to treat the following three cases:

1. blow out of the coefficient μ in Ω_2: we get as a limit the *time-stationary* phase-field model of Plotnikov and Starovoitov [71];

2. blow out of the coefficient ν in Ω_2: we get as a limit the *phase-relaxation* model of Visintin [85];

3. variation of the operators α_1 and α_2: here we still get as a limit a kind of phase-field transmission problem; the interest of this analysis resides indeed in the type of variational convergence which is required for the varying operators.

In all these cases, uniqueness for the limit statements is also discussed.

In the second part of the chapter, we come to the last, and more interesting of the asymptotic studies concerning the transmission problem; this investigation, in a sense, puts all the above described analyses together and gives rise, as a limit, to a mixed formulation accounting for the phase-field equations on Ω_1 and the Stefan problem on Ω_2. One point seems of particular interest in this analysis: it is not difficult to verify directly that the limit statement is solvable without assuming any supplementary condition on the graphs α_1, α_2; so, it is natural to wonder at which step of the limit procedure do these conditions disappear. We precisely prove that a balance of the convergence rates of ν and α_2 in the region Ω_2 is necessary in order to have the convergence of the solutions. However, this result is not straightforward and requires a careful reformulation of the asymptotic problem in the framework of the Γ-convergence theory for convex functionals; in particular, some related results reported among the Preliminaries, and not exploited yet, turn out to be essential for the completion of proofs. The analysis of this problem, which follows the lines of our paper [80], is the final step of the dissertation.

Chapter 1

Preliminaries

In this chapter we provide the main mathematical tools which are required for the treatment and resolution of the concrete problems which we are going to study in the second part of the dissertation. Most results which are reported here are fairly classical; therefore, the proofs are generally omitted; one or more references will be given when necessary. In some cases, anyway, for the convenience of the reader, we preferred to report a brief proof; this is done, for instance, when we require a result in a different or less general form than in the setting which is normally developed on most texts.

1.1 Notation and functional spaces

In this section we report some definitions and properties of functional spaces of Sobolev type. Most results are well known, so that they are just briefly recalled; we shall insist in more detail on trace theorems and on some interpolation results concerning spaces of vector valued functions, which are perhaps more sophisticated topics. The theory is developed in the case of a bounded, connected and sufficiently smooth ($C^{1,1}$ is generally enough; if not, we shall give a remark) open set $\Omega \subset \mathbb{R}^N$; in the following, the capital letter N will always stand for the space dimension.

1.1.1 Lebesgue and Sobolev spaces

Given a number $1 \leq p \leq \infty$, we shall denote by $L^p(\Omega)$ the Lebesgue space of real-valued measurable functions on Ω of summable p-th power, endowed with the usual norm; moreover, for integer $k \geq 0$, $W^{k,p}(\Omega)$ will be the Sobolev space of real-valued functions with all (partial) derivatives (in the sense of distributions on Ω), up to the k-th order, of summable p-th power; naturally, it is $W^{0,p}(\Omega) = L^p(\Omega)$. We also set (also this notation is of common use) $H^k(\Omega) := W^{k,2}(\Omega)$. We recall that the $W^{k,p}(\Omega)$'s are Banach spaces with respect to the natural norms (for $p < +\infty$)

$$\|v\|^p_{W^{k,p}(\Omega)} := \sum_{0 \leq |\alpha| \leq k} \int_\Omega |D^\alpha v(x)|^p \, dx, \tag{1.1}$$

which is hilbertian in the case of $p = 2$. The usual modifications are required in the case of $p = +\infty$.

Given $\sigma \in]0, 1[$, $p < \infty$, we also introduce the Sobolev space (of "fractional order") $W^{\sigma,p}(\Omega)$ as the space of functions $v \in L^p(\Omega)$ such that

$$p_\sigma(v)^p := \int_\Omega \int_\Omega \frac{|v(y) - v(x)|^p}{|y - x|^{N+\sigma p}} \, dx \, dy < +\infty, \tag{1.2}$$

endowed with the graph norm $\|v\|^p_{W^{\sigma,p}} := \|v\|^p_{L^p} + p_\sigma(v)^p$ (modify as usual for $p = \infty$). Finally, if $s > 0$ is an arbitrary noninteger, decompose it as $s = k + \sigma$, with $k \in \mathbb{N}$ and $\sigma \in]0, 1[$ and define $W^{s,p}(\Omega)$ as the space of the $W^{k,p}(\Omega)$-functions with all derivatives up to the k-th order in $W^{\sigma,p}(\Omega)$ (put again the graph norm on it). We point out that, if $0 < s_1 < s_2$, we have that $W^{s_2,p}(\Omega) \subset W^{s_1,p}(\Omega)$ with compact immersion.

In the following, we shall denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable real-valued functions on Ω with compact support. We recall that its dual $\mathcal{D}'(\Omega)$ is the space of distributions on Ω, for whose properties we refer for instance to [75].

We remind that $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^n)$ for every $s \geq 0$ and $p \in [1, \infty[$. This is in general no longer true if $\mathbb{R}^N$ is substituted with Ω; so, we can denote by $W^{s,p}_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{s,p}(\Omega)$. We have anyway

Proposition 1.1.1. *Let $p \in [1, +\infty[$. It is $W^{s,p}(\Omega) = W^{s,p}_0(\Omega)$ if and only if $s \leq 1/p$ and $(s, p) \neq (1, 1)$.* ∎

Next step is to introduce the Sobolev spaces of negative order: if it is again $s > 0$ and $p > 1$, we can denote by $W^{-s,p}(\Omega)$ the dual space of $W^{s,p'}_0(\Omega)$, where p' is the *conjugate exponent* of p (i.e. $1/p + 1/p' = 1$). Since the embedding $\mathcal{D}(\Omega) \subset W^{s,p'}_0(\Omega)$, by definition of $W^{s,p'}_0$, is continuous and dense, we can see $W^{-s,p}(\Omega)$ as a subspace of $\mathcal{D}'(\Omega)$. For every $s \in \mathbb{R}$, we set again $H^s(\Omega) := W^{s,2}(\Omega)$ and, if $s > 0$, $H^s_0(\Omega) := W^{s,2}_0(\Omega)$.

We recall now the Sobolev embedding theorem only in a particular case (the Hilbert one), which will be used in the sequel; for a more general setting, we refer to the text [1] (where indeed all the main properties of Sobolev spaces are presented and deeply discussed).

Theorem 1.1.2. *Given $s > 0$, we have that, if $2s < N$, then $H^s(\Omega) \subset L^{2^*(s)}(\Omega)$, where $2^*(s) := 2N/(N - 2s)$; if $2s = N$, then $H^s(\Omega) \subset L^p(\Omega)$ for all $p \in [1, \infty[$. Finally, if $2s > N$, then we have $H^s(\Omega) \subset L^\infty(\Omega)$. All these embeddings are continuous and still hold if we substitute H^s with H^s_0.* ∎

Finally, we introduce another family of Sobolev spaces which will turn out to be very important in the statement of most trace theorems. Given a function u defined on Ω, we shall denote by $\tilde{u}$ its *trivial extension* to $\mathbb{R}^N$, that is the function which coincides with u in Ω and is identically 0 in $\mathbb{R}^N \setminus \Omega$.

Take now $s \geq 0$, $p \in [1, \infty[$ and $u \in W^{s,p}(\Omega)$: recalling Prop. 1.1.1, one could wonder when it is $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$. Indeed, it is not difficult to see that this is true for instance if $s = 0$ or if $s = 1$, $p \neq 1$ and $u \in W^{1,p}_0(\Omega)$. For intermediate exponents, the problem is more difficult:

Proposition 1.1.3. *Assume $s \geq 0$, $p \in [1,+\infty[$, and $u \in W^{s,p}(\Omega)$. If $s < 1/p$, then $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$. If $s > 1/p$, then $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$ if and only if $u \in W_0^{s,p}(\Omega)$.* ■

The preceding result leads naturally to the

Definition 1.1.4. *Given a function $u \in W^{1/p,p}(\Omega)$ (with $p \in]1,\infty[$), we say that $u \in W_{00}^{1/p,p}(\Omega)$ if and only if $\tilde{u} \in W^{1/p,p}(\mathbb{R}^N)$.* ■

The space $W_{00}^{1/p,p}(\Omega)$ is endowed with the graph norm with respect to the trivial extension operator; it could be proved that it is a Banach space which is continuously and *densely* embedded into $W^{1/p,p}(\Omega)$; in particular, owing also to Prop. 1.1.1, we deduce that $W^{-1/p,p'}(\Omega) = (W_0^{1/p,p})'(\Omega) = (W^{1/p,p})'(\Omega) \subset (W_{00}^{1/p,p})'(\Omega)$ (with proper inclusion). Set also (as usual) $H_{00}^{1/2}(\Omega) := W_{00}^{1/2,2}(\Omega)$.

1.1.2 Traces and integration formulas

We report here some important trace theorems for Sobolev-regular functions. While some results are well known and of common use in the theory of boundary value problems for PDE's and consequently will be only briefly recalled, we also need to present some more refined situations, especially concerning traces on proper subsets of a boundary or optimal regularity estimates for extension operators. Strictly connected with the theory of traces are some extensions of the Gauss-Green formula which will also be recalled.

In order to fit the notation of the subsequent parts of the thesis, we consider here a smooth, bounded and *connected* open set $\Omega_1 \subset \mathbb{R}^N$; Γ and Γ_1 will be proper submanifolds of its boundary $\partial\Omega_1$ of strictly positive $(N-1)$-dimensional measures and also verifying $\overline{\Gamma} \cup \overline{\Gamma}_1 = \partial\Omega_1$.

We remark that in the following we will be concerned with Sobolev spaces defined on the boundaries $\partial\Omega_1$, Γ and Γ_1. In our (smooth) case, they can be introduced essentially as in the previous subsection, by means of a system of local charts; in a more general setting (i.e. nonsmooth manifolds or unbounded ones), instead, the definitions are much more complicated [6]. We only point out that, essentially because $\partial\Omega_1$ is a compact manifold (where no boundary is present), it could be proved that $W^{s,p}(\partial\Omega_1) = W_0^{s,p}(\partial\Omega_1)$ for any (suitable) choice of (s,p). This is no longer true for the spaces living on Γ and Γ_1, where the exponents must satisfy conditions as those of Prop. 1.1.1; so, also Prop. 1.1.3 and the subsequent Def. 1.1.4 can be extended to this case, provided that now $\tilde{}$ denotes the trivial extension to $\partial\Omega_1$.

Theorem 1.1.5. (a) *For any $p \in [1,+\infty[$, there exists a linear, continuous and* surjective *trace operator*

$$\gamma_{\partial\Omega_1} : W^{1,p}(\Omega_1) \to W^{1-1/p,p}(\partial\Omega_1), \tag{1.3}$$

coinciding with the restriction operator (in a classical sense) on the functions of $C^0(\overline{\Omega}_1) \cap W^{1,p}(\Omega_1)$ regularity.

(b) *For any $p \in [1,+\infty[$, there exists an extension operator, that is a linear and continuous operator*

$$\mathcal{E} : W^{1-1/p,p}(\partial\Omega_1) \to W^{1,p}(\Omega_1) \tag{1.4}$$

such that $\gamma_{\partial\Omega_1}\mathcal{E}v = v$ *for every* $v \in W^{1-1/p,p}(\partial\Omega_1)$.
(c) *The conclusions of* (a) *and* (b) *are still valid by substituting therein* $\partial\Omega_1$ *with* Γ *or* Γ_1 *(denote the related trace operators by* γ_Γ *and* γ_{Γ_1} *respectively).* ∎

We point out that the operator $\mathcal{E}$ in (b) is not unique; in the sequel we shall meet a case where a particular construction of $\mathcal{E}$ is needed. Also, in order to lighten the notation, when no danger of confusion is present, we shall avoid to write explicitely the trace operators, actually denoting by the same symbol a function and its trace(s).

In order to state the next theorem, we now have to introduce a new family of Sobolev spaces: set, for $1 \le p \le \infty$,

$$L^p_{\mathrm{div}}(\Omega_1) := \{\mathbf{v} \in L^p(\Omega_1)^N : \operatorname{div}\mathbf{v} \in L^p(\Omega_1)\}, \tag{1.5}$$

endowed with the natural norm; for $p = 2$, define also $H(\mathrm{div}, \Omega_1) := L^2_{\mathrm{div}}(\Omega_1)$ (which is a Hilbert space). The divergence operator in the definition is clearly to be intended in the sense of distributions. It is also obvious that $W^{1,p}(\Omega_1)^N \subset L^p_{\mathrm{div}}(\Omega_1)$ with continuous immersion and equality holding for $N = 1$.

In the following, we shall denote as $\mathbf{n}$ the outer normal unit vector to Ω_1. We have:

Theorem 1.1.6. (a) *For any* $p \in]1, +\infty[$*, there exists a linear, continuous and* surjective *trace operator*

$$\gamma_{\mathbf{n}} : L^p_{\mathrm{div}}(\Omega_1) \to W^{-1/p,p}(\partial\Omega_1), \tag{1.6}$$

such that $\gamma_{\mathbf{n}}(\mathbf{v}) = \mathbf{v}\cdot\mathbf{n}$ *for any* $\mathbf{v} \in C^0(\overline{\Omega_1})^N \cap L^p_{\mathrm{div}}(\Omega_1)$.
(b) *For any* $p \in]1, +\infty[$*, there exists an extension operator, that is a linear and continuous operator*

$$\mathcal{E}_{\mathbf{n}} : W^{-1/p,p}(\partial\Omega_1) \to L^p_{\mathrm{div}}(\Omega_1) \tag{1.7}$$

such that $\gamma_{\mathbf{n}}\mathcal{E}_{\mathbf{n}}v = v$ *for every* $v \in W^{-1/p,p}(\partial\Omega_1)$.
(c) *If* $v \in W^{1,p'}(\Omega_1)$, $\mathbf{w} \in L^p_{\mathrm{div}}(\Omega_1)$*, then we have*

$$\int_{\Omega_1} \nabla v \cdot \mathbf{w}\, dx = -\int_{\Omega_1} v \operatorname{div}\mathbf{w}\, dx + {}_{W^{-1/p,p}}\langle \gamma_{\mathbf{n}}(\mathbf{w}), v\rangle_{W^{1/p,p'}}. \tag{1.8}$$ ∎

The last expression provides a generalization of the Gauss-Green formula; notice that all the terms appearing therein make sense.

We now want to adapt the last result to the case of traces on Γ (or Γ_1). With this aim, given $s > 0$ and $p \in [1, \infty[$, we note as $C^\infty_{0,\Gamma}(\Omega_1)$ the space of the $C^\infty(\Omega_1)$-functions which are identically 0 in *a neighbourhood* of Γ and we define $W^{s,p}_{0,\Gamma}(\Omega_1)$ as the closure of $C^\infty_{0,\Gamma}(\Omega_1)$ in $W^{s,p}(\Omega_1)$. Clearly, it is $C^\infty_{0,\partial\Omega_1}(\Omega_1) = \mathcal{D}(\Omega_1)$ and $W^{s,p}_{0,\partial\Omega_1}(\Omega_1) = W^{s,p}_0(\Omega_1)$. Recalling Prop. 1.1.3 and Def. 1.1.4, it is immediate to derive (the same obviously holds if we interchange Γ and Γ_1):

Proposition 1.1.7. *If* $v \in W^{1,p}_{0,\Gamma}(\Omega_1)$*, then* $\gamma_{\Gamma_1}v \in W^{1-1/p,p}_0(\Gamma_1)$ *if* $p \neq 2$ *and* $\gamma_{\Gamma_1}v \in H^{1/2}_{00}(\Gamma_1)$ *if* $p = 2$. ∎

So, the most difficult case is also the most common, i.e. the Hilbert one. Note, for $p = 2$, another interesting consequence: taking $v \in H^1_{0,\Gamma}(\Omega_1) = W^{1,2}_{0,\Gamma}(\Omega_1)$ and $\mathbf{w} \in H^2(\Omega_1)^N$, we have (compare with (1.8))

$$(1.9) \qquad {}_{H^{-1/2}}\langle \mathbf{w}\cdot\mathbf{n}, v\rangle_{H^{1/2}} = \int_{\partial\Omega_1} \mathbf{w}\cdot\mathbf{n}\, v\, d\mathcal{H}^{N-1} = \int_{\Gamma_1} \mathbf{w}\cdot\mathbf{n}\, v\, d\mathcal{H}^{N-1}$$

since the trace of v on the remaining part (Γ) of $\partial\Omega$ is 0. If we try and extend the above expression to the case of $\mathbf{w} \in H(\mathrm{div}, \Omega_1)$, we notice that, in order it makes sense, it is enough that $\mathbf{w}\cdot\mathbf{n} \in (H^{1/2}_{00})'(\Gamma_1)$, since $\gamma_{\Gamma_1} v \in H^{1/2}_{00}(\Gamma_1)$ which is a proper subspace of $H^{1/2}(\Gamma_1)$. Moreover, if it is clear what means to take the restriction of a L^p function, the same is far from obvious in the case of functionals belonging to Sobolev spaces of negative order.

The following result (find the proof and more details in [39]) explains the sense of the restriction operator in these cases and also summarizes the behaviour of operator $\gamma_{\mathbf{n}}$ on proper subsets of the boundary:

Proposition 1.1.8. *Given two functional spaces $X(\partial\Omega_1)$ and $Y(\Gamma)$ such that the extension operator $\tilde{}$ is continuous from Y to X, we define the generalized restriction operator as $(\ )_{|\Gamma} : X'(\partial\Omega_1) \to Y'(\Gamma)$ as the adjoint of operator $\tilde{}$.*

For instance, given $\phi \in H^{-1/2}(\partial\Omega_1)$, we can set, for $v \in H^{1/2}_{00}(\Gamma)$,

$$(1.10) \qquad \langle \phi_{|\Gamma}, v\rangle := \langle \phi, \tilde{v}\rangle,$$

where $\tilde{}$ is now the trivial extension operator from $H^{1/2}_{00}(\Gamma)$ to $H^{1/2}(\partial\Omega_1)$. Thus, we have that $\phi_{|\Gamma} \in (H^{1/2}_{00})'(\Gamma)$ (in general, it is not *$\phi_{|\Gamma} \in H^{-1/2}(\Gamma)$); moreover, we have that $\phi_{|\Gamma} \in H^{-1/2}(\Gamma)$ if and only if $\phi_{|\Gamma_1} \in H^{-1/2}(\Gamma_1)$. Finally, if this is true, for some $C > 0$ depending only on the domain, it results*

$$(1.11) \qquad \|\phi_{|\Gamma}\|_{H^{-1/2}(\Gamma)} \le C(\|\phi\|_{H^{-1/2}(\partial\Omega_1)} + \|\phi_{|\Gamma_1}\|_{H^{-1/2}(\Gamma_1)}),$$

and the same of course continues holding if we interchange Γ and Γ_1. ■

Remark 1.1.9. *Take now $\phi \in H^{-1/2}(\partial\Omega_1)$ and $v \in H^{1/2}(\partial\Omega_1)$; we see that in general it is* not *possible to write in some suitable sense*

$$(1.12) \qquad \langle \phi, v\rangle = \langle \phi_{|\Gamma}, v_{|\Gamma}\rangle + \langle \phi_{|\Gamma_1}, v_{|\Gamma_1}\rangle.$$

One *situation when this decomposition is allowed is when the (for instance) first duality on the right hand side would result equal to 0; this is the case of either $v_{|\Gamma} = 0$ (so that it results $v_{|\Gamma_1} \in H^{1/2}_{00}(\Gamma_1)$ and the second duality makes sense in $H^{1/2}_{00}$), or $\phi_{|\Gamma} = 0$ (and, owing to the preceding proposition, this implies $\phi_{|\Gamma_1} \in H^{-1/2}(\Gamma_1)$ and the second duality works now in $H^{1/2}$).*

1.1.3 Hilbert triplets

In the course of the thesis, we shall face several different physical situations; so, in order to unify the computations as much as possible, it is necessary to introduce a general abstract setting which turns out to be convenient for almost all the physical problems.

Let us suppose V and H be Hilbert spaces, with V densely and compactly embedded into H; name I the embedding (its compactness is actually required only for the last property). In this situation, it is customary to identify H with H' through the Riesz-Fréchet theorem; it is also a common notation that of $(\cdot,\cdot)$ for the scalar product of H and of $((\cdot,\cdot))$ for that of V. Also, the norm of H is indicated by $|\cdot|$ and that of V by $\|\cdot\|$.

Observe now that, for any $h \in H$, $v \in V$, we have

$$|(h, Iv)| \leq |h||Iv| \leq |h|\|I\|_{\mathcal{L}(V,H)}\|v\|, \tag{1.13}$$

so that h can be seen as a linear and continuous functional on V, i.e. an element of V'. Moreover, if we indicate by $\langle\cdot,\cdot\rangle$ the duality between V' and V and by I^*h the above introduced functional, it is immediate to verify that

$$\langle I^*h, v\rangle = (h, Iv) \quad \text{for all} \;\; h \in H \;\; \text{and} \;\; v \in V; \tag{1.14}$$

moreover the operator I^* is linear, continuous and injective between H and V'. So, the notation I^*h is not accidental, since we see from the preceding relation that I^* is just the adjoint operator of I. We also have that $\|I^*\|_{\mathcal{L}(H,V')} = \|I\|_{\mathcal{L}(V,H)}$.

In this situation, the identification of V with V' becomes misleading; indeed, it is related to a different scalar product (that of V instead of that of H which allowed to identify H and H'); consequently, it is generally preferred to avoid it; anyway, the *Riesz operator* on V can still be introduced as

$$\mathcal{R} : V \to V'; \quad \langle \mathcal{R}v, w\rangle := ((v, w)) \quad \text{for} \;\; v, w \in V. \tag{1.15}$$

This allows to define also a dual scalar product on V' as

$$((\phi, \psi))_* := ((\mathcal{R}^{-1}\phi, \mathcal{R}^{-1}\psi)) = \langle \phi, \mathcal{R}^{-1}\psi\rangle = \langle \psi, \mathcal{R}^{-1}\phi\rangle \quad \text{for} \;\; \phi, \psi \in V'; \tag{1.16}$$

the corresponding norm on V' will be called $\|\cdot\|_*$.

Furthermore, it is immediate to verify that the scalar product of V' and the transpose immersion I^* are linked by the relation

$$(I\mathcal{R}^{-1}\psi, h) = \langle I^*h, \mathcal{R}^{-1}\psi\rangle = ((I^*h, \psi))_* = \langle \psi, \mathcal{R}^{-1}I^*h\rangle, \tag{1.17}$$

holding for $h \in H$, $\psi \in V'$, which entails that $I\mathcal{R}^{-1}$ is the adjoint of $\mathcal{R}^{-1}I^*$ and will result useful in the sequel.

In this situation, we say that the spaces (V, H, V') form a *Hilbert triplet.* We recall a further property (easily following from the Hahn-Banach theorem):

Proposition 1.1.10. *$I^*(I(V))$ is dense in V'.* ■

In particular, we have that $I^*(H)$ is dense in V'.

Notice that we have not used yet the compactness hypothesis on I. The only point where it is required is the following (and easy too)

Proposition 1.1.11. *The transpose embedding $I^* : H \to V'$ is compact.* ■

We remark that in most cases, V is a proper subspace of H, so the embedding I is a true inclusion and it is generally omitted in the notation. The same is done for the transpose embedding I^*, so that we will generally write $V \subset H \subset V'$ and consider also the elements of H as functionals of V'. Of course, the sense is still that given by (1.14)); in particular, for $h \in H$ and $v \in V$, we can now write $\langle h, v\rangle = (h, v)$.

1.1.4 Spaces of vector valued functions

We now want to extend the definitions of Lebesgue and Sobolev space to the case of functions with values in an arbitrary Banach space B. We will limit ourselves to giving the definitions and some properties in a particular case; we refer to [14, Appendix] for a much more general overview of the matter (and also for all the proofs). In the following, $T > 0$ will be a given number (to be thought as the *final time* of some process).

First of all, we recall that the space $C^0([0,T];X)$, defined in the habitual way, is a Banach space with the norm $\|v\|_{C^0([0,T];X)} := \max_{0\le t\le T} \|v(t)\|_X$. Given a function $v : [0,T] \to X$, we can examine the behaviour of its derivative v'; if it is $v' \in C^0([0,T];X)$, then we naturally say that $v \in C^1([0,T];X)$; by induction, we can introduce the spaces $C^k([0,T];X)$ for $k \in \mathbb{N}$, which are all Banach with the natural norms.

More difficult is the definition of the spaces of measurable functions; so, we only give some highlights: we say that a function $v : [0,T] \to X$ belongs to $L^p(0,T;X)$ if and only if it is *measurable* (in some suitable sense, for whose (nontrivial) mathematically precise explanation we refer to [14]) and satisfies

$$\|v\|^p_{L^p(0,T;X)} := \int_0^T \|v(t)\|^p_X \, dt < \infty. \tag{1.18}$$

We point out that it is the condition of measurability that guarantees that also $t \mapsto \|v(t)\|_X$ is a measurable function (in the usual sense). Indeed, $L^p(0,T;X)$ turns out to be a Banach space with respect to the above introduced norm.

Finally, we introduce the Sobolev spaces. The fastest definition is perhaps the following one (see again [14] for the definition of integrals of vector-valued functions):

Definition 1.1.12. *If $p \in [1,\infty]$ and $v \in L^p(0,T;X)$, we say that $v \in W^{1,p}(0,T;X)$ if and only if there exist $c \in X$, $z \in L^p(0,T;X)$ such that*

$$v(t) = c + \int_0^t z(s)\, ds \qquad \text{for a.e. } \ t \in [0,T]. \tag{1.19}$$

Under such a condition we say that z is the derivative of v in the sense of vector-valued distributions and denote it (as usual) by v'. ■

Most properties (but not all!) of standard Sobolev spaces remain true in the vector case. First of all, the above definition can of course be extended to introduce Sobolev spaces of higher integer order k; moreover, if we set $H^k(0,T;X) = W^{k,2}(0,T;X)$ and

X is a Hilbert space H, we still have that $H^k(0,T;H)$ is a Hilbert space with respect to the scalar product

$$(v,w)_{H^k(0,T;H)} := \sum_{j=0}^{k} \int_0^T (v^{(j)}(t), w^{(j)}(t))_H \, dt. \tag{1.20}$$

Let us suppose now that X, Y are Banach spaces such that $X \subset Y$ with continuous inclusion. It is worthwhile to establish if the continuity of the inclusion remains true when we pass to Banach spaces of functions with values in X, Y. We just cite one result:

Proposition 1.1.13. *Let $p, q \in]1, \infty[$ be given numbers. We have:*
(a) *It is $W^{1,p}(0,T;X) \subset C^0([0,T];X)$ with continuous, but not compact, inclusion. Instead, if $X \subset Y$ is compact, then also $W^{1,p}(0,T;X) \subset C^0([0,T];Y)$ is compact.*
(b) *We have that $W^{1,p}(0,T;X) \subset L^q(0,T;Y)$ with continuous inclusion; this is also compact if and only if such is the inclusion $X \subset Y$.* ∎

If a Hilbert triplet (V, H, V') is considered (and here the compactness of the embedding $V \subset H$ is essential!), it is now easy to see that the inclusion $H^1(0,T;V) \subset C^0([0,T];V)$ is continuous but not compact; it is compact instead $H^1(0,T;V) \subset C^0([0,T];H)$, and such is $H^1(0,T;H) \subset C^0([0,T];V')$. In the case of Hilbert triplets we have indeed some other interesting properties, which we can summarize in the following proposition. We remark that a linear operator $A : V \to V'$ is said to be *weakly elliptic* if and only if there exist $\alpha, \lambda_0 > 0$ such that

$$\langle Av, v \rangle + \lambda_0 |v|^2 \geq \alpha \|v\|^2 \quad \text{for all} \quad v \in V \tag{1.21}$$

(the notations for norms are as in the previous subsection).

Proposition 1.1.14. (a) *It is $L^2(0,T;V) \cap H^1(0,T;V') \subset C^0([0,T];H)$ with continuous inclusion; moreover, if $v, w \in L^2(0,T;V) \cap H^1(0,T;V')$, then, for any $s, t \in [0,T]$, we have:*

$$\int_s^t \langle v'(r), w(r) \rangle \, dr = -\int_s^t \langle w'(r), v(r) \rangle \, dr + (v(t), w(t)) - (v(s), w(s)). \tag{1.22}$$

(b) *Consider a* self-adjoint *weakly elliptic operator $A : V \to V'$ and two functions $v, w \in L^2(0,T;V) \cap H^1(0,T;H)$ such that $Av, Aw \in L^2(0,T;H)$. Then we have that $v, w \in C^0([0,T];V)$; moreover, for any $s, t \in [0,T]$, it is:*

$$\begin{aligned} &\int_s^t (v'(r), Aw(r)) \, dr + \int_s^t (w'(r), Av(r)) \, dr \\ &\qquad = \langle Av(t), w(t) \rangle - \langle Aw(s), v(s) \rangle. \end{aligned} \tag{1.23}$$ ∎

Find the proofs, e.g., in [7]; just notice here that the right hand sides of relations (1.22–1.23) make sense precisely because of the regularities in the first part of the related statements.

1.1.5 Miscellanea

We present here some notation and some basic inequalities which will be repeatedly used in the sequel of the dissertation; in particular, our purpose is to simplify the computations involved in the deduction of several a priori estimates.

We start by stating the elementary Young inequality:

$$ab \le \sigma a^2 + \frac{1}{4\sigma} b^2 \quad \text{for every} \;\; a, b \in \mathbb{R} \;\; \text{and} \;\; \sigma > 0. \tag{1.24}$$

Provided that C is (as it will be in the deduction of the a priori estimates) a nonnegative constant depending only on some data (it will result clear from the specific situation which ones), we prefer to rewrite the above formula in a modified way, whose notation will be rather convenient in the practical cases:

$$Cab \le \sigma a^2 + C_\sigma b^2 \quad \text{for every} \;\; a, b \in \mathbb{R} \;\; \text{and} \;\; \sigma > 0. \tag{1.25}$$

Here, σ is thought to be a constant which we are interested in keeping small and C_σ a corresponding value, obviously depending also of the (supposed known) constant C.

We continue, by listing a general form of Gronwall inequality, for whose proof we refer to [7, Th. 2.1, page 245] or to [14, Lemmas A.4, A.5, pages 156–157]

Proposition 1.1.15. *Let $a, T > 0$; $m_1, m_2 \in L^1(0,T)$, $\phi \in L^\infty(0,T)$ such that $m_1, m_2, \phi \ge 0$ a.e. in $]0,T[$. Suppose also that*

$$\frac{1}{2}\phi^2(t) \le \frac{1}{2}a^2 + \int_0^t m_1(s)\phi(s)\, ds + \int_0^t m_2(s)\phi^2(s)\, ds \qquad a.e. \ in \ ]0,T[. \tag{1.26}$$

Then, we have that

$$\phi(t) \le C\left(a + \int_0^T m_1(s)\, ds\right) \qquad a.e. \ in \ ]0,T[, \tag{1.27}$$

where C depends only on a, m_1, m_2. Moreover, if $m_1 \equiv 0$, then we can take $C = \exp\left(\int_0^T m_2(s)\, ds\right)$, while, if $m_2 \equiv 0$, then we can take $C = 1$. ∎

From the quoted paper of Baiocchi, we report a well-known regularity result for parabolic systems, which will be often used in the sequel.

Theorem 1.1.16. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $T > 0$ and take $V = H^1(\Omega)$ (or $V = H_0^1(\Omega)$), $H = L^2(\Omega)$, so that (V, H, V') is a Hilbert triplet. Suppose also that $A : V \to V'$ is a weakly elliptic operator, and choose $u_0 \in V$ and $f \in L^2(0,T; L^2(\Omega)) + W^{1,1}(0,T; H^1(\Omega))$. Then, the solution u of the evolution problem*

$$\begin{cases} \partial_t u + Au = f & in \ V', \quad a.e. \ in \ ]0,T[, \\ u(0) = u_0 & in \ \Omega \end{cases} \tag{1.28}$$

fulfills the regularity property $\theta \in C^0([0,T]; V)$. ∎

Finally, we also state a "discrete" form of Gronwall's inequality, which is useful when dealing with finite-differences approximations of differential problems:

Proposition 1.1.17. *Let $a, m > 0$, and $c_j \in \mathbb{R}$ for $j = 0, \dots, n$, satisfying*

$$c_k \leq a + m \sum_{j=1}^{k-1} c_j \quad \text{for all } k = 1, \dots, n. \tag{1.29}$$

Then,

$$c_k \leq a(1+m)^{k-1} \quad \text{for all } k = 1, \dots, n. \ \blacksquare \tag{1.30}$$

1.2 Convex analysis tools

1.2.1 Convex functions and duality

In this section, unless otherwise specified, X will be a real reflexive Banach space and X' its (topological) dual space; the duality between X' and X will be denoted as $\langle \cdot, \cdot \rangle$. Moreover, Ψ will be a convex functional on X with values in $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$. The *domain* of Ψ is defined as follows:

$$D(\Psi) := \{x \in X : \Psi(x) \neq +\infty\}; \tag{1.31}$$

clearly, it is a convex set in X. Moreover, we say that Ψ is proper if and only if $D(\Psi) \neq \emptyset$. In this case, we define, for every $f \in X'$,

$$\Psi^*(f) := \sup_{x \in X} \{\langle f, x \rangle - \Psi(x)\}. \tag{1.32}$$

We remark that the supremum in the above definition could be restricted to the $x \in D(\Psi)$; moreover, we observe that Ψ^* is a superior envelope of linear affine functions of f; so, it is convex and lower semicontinuous (l.s.c. in the following).

We introduce another related definition: the *epigraph* of Ψ is given by:

$$\operatorname{epi}(\Psi) := \{(x, \lambda) \in X \times \mathbb{R} : \lambda \geq \Psi(x)\}. \tag{1.33}$$

It is not difficult to verify [15] that

$$\Psi \ \text{ convex} \Longleftrightarrow \operatorname{epi}(\Psi) \ \text{ convex} \tag{1.34}$$

$$\Psi \ \text{ l.s.c.} \Longleftrightarrow \operatorname{epi}(\Psi) \ \text{ closed} \tag{1.35}$$

$$\Psi \ \text{ proper} \Longleftrightarrow \operatorname{epi}(\Psi) \ \text{ not empty.} \tag{1.36}$$

Take now an arbitrary (i.e., not necessarily convex) functional Ψ on X. We introduce the *relaxed functional* $\operatorname{sc}^- \Psi$, as follows [31, Def. 3.1, page 28]:

$$\operatorname{sc}^- \Psi(x) := \sup_{U \in \mathcal{U}(x)} \inf_{y \in U} \Psi(y); \tag{1.37}$$

here (and in the sequel), $\mathcal{U}(x)$ denotes a base of neighborhoods of x for the topology of X.

Proposition 1.2.1. *The functional* $\mathrm{sc}^{-}\Psi$ *can be also characterized in the following ways:*

(a) $\mathrm{epi}(\mathrm{sc}^{-}\Psi) = \overline{\mathrm{epi}(\Psi)}$*; that is, the closure (in* $X \times \mathbb{R}$*) of the epigraph of* Ψ *is still an epigraph, and precisely that of* $\mathrm{sc}^{-}\Psi$*.*

(b) *For every* $x \in X$*, we have that*

$$\mathrm{sc}^{-}\Psi(x) \le \liminf_{n\to\infty} \Psi(x_n) \quad \text{for every } x_n \to x, \tag{1.38}$$

$$\mathrm{sc}^{-}\Psi(x) = \limsup_{n\to\infty} \Psi(x_n) \quad \text{for some } x_n \to x. \tag{1.39}$$ ■

Remark 1.2.2. *If we consider more general situations, for example if* X *is an arbitrary topological vector space, characterization (a) still holds; (b) instead remains valid provided that* X *is metrizable. For instance, if* X *is a reflexive Banach space endowed with its* weak *topology, then the functional satisfying the properties (1.38–1.39) (with* $x_n \to x$ *intended now in the weak sense) represents the relaxation of* Ψ *in the* sequential weak *topology, and this can be strictly greater than* $\mathrm{sc}^{-}\Psi$*. We observe anyway that, in case* Ψ *is convex, (a) and (b) are still valid, since, thanks to the Hahn-Banach theorem, the closure of* $\mathrm{epi}(\Psi)$ *(which is a convex set) is the same in the strong, weak and sequential weak topologies; in particular, the strong and weak relaxations of* Ψ *give now rise to the same functional.*

The procedure of conjugation for (proper) functionals can be iterated; indeed, for $x \in X$, we can set

$$\Psi^{**}(x) := \sup_{f\in X'} \{\langle f, x\rangle - \Psi^*(f)\}. \tag{1.40}$$

The following result, due to Fénchel and Moreau, is one of the fundamental tools in convex analysis:

Theorem 1.2.3 ([15]). *If* Ψ *is convex, then* $\Psi^{**} \equiv \mathrm{sc}^{-}\Psi$*; in particular, if* Ψ *is convex and l.s.c., we have that* $\Psi^{**} = \Psi$*.* ■

Remark 1.2.4. *A characterization of* Ψ^{**} *can be given also when* Ψ *is not convex; indeed, in this case* Ψ^{**} *is the so-called* Γ*-regularization of* Ψ*, that is the greatest convex and l.s.c. functional majorized by* Ψ *(or, also, the function whose epigraph is the convex closed envelope of* $\mathrm{epi}(\Psi)$*).*

1.2.2 Maximal monotone operators. Subdifferentials

We introduce here a notion which plays an essential role in the statement of the problems of phase transition in their variational version. For instance, the weak formulation of the Stefan problem in terms of monotone graphs goes back to Olenik [69] and Damlamian [32].

For this part, the standard reference texts are Barbu [9] and Brezis [14]; see also Attouch [5] for the connexions with the theory of Γ-convergence. In the following, we suppose X to be a *uniformly convex* [15] Banach space (so that, in particular, it is reflexive). We denote in the following by $\langle\cdot,\cdot\rangle$ the duality between X and X' and by $J : V \to V'$ the related duality map (which is single-valued owing to the uniform convexity of X), i.e.

$$\langle Jx, x\rangle = \|x\|_X^2 \quad \text{and} \quad \|Jx\|_{X'} = \|x\|_X \quad \text{for } x \in X. \tag{1.41}$$

We begin by a fundamental

Definition 1.2.5. *A (nonlinear) operator on X is a subset $A \subset X \times X'$.* ■

We have chosen to identify nonlinear operators with their graphs rather then viewing them as functions of X into X'; indeed, in the following we will be concerned with *multivalued* operators, i.e. operators A such that, given $x \in X$, it can happen that $Ax := \{y \in X' : (x, y) \in A\}$ is not a singleton. In the sequel, we will use indifferently either of notations $y \in Ax$ and $(x, y) \in A$.

Moreover, we define the *domain* of A as the set $D(A) := \{x \in X : Ax \neq \emptyset\}$ and its *range* as $R(A) := \cup_{x\in X} Ax$.

Remark 1.2.6. *We point out that, in case X is a Hilbert space H identified with its dual, A will be naturally be seen as a subset of $H \times H$; if a Hilbert triplet (V, H, V') is considered instead, an operator A of V will be seen as a subset of $V \times V'$. It is easy to verify that, in this case, setting $(x, y) \in A_V \iff (x, \mathcal{R}y) \in A$, where $\mathcal{R}$ is the Riesz operator of V, we canonically associate to A another operator $A_V \subset V \times V$ (related to the identification of V with V').*

Definition 1.2.7. *An operator $A \subset X \times X'$ is said to be*

(a) monotone*, if and only if, for any $(x_1, y_1) \in A$, $(x_2, y_2) \in A$, we have that $\langle y_2 - y_1, x_2 - x_1\rangle \geq 0$*

(b) maximal monotone*, if and only if it is monotone and also fulfils the surjectivity property $R(J + A) = X'$.* ■

The following result accounts for other equivalent definitions of maximal monotone operators; while the proof of implication (b)$\iff$(c) is trivial, the equivalence of (a) and (b) is a consequence of a difficult theorem of Rockafellar [73]. For the proof, you can also refer to the text [14].

Proposition 1.2.8. *Let A be a monotone operator on X. Then the following conditions are equivalent:*

(a) *A is maximal monotone;*

(b) *for every monotone operator $B \subset X \times X'$ such that $A \subset B$ (in sense of graph inclusion), it results $A = B$;*

(c) *if $(\xi, \eta) \in X \times X'$ fulfils $\langle y - \eta, x - \xi\rangle \geq 0$ for every $(x, y) \in A$, then it is also $(\xi, \eta) \in A$.* ■

Also, it is not difficult to verify that, if A is maximal monotone, then Ax is a convex closed set of X'. Given $x \in D(A)$, we denote by A^0x the element of minimum norm in Ax, whose existence and uniqueness can be proved by exploiting the reflexivity of X and proceeding by weak compactness methods, see [5].

Proposition 1.2.9. *Let A be a maximal monotone operator, J the duality map between X and X', $\epsilon > 0$. Then, the function $(J + \epsilon A)^{-1}J : X \to X$ is single-valued, has the whole X as domain, $D(A)$ as range and is continuous (if X is a Hilbert space, it is Lipschitz continuous of Lipschitz constant 1). We name it (ϵ-)resolvent of A and denote it by $\mathcal{J}_\epsilon$.* ∎

Definition 1.2.10. *Under the same hypotheses as above, we set, for $x \in X$, $A_\epsilon x := \epsilon^{-1}J(x - \mathcal{J}_\epsilon x)$; this is the Yosida-regularization of operator A.* ∎

The main properties of Yosida-regularizations are stated in the following

Proposition 1.2.11. (a) *A_ϵ is a single-valued maximal monotone operator on X which has X as domain; moreover, it is continuous (in the Hilbert case Lipschitz continuous, with ϵ^{-1} as a Lipschitz constant).*
(b) *For every $x \in D(A)$, we have that $A_\epsilon x \to A^0 x$ as $\epsilon \to 0$; moreover, we have the monotonicity $\|A_{\epsilon_1} x\|_{X'} \leq \|A_{\epsilon_2} x\|_{X'}$ for $\epsilon_2 \leq \epsilon_1$. Finally, for $x \in X \setminus D(A)$, we have that $\|A_\epsilon x\|_{X'} \to \infty$ as $\epsilon \to 0$.* ∎

We give now a definition which it is convenient to state in a somehow more general setting than before:

Definition 1.2.12. *Let B, C be Banach spaces, $A : B \to C$ be a (single-valued) operator defined on the whole B. We say that A is*
(a) demicontinuous *if and only if it is continuous with respect to the strong topology of B and the weak one of C;*
(b) hemicontinuous *if and only if, for every $x, y \in B$, we have that*

$$\lim_{t \to 0} A(x + ty) = Ax \quad \text{weakly } \textit{in } C. \tag{1.42}$$ ∎

Naturally, in the above definition, we have (a)⇒(b); moreover, we are particularly interested in the case of $B = X$, $C = X'$ (X as above):

Proposition 1.2.13. *If $A : X \to X'$ is monotone hemicontinuous, then it is maximal monotone.* ∎

The following definition permits to identify another important class of maximal monotone operators.

Definition 1.2.14. *Given $\Psi : X \to \mathbb{R}_\infty$ convex, l.s.c. and proper, we call* subdifferential *of Ψ the operator of X defined, for $x \in X$, by*

$$y \in \partial\Psi(x) \iff \langle y, z - x \rangle \leq \Psi(z) - \Psi(x) \quad \textit{for all } z \in X. \tag{1.43}$$ ∎

We remark that it is enough to verify the above property for all $z \in D(\Psi)$ (otherwise the inequality is trivial); moreover, we observe that $D(\partial\Psi) \subset D(\Psi)$. Under the preceding hypotheses, we also have:

Theorem 1.2.15. *$\partial\Psi$ is a maximal monotone operator.* ∎

An important example of subdifferential operator is the following: in a Hilbert space $\mathcal{H}$, take a convex, closed, not empty subset $K \subset \mathcal{H}$ and denote by I_K the *indicator function* of K; i.e. the function which is identically 0 on K and $+\infty$ elsewhere. It is easy to see that the properties of K entail I_K to be convex, l.s.c. and proper. Moreover, the subdifferential of I_K is a maximal monotone operator which has K as domain and takes identically the value 0 in the interior of K.

Yosida regularizations of subdifferential operators can be characterized in the following additional way:

Proposition 1.2.16. (a) *For all $\epsilon > 0, x \in X$, we can set*

$$\Psi_\epsilon(x) := \min_{y \in X} \left\{ \frac{1}{2\epsilon} \|y - x\|_X^2 + \Psi(y) \right\}; \tag{1.44}$$

then, the minimum on the right hand side of the above expression is achieved in an unique point $\bar{y}$ and precisely in $\bar{y} = \mathcal{J}_\epsilon x$. The functional Ψ^ϵ is called Moreau-Yosida approximation *of Ψ.*

(b) *The function Ψ_ϵ is convex, proper (with X as effective domain) and Fréchet-differentiable for all $x \in X$. Moreover, we have that*

$$\partial \Psi_\epsilon(x) = \{\Psi'_\epsilon(x)\} = A_\epsilon, \tag{1.45}$$

where A_ϵ is the Yosida approximation of operator $A := \partial\Psi$. Finally, $\Psi_\epsilon(x) \nearrow \Psi(x)$ for every $x \in X$. ■

We give now some sufficient condition for a maximal monotone operator A to be surjective; recalling Def. 1.2.7 (b), intuitively it will be necessary for A to satisfy some more coerciveness property than the only monotonicity. Here we only report one result [14, Cor. 2.4, page 31], which will be exploited in the sequel:

Theorem 1.2.17. *Let X be a Hilbert space (identified with its dual) of scalar product $(\cdot, \cdot)$, $A \subset X \times X$ maximal monotone; suppose that there exists $x_0 \in X$ such that*

$$\lim_{\|x\|_X \to \infty} \frac{(A^0 x, x - x_0)}{\|x\|_X} = +\infty. \tag{1.46}$$

Then, A is surjective (i.e., $R(A) = X$). ■

We shall also need a result ensuring that, under some additional hypotheses, the sum of two maximal monotone operators A and B is still maximal monotone; naturally, this cannot be true in the general case, since, for instance, it can happen that $D(A) \cap D(B) = \emptyset$. The following result holds again in the Hilbert space setting.

Theorem 1.2.18 ([14, page 37]**).** *If A is monotone hemicontinuous and B is any maximal monotone operator, then $A + B$ is still maximal monotone.* ■

1.2.3 Convex integrals

We introduce here a particular class of maximal monotone operators, which plays an important role in the statement of phase transition problems. We shall develop this part of the theory in a more general form than it will be effectively needed in the sequel; we think anyway that this kind of approach could result clearer.

Given a smooth and bounded open set $\Lambda \subset \mathbb{R}^N$ (we could also take Λ as an arbitrary measure space of finite regular measure), we consider here a function $\phi : \Lambda \times \mathbb{R} \to \mathbb{R}_\infty$.

Definition 1.2.19. *Let us suppose ϕ to verify the following assumptions:*

(a) *The function $r \mapsto \phi(x, r)$ is convex, l.s.c., proper and* nonnegative *for a.e. $x \in X$ (the last condition could be weakened, but this form is sufficiently general for the sequel).*

(b) *There exists a Borel-measurable function $f : \Lambda \times \mathbb{R} \to \mathbb{R}_\infty$ such that, for a.e. $x \in \Lambda$, it is $f(x, \cdot) = \phi(x, \cdot)$ on $\mathbb{R}$.*

Under these conditions, we say that ϕ is a convex integrand *on $\Lambda \times \mathbb{R}$.* ■

Our aim is to prove that, at least for $1 < p < \infty$, the functional $\Phi : L^p(\Lambda) \to \mathbb{R}_\infty$ given by

$$\Phi : u \mapsto \begin{cases} \int_\Lambda \phi(x, u(x))\, dx & \text{if } \phi(\cdot, u(\cdot)) \in L^1(\Lambda), \\ +\infty & \text{otherwise} \end{cases} \tag{1.47}$$

is still convex, l.s.c., proper (and nonnegative) on $L^p(\Lambda)$.

Remark 1.2.20. *We point out that the above assumptions* (a), (b) *guarantee in particular that the function $\phi(\cdot, u(\cdot))$ is measurable on Λ for any $u \in L^p(\Lambda)$. Moreover,* (b) cannot *be substituted with the (apparently) more natural weaker condition of Carathéodory type*

(b′) *The function $x \mapsto \phi(x, r)$ is measurable on Λ for every $r \in \mathbb{R}$,*

unless we suppose the continuity of ϕ in (a) *(instead of the lower semicontinuity); however, this hypothesis is not verified in most applications (see also [36, Prop. 1.1, page 218]).*

We now denote as $\phi^*(x, r) : \Lambda \times \mathbb{R} \to \mathbb{R}_\infty$ the convex conjugate of ϕ, for fixed $x \in \Lambda$, *with respect to the second variable.* We have:

Proposition 1.2.21. *ϕ^* is still a convex integrand.* ■

So, it can be extended to $L^{p'}(\Lambda)$ too, by setting:

$$G : u \mapsto \begin{cases} \int_\Lambda \phi^*(x, u(x))\, dx & \text{if } \phi^*(\cdot, u(\cdot)) \in L^1(\Lambda), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.48}$$

The following important result summarizes the properties of Φ^* and $\partial\Phi$.

Theorem 1.2.22 ([36, pages 251 and 318]**).** (a) *If there exists $u_0 \in L^\infty(\Lambda)$ verifying $\Phi(u_0) < +\infty$, then it is $\Phi^* = G$.*

(b) *Given $u \in L^{p'}(\Lambda)$, $v \in L^p(\Lambda)$, it is $v \in D(\partial\Phi)$ and $u \in \partial\Phi(v)$ if and only if $v(x) \in D(\partial\phi(x,\cdot))$ and $u(x) \in \partial\phi(x, v(x))$ for a.e. $x \in \Lambda$ (where the last subdifferential is naturally taken with respect to the sole second variable).*

(c) *Denoting as $\phi_\epsilon(x,r)$ the Moreau-Yosida approximation of $\phi(x,\cdot)$ (with respect to the second variable), we have that ϕ_ϵ is a convex integrand; moreover, setting*

$$G_\epsilon : u \mapsto \begin{cases} \displaystyle\int_\Lambda \phi_\epsilon(x, u(x))\, dx & \text{if } \ \phi_\epsilon(\cdot, u(\cdot)) \in L^1(\Lambda), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.49}$$

it results that $G_\epsilon = \Phi_\epsilon$; finally, for $v \in L^p(\Lambda)$, $u \in L^{p'}(\Lambda)$, we have that $u \in \partial\Phi_\epsilon(v)$ if and only if $u(x) \in \partial\phi_\epsilon(x, v(x))$ for a.e. $x \in \Lambda$. ■

Proof. We show only (c), which is not in [36], in the Hilbert case $p = p' = 2$. First of all, let us take f as in condition (b) of definition 1.2.19; set now:

$$f_\epsilon(x,r) := \inf_{s\in\mathbb{R}} \left\{ \frac{1}{2\epsilon}(s-r)^2 + f(x,s) \right\}. \tag{1.50}$$

Recalling Prop. 1.2.16 (a), we see that, for a.e. $x \in \Lambda$ (and precisely where f coincides with ϕ), the infimum in the above expression is achieved for every $r \in \mathbb{R}$; moreover, it is also obvious that $f_\epsilon(x,\cdot) \equiv \phi_\epsilon(x,\cdot)$ for a.e. $x \in \Lambda$. Finally, f_ϵ is Borel-measurable on $\Lambda\times\mathbb{R}$, since, looking back to the right hand side of (1.50), we notice that it can be seen as an inferior envelope of a countably infinite family of Borel-measurable functions of (x, r).

Let us name now $\mathcal{J}_\epsilon$ the resolvent of operator $\partial\Phi$ and $j_\epsilon(\cdot)$ that of $\partial\phi(x,\cdot)$ (as an operator on $\mathbb{R}$); Owing to part (b) of this theorem, given $v, z \in L^2(\Lambda)$, we deduce

$$\begin{aligned} z(x) = j_\epsilon(x, v(x)) \ \text{ a.e.} &\iff v(x) \in (z(x) + \epsilon\partial\phi(x, z(x)) \ \text{ a.e.} \\ &\iff \frac{v(x) - z(x)}{\epsilon} \in \partial\phi(x, z(x)) \ \text{ a.e.} \iff \frac{v-z}{\epsilon} \in \partial\Phi(z) \\ &\iff v \in z + \epsilon\partial\Phi(z) \iff z = \mathcal{J}_\epsilon v; \end{aligned} \tag{1.51}$$

so, also the resolvent of $\partial\Phi$ can be obtained "by integration" from that of ϕ.

Now, recalling the definition of Moreau-Yosida approximation and exploiting again Prop. 1.2.16 (a), for $v \in L^2(\Lambda)$, we infer:

$$\begin{aligned} \Phi_\epsilon(v) &= \frac{1}{2\epsilon}\|v - \mathcal{J}_\epsilon v\|^2_{L^2(\Lambda)} + \Phi(\mathcal{J}_\epsilon v) \\ &= \int_\Lambda \left[\frac{1}{2\epsilon}(v(x) - \mathcal{J}_\epsilon v(x))^2 + \phi(x, \mathcal{J}_\epsilon v(x)) \right] dx \\ &= \int_\Lambda \left[\frac{1}{2\epsilon}(v(x) - j_\epsilon(x, v(x)))^2 + \phi(x, j_\epsilon(x, v(x))) \right] dx \\ &= \int_\Lambda \phi_\epsilon(x, v(x))\, dx = G_\epsilon(v), \end{aligned} \tag{1.52}$$

as desired.

As for the proof of the last statement, using again (1.51), we derive:

$$u = \partial\Phi_\epsilon(v) \iff u = \frac{v - \mathcal{J}_\epsilon v}{\epsilon} \iff v - \epsilon u = \mathcal{J}_\epsilon v \tag{1.53}$$
$$\iff v(x) - \epsilon u(x) = j_\epsilon(x, u(x)) \quad \text{a.e.}$$
$$\iff u(x) = \frac{v(x) - j_\epsilon(x, v(x))}{\epsilon} \quad \text{a.e.}$$
$$\iff u(x) = \partial\phi_\epsilon(x, v(x)) \quad \text{a.e.}. \ \blacksquare$$

To conclude, we give another related result involving convex integrands $\phi(x, r)$ *not depending* of x, but where the second variable r is allowed to vary in an arbitrary Hilbert space (instead of $\mathbb{R}$).

Theorem 1.2.23 ([14, Prop. 2.16, page 47])**.** *Let $\psi : H \to \mathbb{R}_\infty$ be a convex, l.s.c. and proper function on a Hilbert space H. We define, for $v \in L^2(\Lambda; H)$, the functional:*

$$\Psi : v \mapsto \begin{cases} \displaystyle\int_\Lambda \psi(v(x))\, dx & \textit{if } \ \psi(v(\cdot)) \in L^1(\Lambda), \\ +\infty & \textit{otherwise.} \end{cases} \tag{1.54}$$

Then, Ψ is convex, l.s.c. and proper on $L^2(\Lambda; H)$; its subdifferential is given by $u \in \partial\Psi(v) \iff u(x) \in \partial\psi(v(x))$ a.e. in Λ and its Moreau-Yosida approximation Ψ_ϵ is the convex integral of functional ψ_ϵ (i.e. is still given by a formula analogous to (1.54)). $\blacksquare$

It should be interesting to extend this last result to the case of convex integrands $\psi : \Lambda \times H \to \mathbb{R}_\infty$ by following the lines of Theorem 1.2.22; we really do not know how much of the quoted theorem remains true in this framework.

1.3 Variational convergences

We now introduce some important concepts related to the Γ-convergence of convex functionals and monotone operators. The topic is very wide and we only list here a series of results which will be employed in the sequel; for a systematical exposition of the matter, we refer to the texts [5] and [31] (whence we derived indeed almost all the material of this section).

Let us be given a topological space (X, τ), and functions Ψ^n (for $n \in \mathbb{N}$) on X with values in $\mathbb{R}_\infty$.

Definition 1.3.1. *We set*

$$\Gamma_\tau\text{-}\liminf_{n\to\infty} \Psi^n(x) := \sup_{U \in \mathcal{U}(x)} \liminf_{n\to\infty} \inf_{y \in U} \Psi^n(y), \tag{1.55}$$

$$\Gamma_\tau\text{-}\limsup_{n\to\infty} \Psi^n(x) := \sup_{U \in \mathcal{U}(x)} \limsup_{n\to\infty} \inf_{y \in U} \Psi^n(y). \tag{1.56}$$

If $\Gamma_\tau\text{-}\liminf_{n\to\infty} \Psi^n(x) = \Gamma_\tau\text{-}\limsup_{n\to\infty} \Psi^n(x) =: \Psi(x)$ for every $x \in X$, we say that Ψ^n Γ-converges to Ψ in the topology τ. $\blacksquare$

Let us also recall now a characterization of Γ-limits in the case of first countable topological vector spaces:

Proposition 1.3.2 ([31, Prop. 8.1, pages 86–87])**.** *If X is first countable (in particular if X is a metrizable topological vector space), we have:*

$$\Gamma_\tau\text{-}\liminf_{n\to\infty} \Psi^n(x) = \min_{x_n\to x} \liminf_{n\to\infty} \Psi^n(x_n), \tag{1.57}$$

$$\Gamma_\tau\text{-}\limsup_{n\to\infty} \Psi^n(x) = \min_{x_n\to x} \limsup_{n\to\infty} \Psi^n(x_n). \tag{1.58}$$

In particular, we have that Ψ^n Γ-converges to Ψ if and only if, for every $x \in X$, we have that $\Psi(x) \le \liminf \Psi^n(x_n)$ for every $x_n \to x$ and, in addition, there exists $x_n \to x$ such that $\Psi(x) = \lim_{n\to\infty} \Psi^n(x_n)$. ■

The definition of Γ-convergence has a very wide range of applications; here, anyway, we are interested in a very specifical situation. So, in the following, X will always be a reflexive Banach space and Ψ and Ψ^n will be convex, l.s.c. and proper real-extended valued functionals on X.

In this case, we shall write $\Psi = \Gamma_s\text{-}\lim_{n\to\infty} \Psi^n$ ($\Psi = \Gamma_w\text{-}\lim_{n\to\infty} \Psi^n$) to say that Ψ is the Γ-limit of Ψ^n in the strong (*sequential* weak, respectively) topology of X.

Let us remark a fundamental problem which arises at this point and motivates the introduced terminology. When we endow X with its strong topology, the preceding proposition gives a characterization of $\Gamma_s\text{-}\lim \Psi^n$ in terms of convergent sequences to x; anyway, it cannot be used when we consider the weak topology on X, since this is not metrizable. Anyway, in this case, the most relevant concept is fortunately also the easier to handle, that is the Γ-convergence with respect to the sequential weak topology of X (rather than the (usual) weak one).

We are now ready to recall a

Definition 1.3.3 ([5, page 295])**.** *We say that Ψ^n* M-converges *to Ψ (or* converges to Ψ in the sense of Mosco*) if and only if $\Psi = \Gamma_s\text{-}\lim_{n\to\infty} \Psi^n = \Gamma_w\text{-}\lim_{n\to\infty} \Psi^n$.* ■

Remark 1.3.4. *In the case of $X = \mathbb{R}$, the M-convergence obviously coincides with the standard Γ-convergence.*

Proposition 1.3.5 ([5, Prop. 3.19, page 297])**.** *Under the above hypotheses, the following conditions* (a), (b) *are equivalent:*

(a) *Ψ^n M-converges to Ψ;*

(b) *For all $x \in X$ and for all $x_n \to x$ in X-weak, it is*

$$\Psi(x) \le \liminf_{n\to\infty} \Psi^n(x_n); \tag{1.59}$$

moreover, for all $x \in X$ there exists $x_n \to x$ in X-strong, such that

$$\Psi(x) \ge \limsup_{n\to\infty} \Psi^n(x_n). \tag{1.60}$$ ■

We point out that, owing to Prop. 1.3.2, condition (1.59) above is equivalent to $\Psi \le \Gamma_w\text{-}\liminf \Psi^n$, while (1.60) holds if and only if $\Psi \ge \Gamma_s\text{-}\limsup \Psi^n$.

Definition 1.3.6 ([5, page 360])**.** *Let $\mathcal{A}^n : X \to 2^{X'}$ be a sequence of maximal monotone operators on a reflexive Banach space X. We say that $\mathcal{A}^n$* G-converges *to another maximal monotone operator $\mathcal{A} : X \to 2^{X'}$ (or* converges to $\mathcal{A}$ in the sense of graphs*) if and only if, for every $(x, y) \in \mathcal{A}$, there exists a sequence $(x_n, y_n) \in \mathcal{A}^n$ such that $x_n \to x$ in X-strong and $y_n \to y$ in X'-strong.* ∎

The following fundamental result collects all the notions we have just introduced:

Theorem 1.3.7 ([5, Prop. 3.60 and Th. 3.66])**.** *With the same hypotheses and notation as above, if we set $\mathcal{A}^n = \partial\Psi^n$ and $\mathcal{A} = \partial\Psi$, the following conditions* (a), (b), (c) *are equivalent:*

(a) *Ψ^n M-converges to Ψ.*

(b) *$\mathcal{A}^n$ G-converges to $\mathcal{A}$.*

(c) *$\mathcal{A}^n_\epsilon(x)$ converges to $\mathcal{A}_\epsilon(x)$ in X'-strong for every $x \in X$ and every $\epsilon > 0$. Here we have denoted by $\mathcal{A}^n_\epsilon$ ($\mathcal{A}_\epsilon$) the Yosida-regularization of $\mathcal{A}^n$ ($\mathcal{A}$, respectively).* ∎

The next proposition is an easy extension for instance of [9, Prop. 1.1, page 42] or [5, Prop. 3.59]. We give the elementary proof for the sake of convenience of the reader:

Proposition 1.3.8. *If $\mathcal{A}^n$ G-converges to $\mathcal{A}$, given a sequence $(x_n, y_n) \in \mathcal{A}^n$ such that $x_n \to x$ in X-weak, $y_n \to y$ in X'-weak and*

$$\limsup_{n\to\infty} {}_{X'}\langle y_n, x_n\rangle_X \le {}_{X'}\langle y, x\rangle_X, \tag{1.61}$$

then we have that $(x, y) \in \mathcal{A}$.

Proof. Let us consider an arbitrary element $(\xi, \eta) \in \mathcal{A}$. By the definition of G-convergence, there exists a sequence $(\xi_n, \eta_n) \subset X \times X'$ such that $(\xi_n, \eta_n) \in \mathcal{A}^n$ for all $n \in \mathbb{N}$ and $\xi_n \to \xi$ in X-strong, $\eta_n \to \eta$ in X'-strong. Now, hypothesis (1.61) entails

$$0 \le \limsup_{n\to\infty} {}_{X'}\langle \eta_n - y_n, \xi_n - x_n\rangle_X \le {}_{X'}\langle \eta - y, \xi - x\rangle_X. \tag{1.62}$$

Since (ξ, η) is arbitrary, we can conclude by exploiting the maximality of $\mathcal{A}$ and recalling (c) of Prop. 1.2.8. ∎

We now give some results which guarantee the M-convergence of particular classes of functionals (Ψ^n, Ψ and X are as before).

Proposition 1.3.9 ([5, Prop. 3.20, page 298])**.** (a) *If Ψ^n is nondecreasing, then it converges to $\sup_{n\in\mathbb{N}} \Psi^n$ in the sense of Mosco.*

(b) *If Ψ^n is nonincreasing, then it converges to the relaxed functional* $\mathrm{sc}^-(\inf_{n\in\mathbb{N}} \Psi^n)$ *in the sense of Mosco.* ∎

In particular, recalling (b) of Prop. 1.2.16, we immediately derive:

Corollary 1.3.10. *The Moreau-Yosida approximates Ψ_ϵ of functional Ψ M-converge to it (with respect to ϵ)*[1]*.* ∎

---

[1]Indeed, it is not difficult to show [5, page 360] that for *every* maximal monotone operator A, we have that A_ϵ G-converges to A (even if A is not a subdifferential).

In particular, Prop. 1.3.8 can be applied when $A^n = A_{\epsilon_n}$ for a sequence $\epsilon_n \to 0$.

We conclude by listing a property of Γ-limits of sum of functionals:

Proposition 1.3.11 ([31, Prop. 6.17]). *If Φ^n, Ψ^n are two given sequences of functionals with values in $\mathbb{R}_\infty$, defined on a topological space X, then, for every $x \in X$, we have that:*

$$\Gamma\text{-}\liminf_{n\to\infty} (\Phi^n + \Psi^n)(x) \geq \Gamma\text{-}\liminf_{n\to\infty} \Phi^n(x) + \Gamma\text{-}\liminf_{n\to\infty} \Psi^n(x) \tag{1.63}$$

$$\Gamma\text{-}\limsup_{n\to\infty} (\Phi^n + \Psi^n)(x) \geq \Gamma\text{-}\limsup_{n\to\infty} \Phi^n(x) + \Gamma\text{-}\limsup_{n\to\infty} \Psi^n(x). \tag{1.64}$$ ∎

The converse inequalities are in general false; anyway, such a lack of linearity must not surprise: of course it is essentially due to the "inf" and the "sup" appearing in Def. 1.3.1.

Chapter 2

Phase-field models

2.1 The physical background

This section is intended to provide a short overview of the physical motivations underlying the phase-field problems which we are going to study in the sequel. We start by describing the first mathematical model of heat diffusion inside substances allowed to phase transitions which appeared in history, that is the Stefan one; as a second step, we briefly present in some detail several more modern models extending and improving it. We shall especially insist on the parabolic phase-field model, which provides the physical background for the specifical problems of this dissertation.

2.1.1 The Stefan problem

The starting point of the Stefan phase-transition model is nothing else but the usual heat equation. So, fix (as in the previous chapter) a smooth, bounded and connected (in view of the most general boundary conditions) domain $\Omega \subset \mathbb{R}^N$ (obviously, for the physically most relevant results, we could just assume $N \leq 3$; however, often no more mathematical effort is needed to give the proofs in an arbitrary dimension). Consider also a final time $T < +\infty$ and set $Q := \Omega\times]0,T[$ and $Q_t := \Omega\times]0,t[$ for $t \leq T$. Assuming that Ω is filled with a nonnecessarily homogeneous substance of temperature $\theta(x,t)$ and supposing that a heat source $f(x,t)$ is present in Ω at every time $t \in [0,T]$, it is well known that the heat propagation inside Ω is described by the relation

$$C\,\partial_t\theta + \operatorname{div}\mathbf{q} = f \quad \text{in } \Omega\times]0,T[, \tag{2.1}$$

where C is the thermal capacity of the substance, possibly depending on (x,t) (actually also on θ, but we do not want to discuss this case here) and $\mathbf{q}$ is the heat flux at the point $(x,t) \in Q$. It is well known that, when $\mathbf{q}$ is given by the Fourier law

$$\mathbf{q} = -k\nabla\theta \quad \text{in } \Omega\times]0,T[, \tag{2.2}$$

(2.1) gives rise to the standard heat equation. Here, $k > 0$, nonnecessarily constant, denotes the heat conductivity of the fluid; in an anisotropic substance, k could also stand for a regular field of uniformly elliptic matrices. In the following, we will meet some more sophisticated models where (2.2) is substituted by different constitutive assumptions.

The heat equation provides the starting point to introduce the so-called strong formulation of the two-phase Stefan problem. The underlying idea is essentially due to the physician Josef Stefan [84], who constructed this model at the end of the last century, starting from concrete observations of the solidification and melting of polar ices.

So, suppose now that at every time instant $t \in [0,T]$, the domain Ω is subdivided by a smooth interface $S(t)$ into two subdomains $\Omega_{\mathrm{S}}(t)$ and $\Omega_{\mathrm{L}}(t)$ occupied by the solid and liquid phases, respectively. Assume also that the phase transition temperature is identically equal to 0, so that $\theta > 0$ on $\Omega_{\mathrm{L}}(t)$, $\theta < 0$ on $\Omega_{\mathrm{S}}(t)$ and $\theta = 0$ on $S(t)$. Define now $Q_{\mathrm{S}} := \cup_{t\in]0,T[}(\Omega_{\mathrm{S}}(t) \times \{t\})$, $Q_{\mathrm{L}} := \cup_{t\in]0,T[}(\Omega_{\mathrm{L}}(t) \times \{t\})$ and finally $S := \cup_{t\in]0,T[}(S(t) \times \{t\})$; suppose that the so obtained interface S is a globally smooth N-dimensional hypersuperface of $\mathbb{R}^{N+1}$. In particular, we can assume that S is given in an implicit form as the set $\{(x,t) \in \mathbb{R}^{N+1} : F(x,t) = 0\}$, where $F : Q \to \mathbb{R}$ is a suitably smooth function and we also require that $F(x,t) < 0$ on Q_{S} and $F(x,t) > 0$ on Q_{L} (naturally, if also θ is smooth enough, an obvious parametrization for S is obtained by choosing $F = \theta$).

Let us indicate by $\mathbf{n}$ the unit vector of $\mathbb{R}^{N+1}$ normal to S and pointing outwards Q_{S}; clearly, we have that $\mathbf{n} = \nabla_{(x,t)}F/|\nabla_{(x,t)}F|$. We also denote by $\mathbf{n}_x$ the $\mathbb{R}^N$-vector of the space components and by n_t the component of $\mathbf{n}$ in the direction of the time axis, so that $\mathbf{n} = (\mathbf{n}_x, n_t)$. The strong formulation of the two-phase Stefan problem essentially consists of the equations:

$$C_{\mathrm{S}}\,\partial_t\theta - \operatorname{div}(k_{\mathrm{S}}\nabla\theta) = f \quad \text{in } Q_{\mathrm{S}} \tag{2.3}$$

$$C_{\mathrm{L}}\,\partial_t\theta - \operatorname{div}(k_{\mathrm{L}}\nabla\theta) = f \quad \text{in } Q_{\mathrm{L}} \tag{2.4}$$

$$2\lambda n_t = k_{\mathrm{L}}\,\partial_{\mathbf{n}_x}\theta_{\mathrm{L}} - k_{\mathrm{S}}\,\partial_{\mathbf{n}_x}\theta_{\mathrm{S}} = [k\nabla\theta]_{\mathrm{S}}^{\mathrm{L}} \cdot \mathbf{n}_x \quad \text{on } S \tag{2.5}$$

(∇ denoting, here and below, the gradient in the sole space directions). Here, (2.3–2.4) are standard heat equations and also the notations are the same as before; the coefficients $C_{\mathrm{S}}, C_{\mathrm{L}}, k_{\mathrm{S}}, k_{\mathrm{L}}$, anyway, are allowed to depend on the phase. Relation (2.5), instead, is the so-called Stefan condition, which essentially states that the "normal velocity" of the interface at the time t is proportional to the jump of the gradient of the temperature; the positive coefficient λ accounts for the latent heat of the substance. At the end of this subsection, we shall give a more detailed physical (and geometrical) interpretation of this condition in the case of space dimension $N = 1$.

We point out that the above equations (naturally coupled with suitable initial and boundary conditions) constitute what is called a *free boundary problem*, since also the interface S is an unknown. Indeed, we can define a *strong solution* of the Stefan problem to be a sufficiently smooth couple (S, θ) satisfying (2.4–2.5) and the Cauchy and boundary conditions (we do not insist, at this level, on the precise regularity requests for such a solution).

This *strong formulation* (s.S.p. in the following) of the Stefan problem has been deeply studied and still there are many open problems left, especially as the maximal regularity properties of S are concerned. Here, our purpose is different; indeed, we would like to deduce a *weak formulation* of the above problem; we remark that the following procedure is essentially due to Kamenomostskaja [45] and Oleinik [69] who were the first ones to restate the problem in a weak form. First of all, modify (2.3–2.4) by introducing a new variable, that is the *phase-field* χ, by setting,

$$\left.\begin{aligned} \chi(x,t) &:= 1 \quad \text{if} \quad \theta(x,t) > 0 \\ \chi(x,t) &:= -1 \quad \text{if} \quad \theta(x,t) < 0 \end{aligned}\right\} \quad \text{for} \quad (x,t) \in Q \setminus S. \tag{2.6}$$

Notice that there is no hope of getting in this framework a global continuity property for χ on Q; hence, *at present*, there is no reason to define it on S, i.e. for $\theta = 0$. A purely formal modification of (2.3–2.4) gives anyway

$$C_{\mathrm{S}}\partial_t\theta + \lambda\partial_t\chi - \operatorname{div}(k_{\mathrm{S}}\nabla\theta) = f \quad \text{in } Q_{\mathrm{S}}, \tag{2.7}$$

$$C_{\mathrm{L}}\partial_t\theta + \lambda\partial_t\chi - \operatorname{div}(k_{\mathrm{L}}\nabla\theta) = f \quad \text{in } Q_{\mathrm{L}}. \tag{2.8}$$

The reason of adding the two zero-terms inside the above equations will result clear in a while: suppose indeed to have a sufficiently smooth solution (S, θ) of (2.7–2.8) and take a test function $v \in \mathcal{D}(Q)$ (we try to condense (2.7–2.8) *and the Stefan condition* in an unique relation in $\mathcal{D}'(Q)$). Multiplying (2.7) by the restriction of v to Q_{S} and integrating over Q_{S}, we easily obtain:

$$\begin{aligned} &\iint_{Q_{\mathrm{S}}} C_{\mathrm{S}}\partial_t\theta v\,dx\,dt - \iint_{Q_{\mathrm{S}}} \lambda\chi\partial_t v\,dx\,dt + \iint_{Q_{\mathrm{S}}} \lambda\partial_t(\chi v)\,dx\,dt \\ &\qquad + \iint_{Q_{\mathrm{S}}} k_{\mathrm{S}}\nabla\theta\cdot\nabla v\,dx\,dt - \int_S k_{\mathrm{S}}\nabla\theta\cdot\mathbf{n}_x v\,d\mathcal{H}^N = \iint_{Q_{\mathrm{S}}} fv\,dx\,dt, \end{aligned} \tag{2.9}$$

$\mathcal{H}^N$ denoting here and below the N-dimensional Hausdorff measure on S (note that the Gauss-Green formula has been used only for the space variables on a noncylindrical domain; this explains the occurrence of the nonunit normal vector $\mathbf{n}_x$).

Integrating now by parts in time the third term on the left hand side of the previous relation, we infer

$$\iint_{Q_{\mathrm{S}}} \lambda\partial_t(\chi v)\,dx\,ds = \lambda\int_S \chi v n_t\,d\mathcal{H}^N = \lambda\int_S v n_t\,d\mathcal{H}^N, \tag{2.10}$$

since $\chi \equiv 1$ on Q_{S} (and hence $\chi \equiv 1$ in sense of traces on S). The same procedure can be also applied to equation (2.8): notice that when we perform the integration by parts as in (2.10), we get the same result as above, since $\chi \equiv -1$ on Q_{L}, but now the term n_t appears with the minus sign. Summing the result of computation (2.9) with its analogous for the liquid phase and taking account of (2.10) and of the Stefan condition, it is now easy to derive

$$C\partial_t\theta + \lambda\partial_t\chi - \operatorname{div}(k\nabla\theta) = f \quad \text{in } \mathcal{D}'(Q), \tag{2.11}$$

where we have no longer emphasized the dependence of the coefficients C, k, λ on the phase (indeed, in the new setting, they are seen as χ-dependent functions). The latter is a much weaker relation with respect to (2.3–2.4), since all what we can say is that it holds in the sense of distributions on Q. Indeed, we have proved that every regular solution of the strong Stefan problem is also a solution of (2.11). Moreover, if we consider (2.11) by itself, in order to have a well-posed problem, this equation must be complemented with a relation linking the two unknowns θ and χ, as (2.6). However, in this variational setting, condition (2.6) is often restated in a dfifferent way. Indeed, introducing a maximal monotone graph $\alpha \subset \mathbb{R} \times \mathbb{R}$, as

$$\alpha(s) := \begin{cases}]-\infty, 0] & \text{if } s = -1, \\ \{0\} & \text{if } -1 < s < 1, \\ [0, +\infty[& \text{if } s = 1, \\ \emptyset & \text{if } |s| > 1, \end{cases} \tag{2.12}$$

we can now complement equation (2.11) with the following abstract *constitutive assumption*

$$\theta \in \alpha(\chi) \quad \text{in } Q. \tag{2.13}$$

We remark that, anyway, the above relation is much more general than (2.6); in particular, it makes sense also as χ assumes intermediate values between -1 and 1, which was not allowed in the previous formulation. The couple (2.11)–(2.13), still accompanied by suitable initial and boundary conditions, defines the *weak formulation* (w.S.p.) of the Stefan problem. We also point out that (2.11) can actually be stated in a slightly stronger sense than the distributional one; for instance, in the next section, we shall rewrite it in a natural Hilbert space setting.

Furthermore, we emphasize that the solutions of equations (2.3–2.5) and of (2.11), (2.13) can have deeply different physical properties (cf. also the next subsection for a more detailed analysis). For the present, we have only seen that every *smooth* solution of the strong Stefan problem is also a solution of the weak formulation; not only the converse is not true, but we remark that there are solutions of the strong problem which are not "regular" enough to be interpreted as solutions of the weak one.

Let us give examples of both situations: first, notice that a solution of (w.S.p.) can exhibit "solid" interfaces (which are usually called "mushy regions"), where the phase variable takes values strictly between -1 and 1, and this can happen even if the separating interface $S(0)$ related to the initial values of the temperature and phase has 0-depth, instead. Moreover, new regions of solid (liquid) phase can appear, starting from a certain time t, inside the liquid (solid, respectively) one; think for instance of a cold source acting for some time on a region of liquid phase. This phenomenon, called *nucleation,* cannot happen in the framework of the s.S.p., since the interface $S(t)$, by assumption, evolves through a smooth modification in time of $S(0)$; in particular, no disconnected portions of S can appear *ex novo.*

In the above outlined situation (cold source acting on the liquid phase) the s.S.p. would exhibit instead a so-called supercooling region, i.e. a region of liquid phase below the freezing point; correspondently, this is not possible in the w.S.p.-setting, due to the locality of the constitutive relation (2.13).

Notice anyway that neither situation (mushy region-nucleation or supercooling) is unnatural or "wrong", but the occurrence of either phenomenon strongly depends on some finer physical properties of the material considered, that the Stefan model is not able to balance. Indeed, one of the main features of phase-field models is their good description of the solidification dynamics under rather fine thermodynamical assumptions. Indeed, in the next subsection, we shall see that the solidification patterns of w.S.p. and s.S.p. can be seen as asymptotic limits of the phase-field model for different blowouts of an "interfacial energy coefficient", whose purpose is precisely that of weighing the supercooling and nucleation phenomena.

We conclude this subsection by giving a geometrical interpretation of the Stefan condition (2.5) in the case of one space dimension (so that $\mathbf{n}_x$ is now a number n_x). Here, we want to express the interface as the graph of a function $x(t)$, with $t \in [0,T]$; thus, recalling the general case (where S is viewed as $\{(x,t) \in Q : F(x,t) = 0\}$ for a C^1 function $F : Q \to \mathbb{R}$) and applying the implicit function theorem (or differentiating in t the relation $F(x(t),t) \equiv 0$), we get

$$x'(t) = -\frac{\partial_t F}{\partial_x F} = -\frac{\partial_t F}{|\nabla_{(x,t)} F|}\frac{|\nabla_{(x,t)} F|}{\partial_x F} = -\frac{n_t}{n_x}, \tag{2.14}$$

so that (2.5) can be rewritten as

$$-2\lambda x'(t) = k_{\mathrm{L}}\,\partial_x\theta_{\mathrm{L}} - k_{\mathrm{S}}\,\partial_x\theta_{\mathrm{S}} \quad \text{on } S, \tag{2.15}$$

whence it results clear that $x'(t)$, which evidently can be seen as a *normal velocity* of the interface, is proportional to the jump of the temperature gradient at the interface. Actually, in many texts, it is preferred to introduce the strong Stefan problem in any space dimension by starting from the geometrical considerations which led to (2.15), instead that by following our more analytical approach.

2.1.2 Ginzburg-Landau potentials and relaxed models

The main feature of phase-field systems consists in their capacity of extending the Stefan model to cases where a careful balance of the phenomena of supercooling and nucleation is needed; morover, they also account for the presence of a surface tension in the considered substance. They can be introduced in several ways; the most usual one, which we chose to follow, starts exactly from the weak Stefan problem.

First, let us first consider a simpler thermodynamical situation, anyway: suppose to be in time-stationary conditions and take θ as a datum; also, assume a Hilbert space setting (indeed other choices are possible); i.e., take $\theta \in L^2(\Omega)$. Then, we can define a functional F_θ on $L^2(\Omega)$, as follows

$$F_\theta(v) := \begin{cases} -c\displaystyle\int_\Omega \theta\, v\, dx & \text{if } |v| \le 1 \quad \text{a.e. in } \Omega, \\ +\infty & \text{otherwise;} \end{cases} \tag{2.16}$$

we remark [87] that the above constant $c > 0$ has a precise physical meaning; in general, it is proportional to the latent heat.

Now, it is easy to verify that F_θ is convex, l.s.c., proper and coercive on $L^2(\Omega)$; moreover, it admits a unique minimizer χ, which is precisely characterized by means of the constitutive assumption (2.13). The physical interpretation of F_θ is that of a measure of the (global) internal energy of the fluid when the temperature θ is fixed.

Our aim is now to see that, through a modification of the energy functional F_θ, it is possible to increase the number of thermodynamical situations which can be described this way. The phase-field model only accounts for one of the possible choices; other temptatives have been done and we shall briefly discuss some in the next subsection. Thus, we now introduce the so-called Ginzburg-Landau free energy functional [51, 19], as follows

$$F_\theta^\nu(\chi) := \int_\Omega \frac{c_1\nu}{2}|\nabla\chi|^2\,dx + \frac{c_2}{\nu}\int_\Omega(\chi^2-1)^2\,dx - c\int_\Omega \theta\chi\,dx, \tag{2.17}$$

where c_1, c_2, ν are positive constants depending, as c, on the physical characteristics of the fluid (the most relevant from the thermodynamical viewpoint is ν, whose meaning will be discussed later).

We want to remark at once what is the main mathematical feature of the functional F_θ^ν [87]: looking at its first two integral terms, we observe that they are in competition (note that now χ is "allowed" to assume every real value, not only $-1 \le \chi \le 1$): the first one is smaller when the gradient of χ is not too large; the other, instead, when the phase variable χ is very close either to $+1$ or to -1 (pure liquid or pure solid). Naturally, if a phase transition is really present, this means that the gradient of χ must be very large in a neighbourhood of the (thick) interface.

The parameter ν has the purpose of giving more importance to either of the quoted terms; more precisely [20, 40], we remark that $\nu^{1/2}$ actually comes out to be proportional to the thickness of the interface, which, in most physical cases, has the magnitude order of approximately 10^{-7} cm. [87].

We also observe that the above functional is no more convex; this accounts for the presence of relative maximizers and minimizers, which physically correspond to the supercooling and overheating phenomena (called *metastable* states, [87]). Furthermore, in case ν is very small (so that the interface thickness can be neglected), it can also be proved that the solutions of the above minimum problem enjoy a property which is asymptotically equivalent (as $\nu \to 0$) to the following *Gibbs-Thomson relation*

$$\theta = -c_0\kappa\sigma \qquad \text{on } S; \tag{2.18}$$

here, S is, as before, the phase interface, σ the *surface tension* of the fluid, and κ the mean curvature of S (assumed positive for convex solid phase). This is indeed a difficult theorem, which has been proved by Caginalp [19] by means of a formal limit procedure; an alternative approach, which is due to Luckhaus [55], can be given by assuming (2.18) instead of "$\theta = 0$ on S" in the statement of S.s.p.. Observe that the Gibbs-Thomson relation actually states that the solidification temperature is lower for $\kappa > 0$; this can be justified by observing that, in this case, the solid molecules have a greater number of liquid neighbours, or also by means of chemical potential arguments [19]. Furthermore, notice that this effect becomes more important as the surface tension is larger.

Under the previous assumptions, the fact that the system assumes a stable configuration can be mathematically expressed in the form $0 = \delta_\chi F_\theta^\nu$, where by $\delta_\chi F_\theta^\nu$ we mean the Fréchet derivative of F_θ^ν with respect to χ in the appropriate space $L^2(\Omega)$.

Performing the explicit calculations, it is easy to derive the evolution equation

$$-c_1\nu\Delta\chi + \frac{4c_2}{\nu}(\chi^3 - \chi) - c\theta = 0 \quad \text{in } \Omega. \tag{2.19}$$

Notice anyway that (2.19) must be intended in the variational sense; moreover, the choice of either $H^1(\Omega)$ or $H^1_0(\Omega)$ as the domain of the functional F^ν_θ also yields (implicitely) the related homogeneous boundary condition of Neumann type (or Dirichlet, respectively).

This analysis can be simply extended to the case when θ is also an unknown of the problem; now, equation (2.19) is coupled in a natural way with a time-stationary diffusion relation for the temperature of the usual form

$$-\operatorname{div}(k\nabla\theta) = f \quad \text{in } \Omega \tag{2.20}$$

(other choices are possible in a non-Fourier setting). Again, in order to obtain a well-posed problem, it is necessary to assume suitable boundary conditions.

To complete the discussion of the phase-field model, it remains now to consider the time-dependent case. With this purpose, we first recall a general mathematical definition. Given a Hilbert space $\mathcal{H}$ and a functional $J : \mathcal{H} \to \mathbb{R}$ of (at least) C^1 regularity, we call *gradient flow* associated to J the differential problem:

$$\mu\partial_t\chi(t) = -\delta_\chi J(\chi(t)) \quad \text{in } \mathcal{H}, \text{ for } t \in]0, T[, \tag{2.21}$$

where $T > 0$ is, as usual, an assigned final time and $\mu > 0$ the relaxation parameter.

Here we are not interested in a deep mathematical analysis of this kind of problems (which can be very complicated in the general case); we only emphasize that, if the functional J is convex, μ can be seen, roughly speaking, as a time period which the dynamical system "employs" to move towards a stable state (i.e. the minimizer of J). Indeed, we have that

$$\partial_t J(\chi)(t) = (\delta_\chi J(\chi(t)), \partial_t\chi(t))_\mathcal{H} = -\mu^{-1}\|\delta_\chi J(\chi(t))\|^2_\mathcal{H} \le 0, \tag{2.22}$$

so that the energy J results to be decreasing in time "along" the solution χ of the problem, which is seen as the trajectory of a motion in $\mathcal{H}$, starting from the initial configuration χ_0.

If $\mathcal{H} = L^2(\Omega)$ and $J = F^\nu_\theta$ (with the choice of either $H^1(\Omega)$ or $H^1_0(\Omega)$ as the effective domain for it, on account of the desired boundary conditions), we get the *phase-field equation:*

$$\mu\partial_t\chi = c_1\nu\Delta\chi - \frac{4c_2}{\nu}(\chi^3 - \chi) + c\theta \quad \text{in } \Omega\times]0, T[. \tag{2.23}$$

As in the stationary case, in order to obtain a well-posed problem, it is necessary to couple (2.23) with a diffusion equation for the temperature, for instance in the form (2.11) and, of course, with the suitable Cauchy and boundary conditions for both unknowns θ, χ. In the next subsections, anyway, we shall give some further remarks about this derivation, which seems to be not completely satisfactory under the physical viewpoint.

Summarizing, we have obtained a system of coupled parabolic evolution equations of which the heat one is linear, while the second has a monotone nonlinearity (χ^3). Indeed, a similar system will be the starting point of our study in the next section; however, we shall consider slightly more general equations (at least from the mathematical viewpoint), whose thermodynamical derivation, anyway, does not differ too much from that outlined above.

2.1.3 Extensions of the phase-field model

We conclude the discussion of the physical background by giving a brief description of some extensions of the phase-field models which are also currently studied by physicists and mathematicians. Naturally, the topic is very wide, so that we just outline the most physically relevant cases and give, with no claim of completeness, of course, a related list of references.

As a starting point, we observe that the justification of the (time-dependent) phase-field equation given in (2.21) as a gradient flow with respect to the energy functional F^ν_θ has been provided only in case the temperature θ is a datum. If θ is allowed instead to vary, it is no longer true that F^ν_θ, which depends now also on θ, need to be decreasing along the solution trajectories of the resulting system. Consider that the free energy has now a nontrivial dependence on θ, which might also take account of external factors (e.g, a heat source); hence, solutions with locally nondecreasing energy might exist indeed.

Starting from this consideration, in 1990 Penrose and Fife [70] gave a different justification of the phase field model in terms of a more suitable thermodynamical variable, which is the entropy. Moreover, their approach, which we now outline, can be adapted to derive a much greater variety of physically consistent models.

Hence, as a thermodynamical constitutive assumption, we suppose that the entropy $S(\theta, \chi)$ is a function of the following form:

$$S(\theta, \chi) = \int_\Omega \Big(s(e(\theta, \chi), \chi) - \frac{\kappa}{2} |\nabla \chi|^2 \Big)\, dx, \tag{2.24}$$

where κ is a positive constant, e the space density of enthalpy and s a known constitutive function related to the physical properties of the fluid. Since we are trying to write a gradient flow system accounting for a growth of S along solution paths, (i.e. S should relax towards a maximum), the natural hypothesis we assume on s is to be *concave* with respect to the variables (e, θ); however, since (for example in the case of true phase-field models) a double-well term for χ could be present as a factor of s, we do not require it to be concave with respect to χ. We shall see in the following that this does not prevent the entropy from growing along the trajectories of solutions; however, it might happen that the solution of a gradient flow for S moves, for small times, towards a metastable state (i.e. a relative maximizer, or minimizer, of S, *only with respect to* χ).

We can now consider the following abstract evolution problem (which is again to be seen in a variational framework)

$$\partial_t e = -\operatorname{div}(\kappa_1 \nabla \delta_e S) \quad \text{in } Q, \tag{2.25}$$

$$\mu \partial_t \chi = \kappa_2 \delta_\chi S \quad \text{in } Q, \tag{2.26}$$

where κ_1 and κ_2 are smooth and strictly positive functions of (x, t, θ, χ) (in most cases they are reduced to positive constants).

It is clear that in this very general framework, the phase-field model is only a special case of (2.25–2.26) and precisely that corresponding to the choices

$$e = \theta + \lambda(\chi), \qquad s(e, \chi) = -\frac{1}{2}(e - \lambda(\chi))^2 - (\chi^2 - 1)^2; \tag{2.27}$$

notice that here we allow the latent heat to be nonlinear; hence, we have written $\lambda(\chi)$ in place of $\lambda\chi$. Indeed, it seems that two particular choices are thermodynamically consistent for the function λ: the first assumes it Lipschitz continuous, and this means that the latent heat can vary with respect to the phase, but still behaves similarly as in the Stefan case; another possibility [70] is to allow λ a quadratic growth; actually, this choice complicates the mathematical study of the resulting system [12].

Furthermore, we want to emphasize another feature of system (2.25–2.26): why does equation (2.26) account for a true gradient flow for χ, while (2.25) presents a second order differential term before the Fréchet derivative of S? The reason is that, at least in the case where Neumann boundary conditions are assumed on χ (which seems the most meaningful under the physical viewpoint), the internal energy is a *conserved quantity* in time. This fact can be physically justified by simply observing that we are speaking of insulated systems; on a mathematical viewpoint, this hypothesis can be practically checked from equation (2.25) by integrating it in time, so that, on account of the Gauss-Green formula and of Neumann homogeneous boundary conditions, we get

$$\partial_t \int_\Omega e \, dx = 0 \qquad \text{for } t \in]0, T[. \tag{2.28}$$

Notice that, in this framework, the entropy S really results to be increasing along the solution trajectories; infact (compare with (2.22))

$$\begin{aligned} \partial_t S(e, \chi)(t) &= (\delta_e S(e, \chi), \partial_t e)_{\mathcal{H}} + (\delta_\chi S(e, \chi), \partial_t \chi)_{\mathcal{H}} \\ &= \int_\Omega -\operatorname{div}(\kappa_1 \nabla \delta_e S)\, \delta_e S \, dx + \kappa_2 \mu^{-1} \int_\Omega |\delta_\chi S|^2 \, dx \geq 0 \end{aligned} \tag{2.29}$$

(integrate by parts the first term and exploit again the boundary conditions) and this naturally yields the desired thermodynamical justification of all the models that can be described in this setting (corresponding to the various choices of S and e).

The above considerations suggest to investigate other models, where also the phase-field is a conserved quantity (see again [70] for further physical details): here, relation (2.26) is substituted by

$$\mu \partial_t \chi = -\operatorname{div}(\kappa_2 \nabla \delta_\chi S) \quad \text{in } Q, \tag{2.30}$$

so that, by suitably modifying (2.29), it could be seen that also in this case the entropy is increasing along the trajectories of solutions. Notice also that the presence of the term $|\nabla \chi|^2$ in the expression of S implies that the corresponding evolution equation for the phase variable comes out to be of the *fourth order* in space. These kind of models have been deeply studied by [47, 48]; we also quote the papers [68, 29, 28], which are related to the case where memory effects (see below) are present.

Finally, in view of the study of phase-field equations which will be carried out in the next section, it remains to mention another extension of the model, which can be derived by means of the Penrose-Fife approach (indeed, in their original paper, it is the standard model which is derived as a linearization of the following). In our notation, we have to substitute (2.27) with the following

$$e = -\frac{1}{\theta} - \lambda(\chi), \qquad s(e,\chi) = -\frac{1}{2}(e + \lambda(\chi))^2 - (\chi^2 - 1)^2, \tag{2.31}$$

with the agreement that now θ stays for the absolute temperature (so that it is always $\theta > 0$). Developing as before the computations of (2.25–2.26), it is easy to derive the following system (for the sake of simplicity we omit some coefficients)

$$\partial_t(\theta + \lambda(\chi)) + \Delta\left(\frac{1}{\theta}\right) = f \quad \text{in } Q, \tag{2.32}$$

$$\partial_t \chi - \Delta \chi + \chi^3 - \chi = \frac{\lambda'(\chi)}{\theta} \quad \text{in } Q, \tag{2.33}$$

which has been studied for instance in the papers [83, 52].

Another class of extensions of the phase-field model can be obtained by modifying instead the *heat* equation and in particular the Fourier heat flux law (2.2). Two important choices, accounting for models with thermal *memory* effects, are provided by the following alternative constitutive relations:

$$\mathbf{q}(t) = -\int_{-\infty}^{t} k(t-s)\nabla\theta(s)\,ds - k_0\nabla\theta(t) \qquad \text{(Coleman–Gurtin, [24])}, \tag{2.34}$$

$$\mathbf{q}(t) = -\int_{-\infty}^{t} k(t-s)\nabla\theta(s)\,ds \qquad \text{(Gurtin–Pipkin, [43])}. \tag{2.35}$$

In both cases, k is a known function of time of suitable regularity; in (2.35) k is supposed smooth and such that $k(0) > 0$; in (2.34), instead, the regularity requests on k are in general lower, but we need that $k_0 > 0$ and, moreover, that for some $\alpha > 0$, and for all $v \in L^2(0,T)$,

$$\int_0^t k_0 v^2(s)\,ds + \int_0^t (k * v)(s) v(s)\,ds \ge \alpha \int_0^t v^2(s)\,ds \qquad \text{for all } t \in]0,T[. \tag{2.36}$$

This positivity assumption on the kernel k ensures that, if the past history of the temperature is known up to the instant $t = 0$, substituting (2.34) into the diffusion relation (2.1), we obtain a parabolic heat equation with one more convolution term; in the Gurtin-Pipkin case, instead, since no global positivity is supposed for k, one could check (see [25]) that the resulting heat-like equation, rewritten in terms of the time-integral of θ, assumes a *hyperbolic* character (see also the paper [44] for a thermodynamical survey on hyperbolic heat propagation phenomena). For a further discussion and mathematical results on these models, we refer to [12, 25, 26, 27] and again to [68, 29, 28] (for the phase-conserved case).

We conclude this discussion on the physical framework, by briefly presenting the phase-relaxation model [85], which can be seen as an intermediate situation between the Stefan and phase-field ones (also from a historical point of view; indeed, it is preceding to the phase-field); this model couples the standard heat equation (2.11) with the relation

$$\mu \partial_t \chi + \mathcal{A}(\chi) - \frac{4c_2}{\nu}\chi \ni c\theta \quad \text{in Q} \tag{2.37}$$

(compare with (2.23)); here, $\mathcal{A}$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and the other terms are as in (2.23). In particular, two physically relevant choices are $\mathcal{A} = \partial I_{[-1,1]}$, with $c_2 = 0$ (corresponding to a gradient flow problem for the true Stefan model) and $\mathcal{A}(\chi) = 4c_2\nu^{-1}\chi^3$, accounting for a phase-field equation with negligible spatial diffusion of phase.

It also makes sense to couple the heat equation with a time-stationary phase-diffusion dynamics (think of μ_2 very small or see an accurate thermodynamical justification in [71]) of the form (compare again with (2.23))

$$-c_1\nu\Delta\chi + \frac{4c_2}{\nu}(\chi^3 - \chi) = c\theta \quad \text{in Q.} \tag{2.38}$$

This model has been studied in [71] and [27] (in the thermal memory case).

We finally remark that the interest of this last model, as well as of the phase-relaxation one, is also due to the fact that they can be seen in a natural way as *singular limits* of the true phase-field system for vanishing μ and, respectively, c_1. Later, we shall perform these analyses in the case of transmission systems, which is much more delicate due to the resulting discontinuity of coefficients.

2.2 Known results

We present here a short overview of some more or less classical results concerning the physical models discussed in the previous section. As before, we advance no claim of completeness; we only wish to provide a list of basic results and some references, which should turn out to be useful as elements of background and comparison for the new problems which we will present in the sequel.

2.2.1 Results on the Stefan problem

Due to its stricter connexions with the phase-field models, we only refer here to the weak formulation (w.S.p.); for an overview of the (important!) results about (s.S.p.), we refer to the monograph [64] and to the papers by Caffarelli [18] and Athanasopoulos, Caffarelli and Salsa [2, 3, 4], which are especially concerned with the smoothness of the phase interface S under suitable assumptions on data. With respect to the formulation

(2.11), (2.13), here we prefer to give the mathematically precise statement of (w.S.p.) in terms of θ and of a new variable, that is the enthalpy e, defined as $e := \theta + \lambda\chi$ (indeed these are *the* natural variables for the weak Stefan problem; look for instance at the Cauchy condition below). Hence, we can introduce the following (f and k are as in the previous section)

Problem 2.2.1 (ST). *Given a maximal monotone graph $\beta \subset \mathbb{R} \times \mathbb{R}$ and an initial value $e_0 : \Omega \to \mathbb{R}$, we look for a couple of sufficiently regular functions $(\theta, e) : Q \to \mathbb{R}$, satisfying the following system of equations*

$$\partial_t e - \operatorname{div}(k\nabla\theta) = f \qquad \text{a.e. in } Q, \tag{2.39}$$

$$\theta \in \beta(e) \qquad \text{a.e. in } Q, \tag{2.40}$$

$$e(0) = e_0 \qquad \text{a.e. in } \Omega, \tag{2.41}$$

along with suitable (for instance, Dirichlet or Neumann) boundary conditions. ∎

We remark that, if the enthalpy e is defined exactly as above, it easily results that the graph β corresponding to the α of (2.12) ("physical case") is given by

$$\beta(s) = \begin{cases} s + \lambda & \text{if } s < -\lambda, \\ 0 & \text{if } |s| \le \lambda, \\ s - \lambda & \text{if } s > \lambda. \end{cases} \tag{2.42}$$

Apart from the first results of Kamenostskaja and Oleinik quoted in the previous subsection, the first to address the problem through modern subdifferential techniques and to obtain significant results in a variational setting was Damlamian [32] in 1977. Now we briefly present his approach which fits very well the convex analysis machinery described in Section 1.2.

We begin by specifying in some detail the variational framework. First of all, let us choose (for example) third-type boundary conditions in the form

$$(k\nabla\theta)\cdot\mathbf{n} + p\theta = g \qquad \text{on } \partial\Omega\times]0,T[, \tag{2.43}$$

where it is $p \in L^\infty(\partial\Omega\times]0,T[)$, with $p > p_0 > 0$ a.e., and $g \in L^2(0,T;H^{1/2}(\partial\Omega))$, so that we can restate Problem (ST) in the Hilbert triplet $V = H^1(\Omega)$, $H = L^2(\Omega)$, $V' = H^1(\Omega)'$, where V is endowed with the scalar product (clearly equivalent to the standard one)

$$((v,w)) := \int_\Omega k\nabla v\cdot\nabla w\,dx + \int_{\partial\Omega} pvw\,dx \qquad \text{for } v, w \in V. \tag{2.44}$$

We also denote by $\mathcal{F} : V \to V'$ the Riesz operator associated to the above scalar product (cf. Subsec. 1.1.3).

In order to handle (ST), it is necessary to reinterpret the constitutive relation (2.40) in an abstract form; for this purpose, we need to assume a coercivity property on the graph β, which seems essential for the solution of all this kind of problems (and which fortunately is verified in the physical case (2.42)). Hence, we introduce a notation: here and for the rest of the dissertation, given a maximal monotone graph $\beta \subset \mathbb{R}\times\mathbb{R}$

such that $0 \in \beta(0)$, we shall name *convex primitive* of β the function $j : \mathbb{R} \to \mathbb{R}_\infty$ such that $\beta = \partial j$, and, furthermore, $j(0) = \min\{j\} = 0$ (it is clear that such a function can always be constructed). Now, in this case, we require precisely that

$$\liminf_{r\to\infty} \frac{j(r)}{|r|^2} = m > 0. \tag{2.45}$$

We also point out (compare with Theorem 1.2.17) that the above condition ensures the surjectivity of α (and it is actually stronger).

We now want to reinterpret assumption (2.40) in the framework of the Hilbert space V' (and not of H, where this operation would have been immediate referring to the results of Section 1.2.3). For this purpose, we define a new convex functional on V', that is

$$J_{V'}(w) := \begin{cases} \displaystyle\int_\Omega j(w(x))\,dx & \text{if } w \in L^2(\Omega) \quad \text{and} \quad j(w) \in L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{2.46}$$

We remark that this definition is similar to that of (1.47) in Section 1.2.3; however, since here w is allowed to vary in V', it is a priori not obvious why the functional $J_{V'}$ should be l.s.c. (indeed, the topology of V' is less fine than that of H). We recall that this is true, instead, for $J_H := (J_{V'})_{|H}$. The crucial hypothesis which guarantees the lower semicontinuity of $J_{V'}$ is precisely (2.45), as it is proved in a more general framework in Subsec. 3.2.3 (Prop. 3.2.5) below.

We can now state Damlamian's result in its precise mathematical version (indeed, in the original paper, more elaborate boundary conditions are required; here, we present a somehow simplified statement, in whose proof, anyway, all the main ideas are still present).

Theorem 2.2.2 (Damlamian). *If it is $f \in L^2(0,T;V')$ and $e_0 \in H$, with $j(e_0) \in L^1(\Omega)$, then there exists a unique couple (θ, e) of real-valued functions on Q of the regularity*

$$e \in H^1(0,T;V') \cap L^\infty(0,T;H), \tag{2.47}$$

$$\theta \in L^2(0,T;V), \tag{2.48}$$

such that (2.40–2.41) hold, as well as the following weaker form of (2.39)

$$\partial_t e + \mathcal{F}\theta = f \quad \text{in } V' \quad \text{for a.e. } t \in]0,T[. \tag{2.49}$$

Moreover, if β is Lipschitz continuous, $f \in L^2(0,T;H)$, and $\theta_0 := \beta(e_0) \in V$ (θ_0 is well-defined since β is now single-valued), then we have the additional regularity

$$\theta \in L^\infty(0,T;V) \cap H^1(0,T;H). \;\blacksquare \tag{2.50}$$

Proof. We only give here a short idea of the existence part of the proof; indeed, later we shall insist on similar techniques in a greater detail. Denoting by $\partial_{V'} J_{V'}$ the subdifferential of $J_{V'}$ as a monotone operator of V' (identified here with its dual), we consider the auxiliary abstract system in the space V'

$$\text{(2.51)} \qquad \begin{cases} \partial_t e + w = f & \text{a.e. in }]0,T[, \\ w \in \partial_{V'} J_{V'}(e) & \text{a.e. in }]0,T[, \\ e(0) = e_0 & (\in D(J_{V'})). \end{cases}$$

Owing for instance to [14, Theorem 3.6, pages 72–73], it has a unique solution $e \in H^1(0,T;V')$, such that

$$\text{(2.52)} \qquad J_{V'}(e) \in L^\infty(0,T).$$

Setting now $\theta := \mathcal{F}^{-1}(w)$, we see that (2.49) and (2.48) are immediate, while it remains to prove the second of (2.47) and (2.40). For this purpose, we observe that, owing to the results of Subsec. 3.2.4 (to which we refer, here and below, for further details on the mathematical background), the second of (2.51) is equivalent to

$$\text{(2.53)} \qquad e \in \partial_{V,V'} J^*_{V'}(\theta).$$

Here $\partial_{V,V'} J^*_{V'}$ denotes the subdifferential of $J^*_{V'}$, which is now seen as a monotone graph in $V \times V'$: indeed, we have come back to the identification of H with H'. Now, it is easy to see that (2.40) follows by calculating explicitely $J^*_{V'}$ and recalling Theorem 1.2.22, once observed that, by (2.46), the effective domain of $J_{V'}$ is contained in H. The second of (2.47) is now a consequence of the coercivity property (2.45) (see again Subsec. 3.2.4 below) and of (2.52).

Finally, to prove the additional regularity (2.50), we set $\gamma := \beta^{-1}$ and approximate it with its Yosida-regularization γ_ε; if we write the regularized form of (2.49) as $\partial_t \gamma_\varepsilon(\theta) + \mathcal{F}\theta = f$, we see that this equation can be easily solved by means of standard techniques of resolution for evolution equations with Lipschitz nonlinearities. Now, defining $\beta_\varepsilon := \gamma_\varepsilon^{-1}$, it is not difficult to get some a priori estimates (see [32] for the complete procedure), exploiting the Lipschitz continuity of β, which permit to pass to the limit "inside" β_ε. ∎

The above result can taken as a starting point for the resolution of various types of generalized Stefan-like problems. For instance, one could wonder if the coercivity assumption (2.45) could be dropped; indeed, with a more delicate procedure (see the references in [32]), we can weaken it to

$$\text{(2.54)} \qquad \liminf_{r\to\infty} \frac{j(r)}{|r|} = m > 0$$

(see also [10]); if β is completely arbitrary, the problem seems no longer solvable (at least with Hilbert space techniques) in the general case, unless some compatibility hypotheses are assumed; for instance, in [33], it is shown that (ST) is solvable for every monotone graph β, provided that

$$\text{(2.55)} \qquad p^{-1} g \in \beta(h) \quad \text{a.e. on } \Sigma, \quad \text{for some } h \in L^2(\Sigma);$$

the proof requires in this case a careful approximation of the graph β, which is performed in order to recover some more coercivity from the boundary term.

Another kind of troubles arises when pure Neumann conditions are assumed ($p \equiv 0$); infact, in this case the problem is no more coercive, so that, endowing now V with its usual scalar product (and denoting again by $\mathcal{F}$ the associated Riesz operator), it is easy to see that (2.49) should be modified as

$$\partial_t e + \mathcal{F}\theta = f + \theta \quad \text{in } V', \quad \text{for a.e. } t \in]0,T[. \tag{2.56}$$

It is natural to try to solve this equation through a Banach fixed point argument; anyway, this works only provided that β is Lipschitz [32]. In the general case, it seems that strong compatibility conditions on data must be assumed; for instance, Kenmochi [46] is able to treat the case of $D(\beta)$ bounded, under *nonresonance conditions* on data

$$\frac{1}{|\Omega|}\left(\int_\Omega e_0\,dx + \int_0^t\int_\Omega f\,dx\,ds + \int_0^t\int_\Gamma g\,d\mathcal{H}^{N-1}\,ds\right) \in \operatorname{int} D(\beta); \tag{2.57}$$

for other related results, see also the references of the quoted paper.

Finally, we conclude by observing that the weak Stefan problem can also be attacked by L^1–semigroup techniques [62, 16]; indeed, the operator $-\Delta\beta(\cdot)$ turns out to be *m-accretive* [9] in the space $L^1(\Omega)$. Here, we do not discuss this approach since it has no connexions with the object of the next chapters; we just observe that in this framework an hypothesis like (2.45) is no longer required; anyway, less regular solutions will correspondently be obtained.

2.2.2 The phase-field model: existence and uniqueness of solutions

In this section, we present an approach to the parabolic phase-field system which is based on a Faedo-Galerkin approximation scheme; indeed, this method, which is alternative to the finite-differences argument which will be employed in the next chapter for a more general problem, can be considered almost classical [53], since all the nonlinearities exhibited by the system are of rather standard types (monotone or even Lipschitzian). Nevertheless, we want to present also this approach in order to provide an element of comparison for the less elementary results of the sequel; moreover, the contents of this section will be useful also for the case of transmission problems, since the proofs of several estimates are simpler in this less general framework.

We begin by recalling again some basic hypotheses: let $\Omega \subset \mathbb{R}^N$ be a smooth (say, C^2), bounded and connected domain, $T < +\infty$ a fixed final time. Set $H = L^2(\Omega)$ and $V = H^1(\Omega)$ on account of Neumann boundary conditions (the Dirichlet case is analogous and even simpler) and also $Q := \Omega\times]0,T[$, $Q_t := \Omega\times]0,t[$, $\Sigma := \partial\Omega\times]0,T[$, $\Sigma_t := \partial\Omega\times]0,t[$ for $t \in]0,T]$. Assume also that $\alpha \subset \mathbb{R}\times\mathbb{R}$ is a maximal monotone graph such that $0 \in \alpha(0)$ and name j its convex primitive (see Subsection 2.2.1). Suppose also that

$$f \in L^2(0,T;V'), \tag{2.58}$$

$$\theta_0 \in H, \quad \chi_0 \in V, \quad \chi_0 \in D(j) \quad \text{a.e. in } \Omega, \quad \text{with } j(\chi_0) \in L^1(\Omega), \tag{2.59}$$

$$\gamma \in C^1(\mathbb{R},\mathbb{R}) \quad \text{with } \gamma' \in L^\infty \tag{2.60}$$

$$\lambda \in C^2(\mathbb{R},\mathbb{R}) \quad \text{with } \lambda', \lambda'' \in L^\infty \tag{2.61}$$

$$\mu, \nu > 0 \quad \text{fixed constants.} \tag{2.62}$$

Since we have little regularity on data, the phase-field equations will be addressed in a variational framework; thus, we have to introduce the operator

$$A : V \to V', \qquad \langle Av, z\rangle := \int_\Omega \nabla v \cdot \nabla z \, dx \qquad \text{for } v, z \in V. \tag{2.63}$$

We can now state the existence and uniqueness result for the phase field system

Theorem 2.2.3. *Assuming (2.58–2.62), there exists a triplet* (θ, χ, ξ) *of regularity*

$$\theta \in L^2(0,T;V) \cap H^1(0,T;V') (\subset C^0([0,T];H), \ \textit{by} \ \text{Prop. 1.1.14 (a)}), \tag{2.64}$$
$$\chi \in C^0([0,T];V) \cap H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)) \tag{2.65}$$
$$\xi \in L^2(0,T;H), \tag{2.66}$$

satisfying the following generalized phase-field system

$$\partial_t(\theta + \lambda(\chi)) + A\theta = f \qquad \textit{in } V', \quad \textit{a.e. in }]0,T[, \tag{2.67}$$
$$\mu \partial_t \chi - \nu \Delta \chi + \xi + \gamma(\chi) = \lambda'(\chi)\theta \qquad \textit{a.e. in } Q, \tag{2.68}$$
$$\xi \in \alpha(\chi) \qquad \textit{a.e. in } Q, \tag{2.69}$$
$$\partial_{\mathbf{n}} \chi = 0 \qquad \textit{on } \Sigma, \quad \textit{in sense of traces}, \tag{2.70}$$
$$\theta(\cdot,0) = \theta_0, \qquad \chi(\cdot,0) = \chi_0 \qquad \textit{a.e. in } \Omega. \tag{2.71}$$

Moreover, we have uniqueness of solutions if λ *is linear* ($\lambda(r) = \ell r$ *for all* $r \in \mathbb{R}$, *so that* $\lambda'' = 0$)*; in the general case, uniqueness is ensured between the solutions also fulfilling either*

$$\theta \in L^1(0,T;L^\infty(\Omega)), \qquad \textit{or}, \tag{2.72}$$
$$\textit{only in dimension } N = 3, \qquad \theta \in L^\infty(0,T;L^3(\Omega)). \ \blacksquare \tag{2.73}$$

Proof of uniqueness. Suppose to have a couple of solutions, say $(\hat\theta, \hat\chi, \hat\xi)$ and $(\check\theta, \check\chi, \check\xi)$ to (2.67–2.71). Adding θ to both hands sides of (2.67) and calling $\mathcal{F}$ the Riesz operator of V (which is now endowed with the standard scalar product), we can rewrite that equation as

$$\partial_t(\theta + \lambda(\chi)) + \mathcal{F}\theta = f + \theta \qquad \text{in } V', \quad \text{a.e. in }]0,T[, \tag{2.74}$$

Set now $\theta := \hat\theta - \check\theta$, $\chi := \hat\chi - \check\chi$ and $\xi := \hat\xi - \check\xi$; substitute first $(\hat\theta, \hat\chi, \hat\xi)$, then $(\check\theta, \check\chi, \check\xi)$, into equation (2.74) and take the difference; multiply it by $\mathcal{F}^{-1}(\theta + \lambda(\hat\chi) - \lambda(\check\chi))$; then, integrate over $]0,t[$, $t \le T$. Exploiting the usual properties of the Riesz operator (see Subsec. 1.1.3), it follows

$$\begin{aligned} &\frac{1}{2}\|\theta(t) + (\lambda(\hat\chi) - \lambda(\check\chi))(t)\|_*^2 + \|\theta\|^2_{L^2(0,t;H)} \\ &\quad = \int_0^t \big(\theta, \mathcal{F}^{-1}(\theta + \lambda(\hat\chi) - \lambda(\check\chi))\big)\, ds - \int_0^t \big(\theta, \lambda(\hat\chi) - \lambda(\check\chi)\big)\, ds =: I_1 + I_2 \end{aligned} \tag{2.75}$$

We now estimate the integrals on the right hand side. We have (σ and C_σ are, here and below, as in (1.25))

$$I_1 \leq \|\theta\|_{L^2(0,T;V')}\|\theta + \lambda(\hat{\chi}) - \lambda(\check{\chi})\|_{L^2(0,T;V')} \tag{2.76}$$
$$\leq \sigma\|\theta\|^2_{L^2(0,T;H)} + C_\sigma\|\theta + \lambda(\hat{\chi}) - \lambda(\check{\chi})\|^2_{L^2(0,T;V')},$$

$$I_2 \leq \frac{1}{2}\|\theta\|^2_{L^2(0,T;H)} + \frac{\|\lambda'\|^2_{L^\infty(\mathbb{R})}}{2}\|\chi\|^2_{L^2(0,T;H)}. \tag{2.77}$$

Let us now put first $(\hat{\theta}, \hat{\chi}, \hat{\xi})$, then $(\check{\theta}, \check{\chi}, \check{\xi})$ into equation (2.68), take the difference and multiply it by χ. Proceeding in a similar way as before and exploiting the monotonicity of α, we infer ($C_\gamma > 0$ below depending only on γ)

$$\frac{\mu}{2}\|\chi(t)\|^2_H + \nu\|\nabla\chi\|^2_{L^2(0,t;H)} \leq C_\gamma\|\chi\|^2_{L^2(0,t;H)} + \int_0^t\int_\Omega \lambda'(\hat{\chi})\theta\chi \, dx\, ds \tag{2.78}$$
$$+ \int_0^t\int_\Omega (\lambda'(\hat{\chi}) - \lambda'(\check{\chi}))\check{\theta}\chi \, dx\, ds =: C_\gamma\|\chi\|^2_{L^2(0,t;H)} + I_3 + I_4.$$

Thanks to the boundedness of λ' and to (1.24), we immediately have

$$I_3 \leq \frac{1}{4}\|\theta\|^2_{L^2(0,t;H)} + C\|\chi\|^2_{L^2(0,t;H)}, \tag{2.79}$$

(here and below, $C > 0$ is a constant only depending on data) while the estimate of I_4 is more delicate; indeed, in case λ is linear, we have $I_4 = 0$; otherwise,

$$I_4 \leq \|\lambda''\|_{L^\infty(\mathbb{R})}\int_0^t\int_\Omega |\check{\theta}|\chi^2 \, dx\, ds. \tag{2.80}$$

In any case, we sum together (2.75) and (2.78): if (2.72) holds, then we conclude by means of simple considerations based on the Gronwall inequality; in the case of (2.73), instead, thanks to the three-dimensional Sobolev embedding $H^1 \subset L^6$ (see Theorem 1.1.2), we have

$$I_4 \leq C\|\check{\theta}\|_{L^\infty(0,t;L^3(\Omega))}\|\chi\|_{L^2(0,t;H)}\|\chi\|_{L^2(0,t;V)} \leq \sigma\|\chi\|^2_{L^2(0,t;V)} + C_\sigma\|\chi\|^2_{L^2(0,t;H)}, \tag{2.81}$$

and the uniqueness follows again from the Gronwall lemma.

Proof of existence. First of all, we introduce a double approximation of the phase-field system, through the substitution of the graph α with its Yosida-regularization α_ϵ, joint with a Faedo-Galerkin scheme. Since we need a particular family of allowable test functions (see [53]), we consider the following eigenvalue problem

$$\begin{cases} v_n \in V \\ Av_n = \lambda_n v_n \quad \text{in } V', \end{cases} \tag{2.82}$$

where the eigenvalues λ_n are ordered in an increasing sequence and counted according to their respective multiplicities, so that we can assume that their associated eigenvectors v_n form an orthonormal basis of H and an orthogonal system of V; if we define $V_n := \text{span}\{v_1, \ldots, v_n\}$ and $V_\infty := \cup_{n=1}^\infty V_n$, we get that V_∞ is dense in V.

For every $n \in \mathbb{N}$, we now look for a couple of functions

$$\theta^n = \sum_{j=1}^{n} a_{jn}(t) v_j, \qquad \chi^n = \sum_{j=1}^{n} b_{jn}(t) v_j, \tag{2.83}$$

where a_{jn} and b_{jn} are real-valued functions of time, solving the following finite-dimensional approximation of system (2.67–2.68), which we state in a variational form in order to avoid use of projection operators:

$$\int_\Omega \partial_t(\theta^n + \lambda(\chi^n))\, v\, dx + \int_\Omega \nabla\theta^n \cdot \nabla v\, dx = \int_\Omega f\, v\, dx \tag{2.84}$$

$$\mu \int_\Omega \partial_t \chi^n\, v\, dx + \nu \int_\Omega \nabla\chi^n \cdot \nabla v\, dx = \int_\Omega (\lambda'(\chi^n)\theta^n - \alpha_\varepsilon(\chi^n) - \gamma(\chi^n))\, v\, dx \tag{2.85}$$

holding for any $v \in V_n$ and a.e. $t \in]0, T[$.

Moreover, choosing successively $v = v_h$, for $h = 1, \ldots, n$ in (2.84–2.85), taking into account the substitution of α with α_ϵ, and using the expression (2.83) of the unknowns, we are able to interpret the previous system as a system of vectorial ODE's

$$\begin{cases} \mathbf{a}' + \Lambda_n \mathbf{a} = \mathbf{f} - \widetilde{\lambda}'(\mathbf{b})\mathbf{b}' \\ \mu\mathbf{b}' + \nu\Lambda_n \mathbf{b} + \widetilde{\alpha}_\epsilon(\mathbf{b}) + \widetilde{\gamma}(\mathbf{b}) = \widetilde{\lambda}'(\mathbf{b})\mathbf{a}. \end{cases} \tag{2.86}$$

Here, $(\mathbf{a}, \mathbf{b})$ are the vectors $(a_{hn}(t), b_{hn}(t))_{h=1,\ldots,n}$, Λ_n the diagonal matrix of the first n eigenvalues of (2.82), $f_h := \langle f, v_h \rangle$ and $\widetilde{\lambda}', \widetilde{\alpha}_\epsilon, \widetilde{\gamma}$ are Lipschitz continuous functions of the unknowns deriving from $\lambda', \alpha_\epsilon, \gamma$, respectively. To solve the above system, we also have to adapt the Cauchy conditions (2.71) to it; thus, we approximate (θ_0, χ_0) with a sequence

$$(\theta_{0,n}, \chi_{0,n}) \subset V_n^2, \qquad \text{with} \qquad (\theta_{0,n}, \chi_{0,n}) \to (\theta_0, \chi_0) \quad \text{in} \ \ V^2, \tag{2.87}$$

that is, for some $a_{jn,0}, b_{jn,0} \in \mathbb{R}^n$,

$$(\theta_{0,n}, \chi_{0,n}) = \left(\sum_{j=1}^{n} a_{jn,0} v_j, \sum_{j=1}^{n} b_{jn,0} v_j \right). \tag{2.88}$$

Moreover, thanks to the quadratic growth of j^ε and to (2.87), we can also assume

$$\int_\Omega j_\epsilon(\chi_{0,n})\, dx \le 1 + \int_\Omega j_\epsilon(\chi_0)\, dx \tag{2.89}$$

at least for n sufficiently large (indeed, depending on ϵ), j_ϵ being here the convex primitive of α_ϵ.

We are now ready to solve system (2.86); with this aim, we first have to rewrite it in a normal form. So, it is sufficient to derive $\mathbf{b}'$ in terms of $(\mathbf{a}, \mathbf{b})$ from the second equation and to substitute it into the right hand side of the first equation. Unfortunately, we see that the right hand sides of the resulting system are Lipschitz continuous functions of the unknowns $\mathbf{a}, \mathbf{b}$ only locally; so, Cauchy's theorem only guarantees existence and uniqueness of a local solution. This means that, for any ε, n, we can find a small final time t_n (depending also on ε), such that the system (2.86), with the initial values (2.88), has a solution defined on $[0, t_n]$, of the natural regularity $(\theta^n, \chi^n) \in C^1(0, t_n; V_n)^2$.

A priori estimation. We derive an a priori estimate for the solutions of the approximation of system (2.67–2.68); this will allow us to pass to the limit first for $n \to \infty$ and then for $\varepsilon \to 0$. As before, $C > 0$ will denote a constant independent of ε, n, and also of t_n. For this reason, the limit solutions will turn out to have a global character, being defined in the whole interval $[0, T]$. Hence, for the sake of simplicity, we shall directly perform the next computations with T in place of t_n as a final referring time.

Multiply (2.84) by θ^n and (2.85) by $\partial_t \chi^n$. Summing together the resulting relations, integrating them over Q_t, $t \le T$, noting that two terms cancel, and performing standard integrations by parts, also with the aid of (1.25) we infer the following ($C_\gamma, \sigma, C_\sigma$ are as before)

$$\begin{aligned}(2.90)\quad & \frac{1}{2}\|\theta^n(t)\|_H^2 + \|\nabla\theta^n\|_{L^2(0,T;H)}^2 + \mu\|\partial_t\chi^n\|_{L^2(0,T;H)}^2 + \frac{\nu}{2}\|\nabla\chi^n(t)\|_H^2 \\ & + \int_\Omega j_\varepsilon(\chi^n(t))\,dx \le \frac{1}{2}\|\theta_{0,n}\|_H^2 + \frac{\nu}{2}\|\nabla\chi_{0,n}\|_H^2 + \int_\Omega j_\varepsilon(\chi_{0,n})\,dx \\ & + C_\sigma\|f\|_{L^2(0,t;V')}^2 + \sigma\|\theta^n\|_{L^2(0,t;V)}^2 + C_\sigma C_\gamma\|\chi^n\|_{L^2(0,t;H)}^2 + \sigma\|\partial_t\chi^n\|_{L^2(0,t;H)}^2.\end{aligned}$$

From this relation, using also the Gronwall inequality and the properties (2.87) and (2.89) of the initial data, it is immediate to deduce the priori bounds corresponding to the following convergences (holding up to the extraction of subsequences):

$$(2.91)\qquad \theta^n \to \theta^\varepsilon \quad \text{in } L^\infty(0,T;H)\text{-weak}^* \text{ and in } L^2(0,T;V)\text{–weak},$$

$$(2.92)\qquad \chi^n \to \chi^\varepsilon \quad \text{in } L^\infty(0,T;V)\text{-weak}^* \text{ and in } H^1(0,T;H)\text{–weak},$$

whence, on account of the Aubin compactness theorem (see [82, Cor. 4, Sec. 8]), we also infer:

$$(2.93)\qquad \chi^n \to \chi^\varepsilon \quad \text{in } C^0([0,T];H)\text{-strong}.$$

Thanks to the Lipschitz continuity of $\lambda', \alpha_\varepsilon$, this immediately entails

$$(2.94)\qquad \lambda'(\chi^n) \to \lambda'(\chi^\varepsilon) \quad \text{in } C^0([0,T];H)\text{-strong},$$

$$(2.95)\qquad \alpha_\varepsilon(\chi^n) \to \alpha_\varepsilon(\chi^\varepsilon) \quad \text{in } C^0([0,T];H)\text{-strong},$$

$$(2.96)\qquad \gamma(\chi^n) \to \gamma(\chi^\varepsilon) \quad \text{in } C^0([0,T];H)\text{-strong},$$

and, also on account of the Sobolev immersion theorem,

$$(2.97)\qquad \lambda'(\chi^n)\partial_t\chi^n \to \lambda'(\chi^\varepsilon)\partial_t\chi^\varepsilon \quad \text{in } L^2(0,T;L^p(\Omega))\text{-weak},$$

$$(2.98)\qquad \lambda'(\chi^n)\theta^n \to \lambda'(\chi^\varepsilon)\theta^\varepsilon \quad \text{in } L^\infty(0,T;L^q(\Omega))\text{-weak}^*,$$

for some exponents $p, q > 2$ depending on the space dimension.

All these convergences are enough to pass to the limit in equations (2.84–2.85), so that we get the system

$$(2.99)\qquad \partial_t(\theta^\varepsilon + \lambda(\chi^\varepsilon)) + A\theta^\varepsilon = f \quad \text{in } V'$$

$$(2.100)\qquad \mu\partial_t\chi^\varepsilon + \nu A\chi^\varepsilon + \alpha_\varepsilon(\chi^\varepsilon) + \gamma(\chi^\varepsilon) = \lambda'(\chi^\varepsilon)\theta^\varepsilon \quad \text{in } V'.$$

Now, it remains to pass to the limit with ε; here, the procedure is the same as before for what concerns (2.91–2.94) and (2.96–2.98), while, to pass to the limit in the nonlinear term $\alpha_\varepsilon(\chi^\varepsilon)$, we have to use the monotonicity argument of Prop. 1.3.8 and precisely with (χ_ε, χ) in place of (x_n, x), $(\alpha_\varepsilon, \alpha)$ in place of $(\mathcal{A}_n, \mathcal{A})$, and $\alpha_\varepsilon(\chi_\varepsilon)$ in place of y_n.

To determine what corresponds to y, we have to deduce a weak convergence for $\alpha_\varepsilon(\chi_\varepsilon)$, which, together with the strong convergence (2.93), will guarantee (1.61) and the possibility to pass to the limit.

To get this, it is enough to test equation (2.100) by $\alpha_\varepsilon(\chi_\varepsilon)$: using the Gauss-Green formula, the monotonicity of α_ε, and the Young inequality (1.25), we easily derive the boundedness

$$\|\alpha_\varepsilon(\chi_\varepsilon)\|_{L^2(0,T;H)} \leq C, \tag{2.101}$$

which allows to identify by extraction of a subsequence the desired value y and to pass to the limit in the system (2.99–2.100).

Now the proof of the existence part of Theorem 2.2.3 is complete, except for the $C^0([0,T];V)$ regularity of χ; to obtain this, we first observe that, by comparison in (2.100), $A\chi^\varepsilon$ is bounded in $L^2(0,T;H)$, so that we can apply to equation (2.68) the abstract regularity Theorem 1.1.16. ∎

2.2.3 The phase-field model: regularity

Let us present the regularity theorem for system (2.67–2.71), which naturally holds under stronger assumptions on data.

Theorem 2.2.4. *Suppose that, besides (2.58–2.62), also the additional hypotheses*

$$f \in L^1(0,T;H) + W^{1,1}(0,T;V'), \qquad \theta_0 \in V \tag{2.102}$$

are fulfilled. Then, any solution of (2.67–2.71) satisfies also

$$\theta \in C^0([0,T];V) \cap H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)) \tag{2.103}$$

and equation (2.67) actually holds a.e. in Q, together with the Neumann condition for θ: $\partial_{\mathbf{n}}\theta = 0$ on Σ. Also, notice that in this case, owing to the Sobolev immersion theorem (Theorem 1.1.2), (2.72) is verified in dimension 3, so that now the solution comes out to be unique.

Finally, if we assume we are in 3 space dimensions and also suppose that

$$\chi_0 \in H^2(\Omega), \quad \text{with } \partial_{\mathbf{n}}\chi_0 = 0 \text{ on } \Gamma, \tag{2.104}$$

$$\text{there exists } \xi_0 \in H, \quad \text{such that } \xi_0 \in \alpha(\chi_0) \text{ a.e. in } \Omega, \tag{2.105}$$

then we get the further regularity for the phase variable

$$\chi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)). \tag{2.106}$$

Proof. It is based on further a priori estimates for the solutions of the Faedo-Galerkin approximate problem; since, as we have seen, the basic regularity provided by Theorem 2.2.3 is enough to pass to the limit, for the sake of simplicity, we just perform the estimates in a formal way on the solutions of the original problem. Naturally, in this framework, some passages could be unjustified; in that case, we shall give a remark about the correct procedure.

Regularity of $\boldsymbol{\theta}$. Split f as $f = f_1 + f_2$, with $f_1 \in W^{1,1}(0,T;V')$ and $f_2 \in L^1(0,T;H)$. Multiply (2.67) by $\partial_t\theta$: integrating over Q_t the resulting relation, we easily infer

$$\begin{aligned}\|\partial_t\theta\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla\theta(t)\|^2_H &\\ = \frac{1}{2}\|\nabla\theta_0\|^2_H - \int_0^t\int_\Omega \lambda'(\chi)\partial_t\chi\,\partial_t\theta\,dx\,ds + \int_0^t \langle f_1, \partial_t\theta\rangle\,ds &\\ + \int_0^t\int_\Omega f_2\,\partial_t\theta\,dx\,ds =: I_1 + I_2 + I_3.\end{aligned} \tag{2.107}$$

Now, we estimate the three integrals on the right hand side. As for I_1, it is enough to recall the boundedness of λ', use (1.25) and control the L^2-norm of $\partial_t\chi$ by means of the previously obtained estimates; for the second term, through an integration by parts, we obtain:

$$\begin{aligned} I_2 &= -\int_0^t \langle\partial_t f_1, \theta\rangle\,ds + \langle f_1(t), \theta(t)\rangle - \langle f_1(0), \theta_0\rangle \\ &\le \int_0^t \|\partial_t f_1\|_{V'}\left(\|\theta\|_H + \|\nabla\theta\|_H\right)\,ds + \sigma\|\theta(t)\|^2_{V'} + C_\sigma\|f_1\|^2_{L^\infty(0,t;V')} + C\end{aligned} \tag{2.108}$$

(also on account of the regularity of data), while the third one simply reads

$$I_3 \le \int_0^t \|f_2\|_H\|\partial_t\theta\|_H\,ds; \tag{2.109}$$

so, owing to these estimations of I_1, I_2, I_3, it is easy to see that the first part of the thesis follows now from the Gronwall lemma in the form of Prop. 1.1.15, by also exploiting Theorem 1.1.16 to get the C^0–regularity.

Regularity of $\boldsymbol{\chi}$. Let us formally differentiate relation (2.68) (observe that it can happen that β is not differentiable; this is not a problem, since we should reason on β^ε). We simply get

$$\mu\partial_{tt}\chi - \nu\Delta\partial_t\chi + \beta'(\chi)\partial_t\chi + \gamma'(\chi)\partial_t\chi = \lambda''(\chi)\partial_t\chi\theta + \lambda'(\chi)\partial_t\theta. \tag{2.110}$$

Multiplying the above relation by $\partial_t\chi$, integrating as before, we infer

$$\begin{aligned}\frac{\mu}{2}\|\partial_t\chi(t)\|^2_H + \nu\|\nabla\partial_t\chi\|^2_{L^2(0,t;H)} + \int_0^t\int_\Omega \beta'(\chi)(\partial_t\chi)^2\,dx\,ds &\\ = \frac{\mu}{2}\|\partial_t\chi(0)\|^2_H - \int_0^t\int_\Omega \gamma'(\chi)(\partial_t\chi)^2\,dx\,ds + \int_0^t\int_\Omega \lambda''(\chi)(\partial_t\chi)^2\theta\,dx\,ds &\\ + \int_0^t\int_\Omega \lambda'(\chi)\partial_t\chi\,\partial_t\theta\,dx\,ds =: \frac{\mu}{2}\|\partial_t\chi(0)\|^2_H + I_4 + I_5 + I_6.\end{aligned} \tag{2.111}$$

We now forget for a while of the term with $\partial_t \chi(0)$ on the right hand side and we estimate the integrals. As for I_4 and I_6, it is enough to exploit (1.25) and recall the boundedness of γ', λ' and the $L^2(Q)$-estimate for $\partial_t \chi$. I_5 needs instead a more accurate treatment:

$$I_5 \leq C \int_0^t \|\partial_t \chi\|_H \|\partial_t \chi\|_{L^4(\Omega)} \|\theta\|_{L^4(\Omega)} \, ds \quad \leq \sigma \int_0^t \|\nabla \partial_t \chi\|_H^2 \, ds + C_\sigma \int_0^t \|\partial_t \chi\|_H^2 \, ds,$$

where C_σ, which has the same meaning as in (1.25), depends now on the $L^\infty(0,T;V)$-norm of θ and we have used the 3-dimensional immersion $V \subset L^4$.

This is enough to control all the terms on the right hand side, with the exception of $\partial_t \chi(0)$; first of all, we observe that here $\partial_t \chi$ is not continuous; hence, in this framework such a term does not make sense; however, if this estimate were performed at the level of the Faedo-Galerkin approximation, its meaning would have been guaranteed by $\chi \in C^1([0,t_n],V_n)$.

Anyway, we go on in a formal way, by substituting $t = 0$ into equation (2.68) and deriving

$$\partial_t \chi(0) = \mu^{-1} \left(\nu \Delta \chi_0 - \xi_0 - \gamma(\chi_0) + \lambda'(\chi_0) \theta_0 \right), \tag{2.112}$$

which is an element of H by virtue of the boundedness of λ' and by the supplementary hypotheses (2.104–2.105). This permits to control also the remaining term on the right hand side of (2.111) and to conclude the proof of Theorem 2.2.4 (again, an application of the Gronwall lemma is required). ∎

Chapter 3

Resolution of the phase-field system

We present in this chapter a natural abstract framework for the previously discussed phase-field system; we shall prove in a more general setting some existence, uniqueness and regularity results, corresponding to the Theorems 2.2.3 and 2.2.4 of the previous chapter. In particular, in the first section, we introduce the abstract formulation, and we prove some related existence, uniqueness, and regularity theorems; in the second one we first retrieve as a particular case the results about the concrete phase-field system of the last chapter (indeed only under the hypothesis of a *linear* latent heat); then, we see that the abstract machinery can be adapted also to the case of transmission problems between fluids with different thermodynamical characteristics, at least under some compatibility or growth conditions on the involved thermodynamical potentials. As a final application, we eventually discuss another kind of transmission problems for the phase-field model, which are related to a diffusion dynamics of a *concentrated capacity* type. The results of this chapter are in course of publication in the paper [79], save for the last section (discussing the concentrated capacity problem) whose contents are essentially included in [77].

3.1 An abstract approach

3.1.1 The mathematical problem

We put ourselves at once in the abstract setting of Subsec. 1.1.3. So, let us suppose to have two Hilbert spaces V, H, with V densely and compactly embedded into H, in order that (V, H, V') form a Hilbert triplet, and denote with $(\cdot,\cdot)$ the scalar product of H and with $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V. Moreover, take some $T > 0$ as an arbitrary final time.

Here are the minimal regularity hypotheses which the data of the abstract problem are required to satisfy:

(3.1) $\alpha, \lambda_0, C_0, C_1, C_2, \rho, \mu > 0$ fixed constants

(3.2) $P, L, M, \Lambda \in \mathcal{L}(H)$ symmetric operators

(3.3) $A, B \in \mathcal{L}(V, V')$ symmetric operators

(3.4) $f \in L^2(0, T; V')$

(3.5) $(Ph, h) \geq \rho \|h\|_H^2$ for all $h \in H$

(3.6) $(Mh, h) \geq \mu \|h\|_H^2$ for all $h \in H$

(3.7) $\langle Av, v \rangle \geq \alpha \|v\|_V^2 - \lambda_0 \|v\|_H^2$ for every $v \in V$

(3.8) $\langle Bv, v \rangle \geq \alpha \|v\|_V^2 - \lambda_0 \|v\|_H^2$ for every $v \in V$.

We also suppose

(3.9) $\gamma : V \to H$, nonlinear demicontinuous operator

(recall Def. 1.2.12 (a)), enjoying the following boundedness and coerciveness assumptions:

(3.10) $\|\gamma(v)\|_H^2 \leq C_1 + C_2 \|v\|_V^2$ for all $v \in V$

(3.11) $(\gamma(w) - \gamma(v), w - v) \geq -C_0 \|w - v\|_H^2 - \dfrac{\alpha}{2} \|w - v\|_V^2$ for all $v, w \in V$.

The norms of L and Λ in the space $\mathcal{L}(H)$ will be denoted by ℓ and λ respectively.

Let us also be given a function

(3.12) $J : V \to [0, +\infty]$ convex, lower semicontinuous and proper

On account of Remark 1.2.6, it is natural to consider the subdifferential of J as an operator from V to V'. We repeat the definition also with the purpose of specifying the notation which will be used throughout this chapter: if $v \in V$, $w \in V'$, we set

(3.13) $w \in \partial_{V,V'} J(v) \iff$ for all $z \in D(J)$, it is $\langle w, z - v \rangle \leq J(z) - J(v)$.

We also assume that

(3.14) $0 \in D(\partial_{V,V'} J), \quad 0 \in \partial_{V,V'} J(0),$

(compare with (2.59); in that case, indeed, we stated the problem in a H-setting instead that in the (V, V')-one).

Finally, here are our basic assumptions on the initial data:

(3.15) $\theta_0 \in H,$

(3.16) $\chi_0 \in D(J).$

We are now ready to introduce the main abstract problem:

Problem 3.1.1 (aP). *We look for a triplet of functions* (θ, χ, w) *satisfying the following regularity properties*

$$\theta \in L^2(0,T;V) \cap C^0([0,T];H), \tag{3.17}$$

$$P\theta \in H^1(0,T;V') \cap C^0([0,T];H), \tag{3.18}$$

$$\chi \in L^\infty(0,T;V) \cap H^1(0,T;H), \tag{3.19}$$

$$w \in L^2(0,T;V') \tag{3.20}$$

and such that the equations

$$(P\theta)' + \Lambda\chi' + A\theta = f \quad \textit{in } V', \tag{3.21}$$

$$M\chi' + B\chi + w + \gamma(\chi) = L\theta \quad \textit{in } V', \tag{3.22}$$

$$\chi \in D(\partial_{V,V'}J) \quad \textit{and} \quad w \in \partial_{V,V'}J(\chi) \tag{3.23}$$

hold for almost every $t \in [0,T]$.

Moreover we require the following Cauchy initial conditions:

$$\theta(0) = \theta_0, \tag{3.24}$$

$$\chi(0) = \chi_0. \ \blacksquare \tag{3.25}$$

We now state our related existence and uniqueness result (corresponding to Theorem 2.2.3):

Theorem 3.1.2. *There exists a unique solution to Problem* (aP)*, also fulfilling*

$$J(\chi) \in L^\infty(0,T). \ \blacksquare \tag{3.26}$$

We also have two simple regularity theorems, which are the analogous of the θ and χ-parts of Theorem 2.2.4, respectively:

Theorem 3.1.3. *Under the following additional assumptions:*

$$f \in L^2(0,T;H) + H^1(0,T;V'), \tag{3.27}$$

$$\theta_0 \in V, \tag{3.28}$$

any solution to Problem (aP) *satisfies also:*

$$\theta \in C^0(0,T;V) \cap H^1(0,T;H), \tag{3.29}$$

whence (3.21) can be rewritten in the more usual form

$$P\theta' + \Lambda\chi' + A\theta = f \quad \textit{in } V'. \ \blacksquare \tag{3.30}$$

Theorem 3.1.4. *If, in addition to the former hypotheses, we have:*

$$B\chi_0 \in H, \tag{3.31}$$

$$\chi_0 \in D(\partial_{V,V'}J), \tag{3.32}$$

$$\textit{there exists } w^0 \in H \cap \partial_{V,V'}J(\chi_0), \tag{3.33}$$

any solution to Problem (aP) *also fulfills:*

$$\chi \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H), \tag{3.34}$$

$$w \in L^\infty(0,T;V'). \ \blacksquare \tag{3.35}$$

3.1.2 Approximation and a priori estimates

We address Problem (aP) through a backward finite difference approximation scheme, and, with such a construction in mind, for any $n \in \mathbb{N}$, we subdivide the interval $[0,T]$ into n parts by setting $\tau := T/n$. We also approximate the heat source term in the following way:

$$f^i := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} f(s)\,ds, \qquad \text{for } i = 1,\dots,n, \tag{3.36}$$

so that we can introduce

Problem 3.1.5 (aP$_\tau$). *For every $i = 1,\dots,n$, find (θ^i, χ^i, w^i), with $\theta^i, \chi^i \in V$, $w^i \in V'$, such that the following equations hold:*

$$P\left(\frac{\theta^i - \theta^{i-1}}{\tau}\right) + \Lambda\left(\frac{\chi^i - \chi^{i-1}}{\tau}\right) + A\theta^i = f^i \qquad \text{in } V', \tag{3.37}$$

$$M\left(\frac{\chi^i - \chi^{i-1}}{\tau}\right) + B\chi^i + w^i + \gamma(\chi^i) = L\theta^i \qquad \text{in } V', \tag{3.38}$$

$$\chi^i \in D(\partial_{V,V'}J) \qquad \text{and} \qquad w^i \in \partial_{V,V'}J(\chi^i), \tag{3.39}$$

where we also restate the Cauchy initial conditions in the form:

$$\theta^0 = \theta_0, \qquad \chi^0 = \chi_0. \quad \blacksquare \tag{3.40}$$

The proof of the existence of a solution to the previous problem relies on the surjectivity properties of coercive monotone operators. Actually, we can define $\mathcal{A}_m : V^2 \to (V')^2$, as

$$\mathcal{A}_m(\theta,\chi) := \left(\frac{P\theta}{\tau} + A\theta + \frac{\Lambda\chi}{\tau}, m\frac{M\chi}{\tau} + mB\chi + m\partial_{V,V'}J(\chi) + m\gamma(\chi) - mL\theta\right), \tag{3.41}$$

where m is a positive constant to be chosen later. Clearly, $\mathcal{A}_m$ is a multi-valued operator, since such is $\partial_{V,V'}J$; we now see that it is also maximal monotone and that its domain is $V \times D(\partial_{V,V'}J)$. With this purpose, we can split $\mathcal{A}_m$ as $\mathcal{A}_m = \mathcal{B}_m + (\mathcal{A}_m - \mathcal{B}_m)$, where

$$\mathcal{B}_m(\theta,\chi) := \left(\frac{P\theta}{2\tau}, m\partial_{V,V'}J(\chi)\right) \tag{3.42}$$

is clearly maximal monotone with domain $V \times D(\partial_{V,V'}J)$, while $\mathcal{A}_m - \mathcal{B}_m$, at least for τ sufficiently small, is monotone, single-valued and hemicontinuous (recall again Def. 1.2.12; in particular, here $\mathcal{A}_m - \mathcal{B}_m$ is demicontinuous and this is a stronger property) with domain V^2 (to verify the monotonicity of $\mathcal{A}_m - \mathcal{B}_m$, the same procedure we are now going to apply to $\mathcal{A}_m$ can be followed). The maximality of $\mathcal{A}_m$ is now a consequence of Theorem 1.2.18 (cf. also [14, Cor. 2.7, pp. 36–37] and the subsequent observation).

Moreover, given $(\theta, \chi) \in V \times D(\partial_{V,V'} J)$ and $(\zeta, \xi) \in \mathcal{A}_m(\theta, \chi)$ (with w denoting the element of $\partial_{V,V'} J(\chi)$ corresponding to ξ), we deduce from (3.5–3.8) that

$$
\begin{aligned}
(3.43)\qquad {}_{(V')^2}\langle(\zeta,\xi),(\theta,\chi)\rangle_{V^2} \ge{}& \frac{\rho}{\tau}\|\theta\|_H^2 + \alpha\|\theta\|_V^2 - \lambda_0\|\theta\|_H^2 + \frac{1}{\tau}(\Lambda\chi,\theta) \\
&+ m\frac{\mu}{\tau}\|\chi\|_H^2 + m\alpha\|\chi\|_V^2 - m\lambda_0\|\chi\|_H^2 + m\langle w,\chi\rangle \\
&- m(L\theta,\chi) + m(\gamma(\chi)-\gamma(0),\chi-0) + m(\gamma(0),\chi-0).
\end{aligned}
$$

Working now on the last two terms of the previous relation with the aid of (3.10–3.11) and owing to the monotonicity of $\partial_{V,V'} J$ and to (3.14), we easily derive the following coerciveness property:

$$
\begin{aligned}
(3.44)\ {}_{(V')^2}\langle\mathcal{A}_m^0(\theta,\chi),(\theta,\chi)\rangle_{V^2} \ge{}& \alpha\|\theta\|_V^2 + \left(\frac{\rho}{2\tau} - \frac{\ell^2 m}{2} - \lambda_0\right)\|\theta\|_H^2 \\
&+ \frac{m\alpha}{2}\|\chi\|_V^2 + \left(\frac{2m\mu - \lambda^2\rho^{-1}}{2\tau} - m(C_0+\lambda_0+1)\right)\|\chi\|_H^2 - \frac{m}{2}\|\gamma(0)\|_H^2
\end{aligned}
$$

(here $\mathcal{A}_m^0(\theta,\chi)$ denotes the element of minimum norm in the set $\mathcal{A}_m(\theta,\chi)$, recall Subsec. 1.2.2), which actually permits to solve Problem (aP$_\tau$).

Indeed, choose $m = \lambda^2\mu^{-1}\rho^{-1}$: since (aP$_\tau$) can be rewritten in the form

$$
(3.45)\qquad \left(f^i + \frac{P\theta^{i-1}}{\tau} + \frac{\Lambda\chi^{i-1}}{\tau}, m\frac{M\chi^{i-1}}{\tau}\right) \in \mathcal{A}_m(\theta^i,\chi^i),
$$

for $i = 1, \ldots, n$, and proceeding by induction on i, owing for instance to [14, Cor. 2.4, page 31], at least for τ sufficiently small we can find a solution for it satisfying also the Cauchy conditions (3.40).

In order to remove the approximation, we now derive some a priori estimates for such a solution; in the following computations, relations (1.24–1.25) will be repeatedly used, with natural choices of the parameter σ, without further warning.

First a priori estimate. Multiply equation (3.37) by $\tau\theta^i$; summing the results for $i = 1, \ldots, m$, with $m \le n$, observing also that (σ is as in (1.24))

$$
(3.46)\qquad \tau\langle f^i,\theta^i\rangle = \int_{(i-1)\tau}^{i\tau}\langle f(s),\theta^i\rangle\,ds \le \frac{1}{2\sigma}\int_{(i-1)\tau}^{i\tau}\|f(s)\|_{V'}^2\,ds + \frac{\sigma\tau}{2}\|\theta^i\|_V^2
$$

and owing to the symmetry of operators P, A, to (3.5), and to the elementary relation

$$
(3.47)\qquad \sum_{i=1}^m a_i(a_i - a_{i-1}) = \frac{1}{2}\left[a_m^2 + \sum_{i=1}^m (a_i - a_{i-1})^2 - a_0^2\right] \qquad \text{for } a_0,\ldots,a_m \in \mathbb{R}
$$

(adapted in the natural way to symmetric bilinear forms), we get at once:

$$
\begin{aligned}
(3.48)\qquad \frac{\rho}{2}\|\theta^m\|_H^2 + \sum_{i=1}^m \frac{\rho}{2}\|\theta^i - \theta^{i-1}\|_H^2 + \tau\sum_{i=1}^m\langle A\theta^i,\theta^i\rangle \le{}& \frac{\|P\|_{\mathcal{L}(H)}}{2}\|\theta^0\|_H^2 \\
+ \frac{\sigma\tau}{2}\sum_{i=1}^m\|\theta^i\|_V^2 + \frac{1}{2\sigma}\int_0^{m\tau}\|f(t)\|_{V'}^2\,dt - \sum_{i=1}^m(\theta^i,\Lambda(\chi^i - \chi^{i-1})).&
\end{aligned}
$$

Test now equation (3.38) with $\chi^i - \chi^{i-1}$ and sum again the results for $i = 1, \dots, m$, $m \le n$. Using the definition of subdifferential, recalling also (3.3), (3.6), (3.47), and proceeding as for the "heat" equation, we easily derive

$$\begin{aligned}\mu\tau \sum_{i=1}^{m} \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_H^2 + \frac{1}{2}\langle B\chi^m, \chi^m\rangle \\ + \frac{1}{2}\sum_{i=1}^{m}\langle B\chi^i - B\chi^{i-1}, \chi^i - \chi^{i-1}\rangle + J(\chi^m) \\ \le \frac{1}{2}\langle B\chi^0, \chi^0\rangle + J(\chi^0) + \sum_{i=1}^{m}(L\theta^i, \chi^i - \chi^{i-1}) - \sum_{i=1}^{m}\langle \gamma(\chi^i), \chi^i - \chi^{i-1}\rangle.\end{aligned} \tag{3.49}$$

Adding now relation (3.49) to (3.48), choosing $\sigma = \alpha$ and recalling the coercivity assumptions (3.7), (3.8) and the property (3.10), it is not difficult to obtain, for τ sufficiently small,

$$\begin{aligned}\frac{\rho}{2}\|\theta^m\|_H^2 + \frac{\alpha\tau}{2}\sum_{i=1}^{m}\|\theta^i\|_V^2 + \mu\tau\sum_{i=1}^{m}\left\|\frac{\chi^i - \chi^{i-1}}{\tau}\right\|_H^2 + \frac{\alpha}{2}\|\chi^m\|_V^2 + J(\chi^m) \\ + \frac{\rho\tau^2}{2}\sum_{i=1}^{m}\left\|\frac{\theta^i - \theta^{i-1}}{\tau}\right\|_H^2 + \frac{\alpha\tau^2}{2}\left\|\frac{\chi^i - \chi^{i-1}}{\tau}\right\|_V^2 \\ \le \frac{\|P\|_{\mathcal{L}(H)}}{2}\|\theta^0\|_H^2 + \left(\frac{\lambda^2 + \ell^2}{\mu} + \lambda_0\right)\tau\sum_{i=1}^{m}\|\theta_i\|_H^2 \\ + \frac{1}{2\alpha}\|f\|_{L^2(0,m\tau;V')}^2 + \frac{1}{2}\langle B\chi^0, \chi^0\rangle + J(\chi^0) + \frac{\lambda_0}{2}\|\chi^m\|_H^2 \\ + \left(\frac{3\mu\tau}{4} + \frac{\lambda_0\tau^2}{2}\right)\sum_{i=1}^{m}\left\|\frac{\chi^i - \chi^{i-1}}{\tau}\right\|_H^2 + \frac{T}{\mu}C_1 + \frac{\tau}{\mu}C_2\sum_{i=1}^{m}\|\chi^i\|_V^2.\end{aligned} \tag{3.50}$$

In order to estimate the $\|\chi^m\|_H$–term in the right hand side, take equation (3.38) again and multiply it by $\tau\chi^i$; sum on i and recall the symmetry of M. Thanks to (3.47), it follows

$$\begin{aligned}\frac{\mu}{2}\|\chi^m\|_H^2 + \frac{\mu}{2}\sum_{i=1}^{m}\|\chi^i - \chi^{i-1}\|_H^2 + \alpha\tau\sum_{i=1}^{m}\|\chi^i\|_V^2 + \tau\sum_{i=1}^{m}\langle w^i, \chi^i\rangle \\ \le \frac{\|M\|_{\mathcal{L}(H)}}{2}\|\chi^0\|_H^2 - \tau\sum_{i=1}^{m}(\gamma(\chi^i), \chi^i) + \tau\sum_{i=1}^{m}(L\theta^i, \chi^i) + \lambda_0\tau\sum_{i=1}^{m}\|\chi^i\|_H^2 \\ \le \frac{\|M\|_{\mathcal{L}(H)}}{2}\|\chi^0\|_H^2 + \frac{T}{2}C_1 + \frac{C_2}{2}\tau\sum_{i=1}^{m}\|\chi^i\|_V^2 \\ + (1 + \lambda_0)\tau\sum_{i=1}^{m}\|\chi^i\|_H^2 + \frac{\ell^2}{2}\tau\sum_{i=1}^{m}\|\theta^i\|_H^2.\end{aligned} \tag{3.51}$$

Taking the sum of the previous relation times $2\lambda_0/\mu$ with (3.50), recalling also (3.14) and applying the Gronwall inequality in its discrete formulation, given by Prop. 1.1.17, it is easy to derive the following estimates:

$$\|\theta^m\|_H \le C \qquad \text{for all } n \in \mathbf{N} \ \text{ and } m \le n, \tag{3.52}$$

$$\tau \sum_{i=1}^{n} \|\theta^i\|_V^2 \le C \qquad \text{for all } n \in \mathbf{N}, \tag{3.53}$$

$$\|\chi^m\|_V \le C \qquad \text{for all } n \in \mathbf{N} \ \text{ and } m \le n, \tag{3.54}$$

$$\tau \sum_{i=1}^{n} \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_H^2 \le C \qquad \text{for all } n \in \mathbf{N}, \tag{3.55}$$

$$\tau^2 \sum_{i=1}^{n} \left\| \frac{\theta^i - \theta^{i-1}}{\tau} \right\|_H^2 \le C \qquad \text{for all } n \in \mathbf{N}, \tag{3.56}$$

$$\tau^2 \sum_{i=1}^{n} \left\| \frac{\chi^i - \chi^{i-1}}{\tau} \right\|_V^2 \le C \qquad \text{for all } n \in \mathbf{N} \tag{3.57}$$

$$J(\chi^m) \le C \qquad \text{for all } n \in \mathbf{N} \ \text{ and } m \le n. \tag{3.58}$$

Here and in the following, $C > 0$ is a constant allowed to vary from row to row, but supposed to depend only on the data $\rho, \mu, \alpha, \lambda_0, C_0, C_1, C_2, \theta_0, \chi_0, f, J(\chi_0)$ and, in particular, not on τ.

In view of a limit procedure, we now introduce some interpolating functions obtained from the $\theta^i, \chi^i, w^i, f^i$. First, we define the stair function

$$\theta_\tau(t) := \theta^i \qquad \text{for } t \in](i-1)\tau, i\tau], \qquad i = 1, \ldots, n \tag{3.59}$$

and in the same way we construct $\chi_\tau, w_\tau, f_\tau$. Moreover, we also need a piecewise linear interpolation of the solutions, which is introduced as

$$\widehat{\theta}_\tau(t) := \theta^{i-1} + \frac{\theta^i - \theta^{i-1}}{\tau}(t - (i-1)\tau) \qquad \text{for } t \in [(i-1)\tau, i\tau], \qquad i = 1, \ldots, n \tag{3.60}$$

(and the definition of $\widehat{\chi}_\tau$ is analogous). Observe that the above definitions imply:

$$\|\widehat{\theta}_\tau - \theta_\tau\|_{L^\infty(0,T;H)} \le \tau^{1/2} \|\widehat{\theta}'_\tau\|_{L^2(0,T;H)} \tag{3.61}$$

and similar relations with H and V (or V') or θ and χ interchanged. Moreover, we have that

$$\begin{aligned} \|\widehat{\theta}_\tau - \theta_\tau\|_{L^2(0,T;H)}^2 &= \sum_{i=1}^{n} \int_{(i-1)\tau}^{i\tau} \left\| \theta^i - \theta^{i-1} - \frac{\theta^i - \theta^{i-1}}{\tau}(t - (i-1)\tau) \right\|_H^2 dt \\ &= \frac{\tau^3}{3} \sum_{i=1}^{n} \left\| \frac{\theta^i - \theta^{i-1}}{\tau} \right\|_H^2 = \frac{\tau^2}{3} \|\widehat{\theta}'_\tau\|_{L^2(0,T;H)}^2 \le \frac{2}{3} \|\theta_\tau\|_{L^2(0,T;H)}^2 \end{aligned} \tag{3.62}$$

(as before, θ can be exchanged with χ and H with V or V'). Finally, recalling definition (3.36), standard arguments permit to see that

$$f_\tau \to f \qquad \text{in } L^2(0, T; V')\text{–strong.} \tag{3.63}$$

So, we conclude by rewriting estimates (3.52–3.58) in terms of the interpolating functions:

$$\|\theta_\tau\|_{L^\infty(0,T;H)} \le C, \tag{3.64}$$

$$\|\theta_\tau\|_{L^2(0,T;V)} \le C, \tag{3.65}$$

$$\|\chi_\tau\|_{L^\infty(0,T;V)} \le C, \tag{3.66}$$

$$\|\widehat{\chi}'_\tau\|_{L^2(0,T;H)} \le C, \tag{3.67}$$

$$\tau^{1/2}\|\widehat{\theta}'_\tau\|_{L^2(0,T;H)} \le C, \tag{3.68}$$

$$\tau^{1/2}\|\widehat{\chi}'_\tau\|_{L^2(0,T;V)} \le C, \tag{3.69}$$

$$\|J(\chi_\tau)\|_{L^\infty(0,T)} \le C. \tag{3.70}$$

Moreover, from (3.66), (3.62) (with χ in place of θ) and (3.67), it immediately results that

$$\|\widehat{\chi}_\tau\|_{H^1(0,T;H)} \le C. \tag{3.71}$$

The following supplementary estimates hold under the additional regularity hypotheses stated in Theorems 3.1.3 and 3.1.4 respectively.

Second a priori estimate. Multiply (3.37) by $\theta^i - \theta^{i-1}$ and sum as usual for $i = 1, \dots, m$; $m \le n$. Owing to (3.27), it is possible to split $f = f_1 + f_2$, with $f_1 \in L^2(0,T;H)$ and $f_2 \in H^1(0,T;V')$. Defining also f_1^i and f_2^i in the natural way imitating (3.36), observe now that

$$\begin{aligned}(f_1^i, \theta^i - \theta^{i-1}) &= \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} (f_1(s), \theta^i - \theta^{i-1})\, ds \\ &\le \frac{1}{\rho}\int_{(i-1)\tau}^{i\tau} \|f_1(s)\|_H^2\, ds + \frac{\rho\tau}{4}\left\|\frac{\theta^i - \theta^{i-1}}{\tau}\right\|_H^2.\end{aligned} \tag{3.72}$$

Moreover, we have

$$\sum_{i=1}^m \langle f_2^i, \theta^i - \theta^{i-1}\rangle = \langle f_2^m, \theta^m\rangle + \sum_{i=2}^m \langle f_2^{i-1} - f_2^i, \theta^{i-1}\rangle - \langle f_2^1, \theta^0\rangle, \tag{3.73}$$

where the second term on the right hand side reads

$$\begin{aligned}\sum_{i=2}^m \langle f_2^{i-1} - f_2^i, \theta^{i-1}\rangle &= \sum_{i=2}^m \int_{(i-1)\tau}^{i\tau} \left\langle \frac{f_2(s-\tau) - f_2(s)}{\tau}, \theta^{i-1}\right\rangle ds \\ &\le \frac{1}{2}\sum_{i=2}^m \int_{(i-1)\tau}^{i\tau} \left\|\frac{f_2(s-\tau) - f_2(s)}{\tau}\right\|_{V'}^2 ds + \frac{\tau}{2}\sum_{i=2}^m \|\theta^{i-1}\|_V^2 \\ &\le C_3 + C_3\int_\tau^{m\tau} \|f_2'\|_{V'}^2\, ds + \frac{\tau}{2}\sum_{i=2}^m \|\theta^{i-1}\|_V^2\end{aligned} \tag{3.74}$$

for some positive constant C_3, while we manage the first term by

$$\begin{aligned}(3.75)\qquad \langle f_2^m, \theta^m\rangle &= \frac{1}{\tau}\int_{(m-1)\tau}^{m\tau} \langle f_2(s), \theta^m\rangle\, ds \le \frac{1}{\alpha\tau}\int_{(m-1)\tau}^{m\tau} \|f_2(s)\|_{V'}^2\, ds + \frac{\alpha}{4}\|\theta^m\|_V^2\\ &\le \frac{1}{\alpha}\|f_2\|^2_{L^\infty(0,T;V')} + \frac{\alpha}{4}\|\theta^m\|_V^2\end{aligned}$$

and the third one in an identical way. So, collecting (3.72–3.75), we get after easy computations (notice that the first term in the following relation also accounts for two quantities derived from the right hand side of (3.37) through Young's inequality (1.24)):

$$\begin{aligned}(3.76)\ &\frac{\rho\tau}{2}\sum_{i=1}^m \left\|\frac{\theta^i-\theta^{i-1}}{\tau}\right\|_H^2 + \frac{1}{2}\langle A\theta^m, \theta^m\rangle + \frac{1}{2}\sum_{i=1}^m \langle A(\theta^i-\theta^{i-1}), \theta^i-\theta^{i-1}\rangle\\ &\le \frac{1}{2}\langle A\theta^0, \theta^0\rangle + \frac{1}{\rho}\|f_1\|^2_{L^2(0,m\tau;H)} + \frac{\lambda^2\tau}{\rho}\sum_{i=1}^m \left\|\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_H^2 + C_3\\ &+ C_3\|f_2\|^2_{H^1(0,T;V')} + \frac{\tau}{2}\sum_{i=1}^m \|\theta^i\|_V^2 + \frac{2}{\alpha}\|f_2\|^2_{L^\infty(0,T;V')} + \frac{\alpha}{4}\|\theta^m\|_V^2 + \frac{\alpha}{4}\|\theta^0\|_V^2,\end{aligned}$$

whence, recalling the regularity hypothesis (3.28) on the initial datum, the coercivity of A and relations (3.64–3.65), (3.67), we can derive other two a priori estimates, which we prefer to write at once in terms of the interpolating functions:

$$(3.77)\qquad \|\widehat{\theta}'_\tau\|_{L^2(0,T;H)} \le C,$$

$$(3.78)\qquad \|\theta_\tau\|_{L^\infty(0,T;V)} \le C.$$

Third a priori estimate. Write equation (3.38) for the index i; write it also for the index $i-1$ and take the difference; multiply it by $(\chi^i - \chi^{i-1})\tau^{-1}$ and sum for i from 2 to $m \le n$. Owing to (3.6), to the symmetry of M, to the monotonicity of operator $\partial_{V,V'}J$ and to the elementary relation (3.47), we easily infer:

$$\begin{aligned}(3.79)\ &\frac{\mu}{2}\left\|\frac{\chi^m-\chi^{m-1}}{\tau}\right\|_H^2 + \frac{\mu}{2}\sum_{i=2}^m \left\|\frac{\chi^i-\chi^{i-1}}{\tau} - \frac{\chi^{i-1}-\chi^{i-2}}{\tau}\right\|_H^2\\ &+ \tau\sum_{i=2}^m \left\langle B\frac{\chi^i-\chi^{i-1}}{\tau}, \frac{\chi^i-\chi^{i-1}}{\tau}\right\rangle + \frac{1}{\tau}\sum_{i=2}^m (\gamma(\chi^i)-\gamma(\chi^{i-1}), \chi^i-\chi^{i-1})\\ &\le \frac{\|M\|_{\mathcal{L}(H)}}{2}\left\|\frac{\chi^1-\chi^0}{\tau}\right\|_H^2 + \sum_{i=2}^m \left(L(\theta^i-\theta^{i-1}), \frac{\chi^i-\chi^{i-1}}{\tau}\right).\end{aligned}$$

Observe now that the last two terms on the left hand side can be split by using (3.8) and (3.11); by splitting also the mixed–unknowns term, it is immediate to get the relation

$$(3.80)\qquad \frac{\mu}{2}\left\|\frac{\chi^m-\chi^{m-1}}{\tau}\right\|_H^2 + \frac{\alpha\tau}{2}\sum_{i=2}^m \left\|\frac{\chi^i-\chi^{i-1}}{\tau}\right\|_V^2 \le C_4 + \frac{\|M\|_{\mathcal{L}(H)}}{2}\left\|\frac{\chi^1-\chi^0}{\tau}\right\|_H^2,$$

where C_4 is a positive constant only depending on the bounds (3.67) and (3.77). Now, in order to estimate the $(\chi^1 - \chi^0)\tau^{-1}$–term, let us write equation (3.38) for $i = 1$ and multiply it by $(\chi^1 - \chi^0)\tau^{-1}$. Owing to (3.6) and to the supplementary hypothesis (3.31–3.33) (whose notation is used below), it follows

$$\begin{aligned}(3.81)\quad \mu \left\| \frac{\chi^1 - \chi^0}{\tau} \right\|_H^2 + \frac{1}{\tau}\langle B\chi^1 - B\chi^0, \chi^1 - \chi^0\rangle + \frac{1}{\tau}\langle w^1 - w^0, \chi^1 - \chi^0\rangle \\ + \frac{1}{\tau}(\gamma(\chi^1) - \gamma(\chi^0), \chi^1 - \chi^0) = \left(L\theta^1, \frac{\chi^1 - \chi^0}{\tau}\right) - \frac{1}{\tau}(B\chi^0, \chi^1 - \chi^0) \\ - \frac{1}{\tau}(w^0, \chi^1 - \chi^0) - \frac{1}{\tau}(\gamma(\chi^0), \chi^1 - \chi^0).\end{aligned}$$

Assumptions (3.8), (3.10–3.11), estimate (3.64) and the monotonicity of $\partial_{V,V'}J$ ensure now the boundedness of the $(\chi^1 - \chi^0)\tau^{-1}$–term in (3.79), whence we derive our final "abstract" estimates:

$$(3.82)\qquad \|\widehat{\chi}'_\tau\|_{L^\infty(0,T;H)} \le C,$$

$$(3.83)\qquad \|\widehat{\chi}'_\tau\|_{L^2(0,T;V)} \le C.$$

3.1.3 Proof of the abstract results

Proof of Theorem 3.1.2. Existence. Referring to the weaker regularity hypotheses of Theorem 3.1.2, we now complete the proof of the existence of a solution to the abstract Problem (aP). First, observe that estimates (3.64–3.66) and (3.71) immediately entail the following convergences:

$$(3.84)\qquad \theta_\tau \to \theta \quad \text{in } L^\infty(0,T;H)\text{–weak}^* \cap L^2(0,T;V)\text{–weak},$$

$$(3.85)\qquad \chi_\tau \to \chi \quad \text{in } L^\infty(0,T;V)\text{–weak}^*,$$

$$(3.86)\qquad \widehat{\chi}_\tau \to \chi \quad \text{in } H^1(0,T;H)\text{-weak},$$

holding at least for suitable subsequences. From (3.66), (3.61) (with V and χ in place of H and θ) and (3.69), we derive also

$$(3.87)\qquad \widehat{\chi}_\tau \to \chi \quad \text{in } L^\infty(0,T;V)\text{–weak}^*,$$

whence, owing to the compactness result [82, Sec. 8, Cor. 4], we deduce also

$$(3.88)\qquad \widehat{\chi}_\tau \to \chi \quad \text{in } C^0([0,T];H)\text{–strong},$$

while we have not enough regularity to obtain a strong convergence for the θ.

Let us remark anyway that (3.64), (3.61) and (3.68) yield

$$(3.89)\qquad \widehat{\theta}_\tau \to \theta \quad \text{in } L^\infty(0,T;H)\text{–weak}^*;$$

moreover, thanks to the symmetry of P, it follows

$$(3.90)\qquad P\widehat{\theta}_\tau \to P\theta \quad \text{in } L^\infty(0,T;H)\text{–weak}^*.$$

Besides, equation (3.37) and the above convergences entail that

$$(P\widehat{\theta}_\tau)' = P\widehat{\theta}'_\tau = -\Lambda\widehat{\chi}'_\tau - A\theta_\tau + f_\tau \tag{3.91}$$

is bounded in $L^2(0,T;V')$. It easily follows that

$$(P\widehat{\theta}_\tau)' \to \mathcal{T} \quad \text{in } L^2(0,T;V')\text{–weak} \tag{3.92}$$

for some function $\mathcal{T}$; relation (3.90) permits to see that $\mathcal{T} = (P\theta)'$. So, passing to the limit in (3.91), we get back (3.21).

The above procedure allows to recover easily the regularity requests (3.17–3.19), save for the continuity of θ and $P\theta$ (at present, it is only possible to say that $\theta, P\theta \in L^\infty(0,T;H)$); the 1-dimensional continuous embedding $H^1 \subset C^0$, together with (3.86) and (3.40), yields anyway the Cauchy condition (3.25), and, similarly, from (3.92) and (3.40), we infer that $(P\theta)(0) = P\theta_0 \in H$.

Notice now that equation (3.21) with the above deduced Cauchy condition can be rewritten as

$$\begin{cases} (P\theta)' + A\theta = f - \Lambda\chi' \in L^2(0,T;V') \\ (P\theta)(0) = P\theta_0 \in H \end{cases} \tag{3.93}$$

and consequently we can apply [21, Remark 6.4, page 209], obtaining that it is actually $P^{1/2}\theta \in C^0([0,T];H)$, whence, since P in our simpler case is an isomorphism, it follows that both $P\theta$ and θ lie in $C^0([0,T];H)$. Condition (3.24) makes now sense and can be recovered from (3.93).

Relations (3.20) and (3.22–3.23), as well as (3.26), anyway, cannot be obtained yet, since the nonlinearities in (3.22) require some

Strong convergence. Thanks to assumption (3.10) and to estimate (3.66), we obtain that, at least for subsequences,

$$\gamma(\chi_\tau) \to g \quad \text{in } L^\infty(0,T;H)\text{–weak}^*, \tag{3.94}$$

for some function $g \in L^\infty(0,T;H)$, whence, looking back to equation (3.38), we derive also that, for some function w,

$$w_\tau \to w \quad \text{in } L^2(0,T;V')\text{–weak}, \tag{3.95}$$

which guarantees (3.20).

Equation (3.38), with the introduction of the interpolating functions and the subtraction of the same term $B\chi$ from both hands sides, can now be rewritten as:

$$B(\chi_\tau - \chi) = -w_\tau + L\theta_\tau - B\chi - M\widehat{\chi}'_\tau - \gamma(\chi_\tau), \tag{3.96}$$

which is a relation in $L^2(0,T;V')$. Test it with $\chi_\tau - \chi$ and integrate in time between 0 and T. Recalling (3.8), and using the definition of subdifferential in the form of (3.13), it follows:

$$\begin{aligned} \alpha \int_0^T \|\chi_\tau(s) - \chi(s)\|_V^2 \, ds &\le \int_0^T (J(\chi(s)) - J(\chi_\tau(s)))\, ds \\ &+ \lambda_0 \int_0^T \|\chi_\tau(s) - \chi(s)\|_H^2 \, ds + \int_0^T (L\theta_\tau(s), \chi_\tau(s) - \chi(s))\, ds \\ &- \int_0^T \langle B\chi(s), \chi_\tau(s) - \chi(s)\rangle\, ds - \int_0^T (M\widehat{\chi}'_\tau(s) + \gamma(\chi_\tau), \chi_\tau(s) - \chi(s))\, ds. \end{aligned} \tag{3.97}$$

For $v \in L^2(0,T;V)$, we can introduce the $]0,T[$–extension of J, as

$$J^T(v) := \begin{cases} \int_0^T J(v(s))\,ds & \text{if } J(v) \in L^1(0,T) \\ +\infty & \text{otherwise} \end{cases} \tag{3.98}$$

On account of Theorem 1.2.23, J^T is a convex, lower semicontinuous and proper functional on $L^2(0,T;V)$, whence, thanks also to (3.85),

$$\limsup_{\tau\to 0}\,(J^T(\chi) - J^T(\chi_\tau)) \le 0. \tag{3.99}$$

Recalling (3.84–3.86) and (3.88), looking back at equation (3.97) and observing that (3.62) (with χ in place of θ), (3.88) and (3.67) also entail

$$\|\chi_\tau - \chi\|_{L^2(0,T;H)} \le \|\chi_\tau - \widehat{\chi}_\tau\|_{L^2(0,T;H)} + \|\widehat{\chi}_\tau - \chi\|_{L^2(0,T;H)} \to 0, \tag{3.100}$$

we can conclude that

$$\limsup_{\tau\to 0} \int_0^T \|\chi_\tau(s) - \chi(s)\|_V^2\,ds \le 0, \tag{3.101}$$

that is

$$\chi_\tau \to \chi \quad \text{in } L^2(0,T;V)\text{–strong.} \tag{3.102}$$

End of proof. It is easy to see that the last convergence and the demicontinuity of γ entail:

$$\gamma(\chi_\tau) \to \gamma(\chi) \quad \text{in } L^2(0,T;H)\text{–weak} \tag{3.103}$$

at least for subsequences. It follows that $g = \gamma(\chi)$ (compare with (3.94)) and that equation (3.22) holds. Moreover, (3.95), (3.102) and Prop. 1.3.8 imply the validity of the constitutive relation (3.23). Finally, (3.26) is an immediate consequence of (3.102) and estimate (3.70), if we exploit the semicontinuity of J. This completes the proof of the existence part of Theorem 3.1.2. ∎

Proof of Theorem 3.1.3. Estimates (3.77–3.78) easily entail, after passing to the limit, that

$$\theta \in H^1(0,T;H) \cap L^\infty(0,T;V),$$

whence equation (3.21) can be rewritten in the form (3.30). Moreover, proceeding by comparison, we get that $A\theta \in L^2(0,T;H) + H^1(0,T;V')$, whence assumption (3.7) and Theorem 1.1.16 allow us to conclude that $\theta \in C^0([0,T];V)$, as desired. ∎

Proof of Theorem 3.1.4. Recalling our third a priori estimate, we observe that $\chi \in W^{1,\infty}(0,T;H)$ thanks to (3.82) and (3.86) and that $\chi \in H^1(0,T;V)$ thanks to (3.83) and (3.87), while (3.35) follows now by comparison from equation (3.22). ∎

Remark 3.1.6. *Under the regularity hypotheses of Theorem 3.1.3, subtracting equation (3.30) from (3.37) (written in terms of the interpolating functions) and testing the result by $\theta_\tau - \theta$, it could be proved with standard techniques that*

$$\theta_\tau \to \theta \quad \text{in } L^2(0,T;V)\text{–strong.} \tag{3.104}$$

However, this convergence will not be needed in the following and we omit the (simple) details of its proof.

Uniqueness. It concludes the proof of Theorem 3.1.2. Naturally, it will follow also the uniqueness of the solution of (aP) under the stronger regularity hypotheses of Theorems 3.1.3–3.1.4.

Let us suppose to have two solutions of (aP), denoted with $(\hat\theta, \hat\chi, \hat w)$ and $(\check\theta, \check\chi, \check w)$ respectively. Substitute them successively into equations (3.21–3.22) and take the difference; setting $\theta := \hat\theta - \check\theta$, $\chi := \hat\chi - \check\chi$, $w := \hat w - \check w$, we obtain:

$$(P\theta)' + \Lambda\chi' + A\theta = 0 \tag{3.105}$$

$$M\chi' + B\chi + w + \gamma(\hat\chi) - \gamma(\check\chi) = L\theta. \tag{3.106}$$

Define now $\omega := P\theta + \Lambda\chi$ and $\mathcal{R} := A + \lambda_0 I$ as an operator from V to V'. I denotes here the embedding function of V into H. Observe that, owing to assumption (3.7), it follows

$$\alpha\|v\|_V^2 \le \langle \mathcal{R}v, v\rangle \le \|A\|_{\mathcal{L}(V,V')}\|v\|_V^2 + \lambda_0\|v\|_H^2 \tag{3.107}$$

for any $v \in V$, so that, if we set $a(v,z) := \langle \mathcal{R}v, z\rangle$ for $v, z \in V$, $a(\ ,\ )$ turns out to be a scalar product on V equivalent to the original one; moreover $\mathcal{R}$ is the Riesz operator associated to $a(\ ,\ )$. On account of the machinery of Subsec. 1.1.3, this leads to introduce also the dual scalar product on V' as

$$a_*(v_*, z_*) := a(\mathcal{R}^{-1}v_*, \mathcal{R}^{-1}z_*) = \langle v_*, \mathcal{R}^{-1}z_*\rangle = \langle z_*, \mathcal{R}^{-1}v_*\rangle, \tag{3.108}$$

for $v_*, z_* \in V'$.

Equation (3.105), with the new notation, assumes the form:

$$\omega' + \mathcal{R}\theta = \lambda_0\theta. \tag{3.109}$$

Now, multiply it by $\mathcal{R}^{-1}\omega \in H^1(0,T;V)$ and integrate the result between 0 and $t \in]0,T]$; managing separately its three terms, we first get

$$\begin{aligned}\int_0^t \langle \omega', \mathcal{R}^{-1}\omega\rangle\, ds &= \int_0^t a_*(\omega', \omega)\, ds = \frac{1}{2}a_*(\omega(t), \omega(t)) \\ &\ge \alpha\|\mathcal{R}^{-1}\omega(t)\|_V^2 \ge \frac{\alpha}{\|\mathcal{R}\|^2_{\mathcal{L}(V,V')}}\|\omega(t)\|^2_{V'}\end{aligned} \tag{3.110}$$

(notice that $\omega \in H^1(0,T;V')$, whence $a_*(\omega,\omega) \in W^{1,1}(0,T)$ and the use of the fundamental theorem of calculus is then justified). The second term of (3.109), after multiplication and integration, reads

$$\begin{aligned}\int_0^t \langle \mathcal{R}\theta, \mathcal{R}^{-1}\omega\rangle\, ds &= \int_0^t \langle \omega, \theta\rangle\, ds = \int_0^t (P\theta, \theta)\, ds + \int_0^t (\Lambda\chi, \theta)\, ds \\ &\ge \frac{\rho}{2}\|\theta\|^2_{L^2(0,t;H)} - \frac{\lambda^2}{2\rho}\|\chi\|^2_{L^2(0,t;H)},\end{aligned} \tag{3.111}$$

since it is also $\omega \in C^0([0,T];H)$. Finally, the right hand side term in (3.109), for arbitrary $\sigma > 0$, gives

$$\int_0^t (\lambda_0\theta, \mathcal{R}^{-1}\omega)\,ds \le \sigma\|\theta\|^2_{L^2(0,t;V')} + \frac{\lambda_0^2\|\mathcal{R}^{-1}\|^2_{\mathcal{L}(V',V)}}{4\sigma}\|\omega\|^2_{L^2(0,t;V')}. \tag{3.112}$$

Multiply now (3.106) by χ and integrate in time as before. Owing to (3.11) and to the monotonicity of $\partial_{V,V'}J$, it follows

$$\frac{\mu}{2}\|\chi(t)\|^2_H + \frac{\alpha}{2}\|\chi\|^2_{L^2(0,t;V)} \le \left(\lambda_0 + C_0 + \frac{1}{2}\right)\|\chi\|^2_{L^2(0,t;H)} + \frac{\ell^2}{2}\|\theta\|^2_{L^2(0,t;H)}. \tag{3.113}$$

Summing together the equation resulting from (3.110–3.112) and relation (3.113) times $m > 0$ (to be chosen), owing also to the continuity of the transpose embedding $H \subset V'$, it is immediate to get, for some positive constant C_5 depending only on $\alpha, \mathcal{R}, M, \rho, \lambda_0, C_0, \ell$,

$$\begin{aligned}\|\omega(t)\|^2_{V'} &+ \|\theta\|^2_{L^2(0,t;H)} + m\,\|\chi(t)\|^2_H + m\,\|\chi\|^2_{L^2(0,t;V)}\\ &\le C_5\,\|\chi\|^2_{L^2(0,t;H)} + \sigma C_5\,\|\theta\|^2_{L^2(0,t;H)}\\ &\quad + \sigma^{-1}C_5\,\|\omega\|^2_{L^2(0,t;V')} + mC_5\,\|\chi\|^2_{L^2(0,t;H)} + mC_5\,\|\theta\|^2_{L^2(0,t;H)},\end{aligned} \tag{3.114}$$

so that, choosing m and σ sufficiently small (for instance, $m = \sigma = (4C_5)^{-1}$), through an application of the Gronwall lemma, we easily derive $\omega = \chi = \theta = 0$, whence $\gamma(\hat{\chi}) = \gamma(\check{\chi})$. Substituting into equation (3.106), we obtain also $w = 0$, which completes the proof of Theorem 3.1.2. ∎

3.2 Applications to the phase-field system

In this section, we apply the previously developed abstract machinery to solve the "physical" phase field systems in two different situations: as a first example, we retrieve the existence, uniqueness and regularity results of Subsec. 2.2.2 concerning the standard model of heat diffusion within a substance with change of phase (in the case of a linear latent heat λ) from the abstract theorems of Subsec. 3.1.1. Also in this simple case, anyway, the procedure is not completely trivial; indeed, some difficulties arise from the different functional setting of the abstract theorems, essentially working on the duality (V, V'), while the proof of Theorem 2.2.3 was based on a "concrete" approach in the space $L^2(\Omega)$, providing the equations of the problem directly in the strong sense. Hence, in this framework, we have obtained a weaker solution and consequently it is necessary to exploit some machinery (we shall mainly refer to the work [10]) in order to derive a physical interpretation.

We chose to present in some detail also this "alternative" approach, which is indeed more complicated than the direct one, for two main reasons: first, it serves as a comparison with the procedure of Subsecs. 2.2.2–2.2.3, which gives concordant results; second, it prepares the study of the transmission problem, which is based on a similar, but more delicate argument.

3.2.1 An alternative approach to the phase-field system

In this subsection, we discuss the existence of solutions to system (2.67–2.71) by means of the abstract approach. Since the related results have already be obtained by a direct analysis, we just show here some details of the abstract method, and in particular, we limit ourselves to present the regularity framework provided by Theorem 3.1.3. Hence, let us first recall that the Cauchy conditions (3.24–3.25) are now required with the data θ_0 and χ_0 lying in $H^1(\Omega)$, which we choose as the abstract space V (while we set $H := L^2(\Omega)$); moreover, we suppose that

$$\chi_0(x) \in D(j) \quad \text{for a.e. } x \in \Omega, \quad \text{with } j(\chi_0) \in L^1(\Omega), \tag{3.115}$$

and we add the usual homogeneous Neumann boundary conditions $\partial_{\mathbf{n}}\theta = \partial_{\mathbf{n}}\chi = 0$ on $\partial\Omega\times]0,T[$. This corresponds precisely to the framework of Theorem 2.2.3, with the further regularity for θ provided by hypothesis (2.102) of Theorem 2.2.4. So, what we expect to obtain in this setting is a solution (θ, χ, ξ) enjoying precisely the properties (2.65–2.66) and (2.103).

So, let us address the notation of the abstract theorems (some ambiguity will arise as far as the constants μ, λ are concerned; however, the procedure should not be misleading). With this aim, also accounting for a linear latent heat $\lambda(r) = \lambda r$ with $\lambda > 0$, given $h \in H$ and $v, z \in V$, we can set:

$$(Ph)(x) := h(x), \quad (Mh)(x) := \mu h(x) \tag{3.116}$$

$$(\Lambda h)(x) = (Lh)(x) := \lambda h(x) \tag{3.117}$$

$$\langle Av, z\rangle := \int_\Omega \nabla v(x) \cdot \nabla z(x)\, dx \tag{3.118}$$

$$\langle Bv, z\rangle := \int_\Omega \nu(x)\nabla v(x) \cdot \nabla z(x)\, dx, \tag{3.119}$$

so that, as it is immediate to verify, the above introduced functionals and operators fit assumptions (3.2–3.8), while (3.9–3.11) are trivially satisfied by virtue of (2.60).

Moreover, for $v \in H$, we define,

$$J_H(v) := \begin{cases} \displaystyle\int_\Omega j(v(x))\, dx & \text{if } j(v) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.120}$$

We recall (see Theorem 1.2.22 (b)) that J_H is a convex, lower semicontinuous and proper function on H; we define the functional J of (3.12) as the restriction of J_H to V; since the topology of V is finer than H's one, it is immediate to verify that J is still l.s.c.; furthermore, it evidently satisfies (3.14).

Now, almost all the assertions of Theorem 2.2.3 can be easily deduced from Theorems 3.1.2–3.1.3; in particular, the derivation of equation (2.67) and of the related Neumann condition for θ from the abstract formulation, although not completely immediate, can be performed with rather standard techniques. However, the constitutive relation (2.69) and the regularity property (2.66), which are clearly linked together, cannot be derived yet; moreover, at present, we are only able to obtain L^∞ (instead of C^0) in (2.65). In fact, we can conclude from Theorem 3.1.2 that $w \in \partial_{V,V'} J(\chi)$ and it is not obvious (nor true in the general case) that this implies (2.66) and (2.69). We now briefly outline a possible way to recover these relations: first of all, replace in the formulation of the problem the operator α with its Yosida approximate α^ε and modify the definitions of J_H and J by substituting therein α with α^ε (call J_H^ε and J^ε, respectively, the new functionals).

Applying now Theorem 3.1.3 to the regularized problem, we obtain a solution, that we denote as $(\theta^\varepsilon, \chi^\varepsilon, \xi^\varepsilon)$. Moreover, the analogous of relation (3.23) reads here

$$\xi^\varepsilon \in \partial_{V,V'} J^\varepsilon(\chi^\varepsilon) \qquad \text{a.e. in } [0,T], \tag{3.121}$$

and, due to the Lipschitz continuity of α^ε and to [10, Prop. 2.9], we deduce that $\xi^\varepsilon \in L^2(0,T;H^1(\Omega))$ and $\xi^\varepsilon \in \partial_H J_H^\varepsilon(\chi^\varepsilon)$, which means $\xi^\varepsilon(x,t) \in \alpha^\varepsilon(\chi^\varepsilon(x,t))$ for a.e. $x \in \Omega, t \in [0,T]$. Here $\partial_H J_H^\varepsilon$ denotes the subdifferential of J_H^ε as a maximal monotone graph in $H \times H$.

It is now possible to multiply the Yosida-regularization of (2.68) by $\xi^\varepsilon = \alpha^\varepsilon(\chi^\varepsilon)$; thus, exploiting monotonicity and reasoning by comparison, it is a standard matter to infer that, for some function ξ, it is (for subsequences)

$$\xi^\varepsilon \to \xi \qquad \text{in } L^2(\Omega\times]0,T[)\text{–weak}, \tag{3.122}$$

$$\operatorname{div}(\nu\nabla\chi^\varepsilon) \to \operatorname{div}(\nu\nabla\chi) \qquad \text{in } L^2(\Omega\times]0,T[)\text{–weak}. \tag{3.123}$$

Notice indeed that the constant C in our abstract a priori estimates depends only on the data, which are not modified by the introduction of the Yosida approximation, with the exception of $J(\chi_0)$; anyway, it is $J^\varepsilon(\chi_0) \le J(\chi_0)$ (see Prop. 1.2.16 (b)), so that this dependence does not give troubles.

Now, to prove the convergence of $(\theta^\varepsilon, \chi^\varepsilon)$ to some (θ, χ) solving (2.67–2.68), just adapt the τ–limit procedure already performed for the abstract problem. Also, (2.69) easily follows from the monotonicity argument of Prop. 1.3.8, and, finally, the C^0 in (2.65) is again a consequence of the result of [7] reported as Theorem 1.1.16.

3.2.2 Formulation of the transmission problem

In this subsection, we wish to present in some detail the mathematical statement and outline the technique of resolution of the transmission problem for the phase-field model; the proofs of the related results will be carried out up to the end of this chapter.

We start by describing the physical situation: referring to the previous subsections for most of the notation, we suppose that the domain $\Omega \subset \mathbb{R}^N$ is subdivided by a smooth interface Γ into two subregions Ω_1, Ω_2 of Lipschitz regularity (naturally, we cannot ask more), where are assumed to lie two different and possibly inhomogeneous substances, whose thermal diffusion properties are still described by equations similar to (2.67–2.69). Moreover, transmission conditions for both variables θ and χ are assumed on the common boundary Γ.

The main difficulty of the problem, which is also the main difference with respect to the previous case, lies now in the discontinuity of data; in fact, in this setting all coefficients do depend on the domain, and, in particular, the most relevant trouble is due to the presence of two thermodynamical potentials, corresponding to different graphs α_1 and α_2 in the phase-field equations relative to the two substances. Indeed, in this setting, we have to work with great care on the constitutive relation corresponding to (2.69); we see that, in order that the problem be solvable, some further conditions on the graphs α_1, α_2 need to be assumed. In particular, we are able to treat two different cases: compatibility conditions (i.e., α_1, α_2 must be not too different) and growth conditions (their inverse graphs must be coercive); we emphasize that only in the second case we address the problem by the abstract method; in the compatibility one, an approach similar to that of Subsec. 2.2.2 seems preferable. Hence, in that case, we do not repeat the whole argument and we limit ourselves to develop in detail the only different (and troublesome) estimate, referring to that subsection for more details.

To start with the mathematical formulation of the problem, we need to introduce a notation which will turn out to be very useful in this setting and will be implicitely kept throughout all the remainder of the dissertation. So, for any function $v \in L^p(\Omega)$ $(1 \leq p \leq +\infty)$, we set $v_i := v_{|\Omega_i}$; conversely, given $v_1 \in L^p(\Omega_1)$ and $v_2 \in L^p(\Omega_2)$, we denote by $v \in L^p(\Omega)$ the function coinciding with v_i in Ω_i, $i = 1, 2$. Naturally, this construction cannot be arbitrarily extended to spaces different from the L^p; anyway, given $v \in H^1(\Omega)$, it is mathematically justified to write $v_i \in H^1(\Omega_i)$ to denote its restrictions to the sets Ω_i, and, given $w_i \in H^1(\Omega_i)'$, it is possible to "glue" them by setting $\langle w, v\rangle := \langle w_1, v_1\rangle + \langle w_2, v_2\rangle$ for all $v \in H^1(\Omega)$. Clearly, we obtain that $w \in H^1(\Omega)'$

Let us now state the transmission system, which is (compare with (2.67–2.69)):

$$\rho_i \partial_t \theta_i + \lambda_i \partial_t \chi_i - \operatorname{div}(\kappa_i \nabla \theta_i) = \phi_i \quad \text{a.e. in } \Omega_i \times]0, T[, \tag{3.124}$$

$$\mu_i \partial_t \chi_i - \operatorname{div}(\nu_i \nabla \chi_i) + w_i + \gamma_i(\chi_i) = \ell_i \theta_i \quad \text{a.e. in } \Omega_i \times]0, T[, \tag{3.125}$$

$$w_i \in \alpha_i(\chi_i), \quad \text{a.e. in } \Omega_i \times]0, T[\tag{3.126}$$

(here, and in the rest of the dissertation, the index i will be always supposed to assume the values $1, 2$). As for the hypotheses on the data, we point out that they are very similar to the corresponding assumptions for the one-domain problem (in some case, we could just substitute Ω with Ω_i or add the subscript i); however, in that setting, some coefficients were supposed equal to one; here we manage the system in full generality, instead. Hence, we require $\rho_i, \lambda_i, \kappa_i, \mu_i, \nu_i$, and ℓ_i be strictly positive constants, $\gamma_i : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous functions, ϕ_i source terms satisfying $\phi_i \in L^2(0, T; L^2(\Omega_i))$, and α_i be maximal monotone graphs with $0 \in \alpha_i(0)$. As usual, we take j_i as convex primitives of α_i.

Moreover, we choose, as before, $V = H^1(\Omega)$ and $H = L^2(\Omega)$, and, given $\theta_0 \in H$ and $\chi_0 \in V$ verifying the relation

$$\chi_{0,i}(x) \in D(j_i) \quad \text{for a.e. } x \in \Omega_i, \quad \text{with } j_i(\chi_{0,i}) \in L^1(\Omega) \tag{3.127}$$

(which is naturally the analogous of (3.16) and (3.115)), we can require the Cauchy conditions in the habitual form (3.24–3.25).

To obtain a well-posed problem, we add homogeneous Neumann boundary conditions for the χ (i.e. $\partial_{\mathbf{n}}\chi = 0$ on $\partial\Omega \times]0, T[$) and third-type ones for the temperature: taken $p \in L^\infty(\Sigma)$, and

$$g \in L^2(0, T; H^{-1/2}(\partial\Omega)), \tag{3.128}$$

we require

$$(\kappa\nabla\theta)\cdot\mathbf{n}+p\theta=g \quad \text{on } \partial\Omega\times]0,T[; \tag{3.129}$$

in a variational setting, this leads to introduce a generalized source datum as follows: for $v\in V$, $t\in]0,T[$, we define

$$\langle f(t),v\rangle := \int_\Omega \phi(t)v\,dx+\langle g(t),v\rangle, \tag{3.130}$$

so that the regularity request (3.4) is clearly fulfilled.

Moreover, we state the compatibility and transmission relations at the interface:

$$\theta_1=\theta_2 \quad \text{and} \quad \chi_1=\chi_2 \quad \text{on } \Gamma\times]0,T[\tag{3.131}$$

$$(\kappa_1\nabla\theta_1)\cdot\mathbf{n}=(\kappa_2\nabla\theta_2)\cdot\mathbf{n} \quad \text{on } \Gamma\times]0,T[\tag{3.132}$$

$$(\nu_1\nabla\chi_1)\cdot\mathbf{n}=(\nu_2\nabla\chi_2)\cdot\mathbf{n} \quad \text{on } \Gamma\times]0,T[, \tag{3.133}$$

where $\mathbf{n}$ denotes the normal unit vector to Γ pointing (e.g.) outwards Ω_1. As we have already remarked in the Introduction, while assumption (3.132) is standard, accounting for the diffusion of heat across Γ, the corresponding hypothesis (3.133) concerning the order parameter χ can be physically justified by observing that the diffusion coefficient ν_i is proportional to the *interfacial energy* of the substance in Ω_i; with relation (3.133), we mean that no source of interfacial energy is present on the common boundary.

We now discuss the first case when existence and uniqueness of solutions to the transmission system are assured, i.e. that of *growth conditions:*

(GC) For some $m>0$, we have that

$$\liminf_{|r|\to\infty}\frac{j_i^*(r)}{|r|^{2_*}}\geq 2m>0, \qquad \text{for } i=1,2, \tag{3.134}$$

where j_i^* is the *convex conjugate* function to j_i (cf. Subsection 1.2.1).

Here (and this notation will be kept for the rest of the thesis), supposing $N\geq 3$ (the cases $N=1,2$ are clearly simpler and require only minor adjustments), we have denoted as $2_*:=2N/(N+2)$ and $2^*:=2N/(N-2)$ the Sobolev embedding exponents, i.e. the best ones verifying $V\subset L^{2^*}(\Omega)$ and $L^{2_*}(\Omega)\subset V'$ (recall Theorem 1.1.2). We point out that condition (3.134) entails in particular that $D(j_i)=D(\alpha_i)=\mathbb{R}$ for $i=1,2$ (see [14, Remarque 2.3, page 43]) and it is slightly stronger than the corresponding assumption (2.29) of the paper [10].

Moreover, comparing (3.134) with hypothesis (2.45) related to the Stefan problem (ST), we observe that it is exactly the opposite: a lower bound is imposed to the growth of j_i^* and not to that of j_i; here, anyway, the role played by the α_i's is different, since they do not appear under the Laplace operator.

As we have already mentioned before, the graphs α_i satisfy assumption (3.134) when the constitutive relation (3.126) is governed by a double-well type potential, but not in the case of the relaxed Stefan problem which cannot be managed by these means (actually, the choice $\alpha_i(x)=c_2x^3$, as stated in the introduction, requires $N<4$; in a greater spatial dimension, we can take $\alpha_i(x) = c_2|x|^k x$ for some small $k > 0$, which seems easily justifiable also from the physical point of view). In this case, the only way to treat the problem seems that of assuming the α_i's be not too different, i.e. the following *compatibility conditions:*

(CC) We have that

$$D(\alpha_1) = D(\alpha_2) =: D; \tag{3.135}$$

moreover,

$$\begin{cases} |y_1| \le C_{\alpha,1}(1 + |y_2|) \\ |y_2| \le C_{\alpha,2}(1 + |y_1|) \end{cases} \tag{3.136}$$

for every $x \in \operatorname{int} D$ and for every $y_1 \in \alpha_1(x)$, $y_2 \in \alpha_2(x)$; $C_{\alpha,1}, C_{\alpha,2} \ge 1$ being constants independent of x, y_1, y_2.

Under either of the conditions (GC), (CC), we need to introduce the elliptic operator $(v, w \in V)$

$$A : V \to V', \qquad \langle Av, w\rangle := \int_\Omega \kappa \nabla v \cdot \nabla w \, dx + \int_\Gamma p v w \, d\mathcal{H}^{N-1}, \tag{3.137}$$

which allows to state in the mathematically precise form the existence and uniqueness result for the transmission problem, collecting all the regularity instances of the "abstract" Theorems 3.1.2–3.1.4:

Theorem 3.2.1. *Under the above listed regularity assumptions and either of conditions* (GC) *or* (CC), *there exists a unique triplet of functions* (θ, χ, w) *of regularity*

$$\theta \in L^2(0,T; H^1(\Omega)) \cap C^0([0,T]; L^2(\Omega)) \tag{3.138}$$

$$\rho\theta \in H^1(0,T; H^1(\Omega)') \cap C^0([0,T]; L^2(\Omega)) \tag{3.139}$$

$$\chi \in L^\infty(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)), \tag{3.140}$$

which satisfy equations (3.125–3.126) together with

$$\partial_t(\rho\theta) + \lambda\partial_t\chi + A\theta = f \quad \text{in } H^1(\Omega)' \quad \text{for a.e. } t \in [0,T] \tag{3.141}$$

and also fulfill the Cauchy conditions (3.24–3.25) and the Neumann boundary ones $\partial_{\mathbf{n}}\chi = 0$ *and (3.133) (in the appropriate trace spaces).*

Moreover, in the (GC) *case, we have*

$$w \in L^{2\cdot 2_*}(0,T; L^{2_*}(\Omega)), \tag{3.142}$$

while, when (CC) *are assumed instead, the regularity of* w *is given by*

$$w \in L^2(0,T; H). \tag{3.143}$$

If, in addition, it is $g \in H^1(0,T; H^{-1/2}(\partial\Omega))$ *and* $\theta_0 \in H^1(\Omega)$, *then the following ulterior regularity is fulfilled*

$$\theta \in C^0([0,T]; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)), \tag{3.144}$$

whence (3.141) holds in its more usual form (3.124) and we also get back the remaining boundary conditions for θ (3.129) and (3.132) (in sense of traces).

Finally, if $\nu\nabla\chi_0 \in H(\mathrm{div},\Omega)$ (that is, if $\mathrm{div}(\nu\nabla\chi_0) \in L^2(\Omega)$) and there exist $w_{0,i} \in L^2(\Omega_i)$ such that

$$\chi_{0,i} \in D(\alpha_i) \quad \textit{and} \quad w_{0,i} \in \alpha_i(\chi_{0,i}) \quad \textit{for a.e. } x \in \Omega_i, \tag{3.145}$$

then we have also

$$\chi \in H^1(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)), \tag{3.146}$$

$$w \in L^\infty(0,T;L^{2_*}(\Omega)). \ \blacksquare \tag{3.147}$$

The proof of this result is postponed to the final subsection of this chapter, while we now develop some further theoretical background.

3.2.3 Further monotone operators techniques

Our aim is now to extend the construction of the paper [10] to our setting concerning the transmission problems, in case that the growth conditions (GC) are imposed. Since we refer to that work again in the following, we try and use, when possible, a similar terminology.

First of all, here and in the sequel, we assume to be $1 < p < \infty$ and $q = p/(p-1)$ its conjugate exponent; moreover, the notation $(\cdot,\cdot)$ will now be used indifferently to denote the scalar product of H or the duality pairing between L^p and L^q.

As a particular case of the construction of convex integrals performed in Subsec. 1.2.3, given $v \in L^p(\Omega)$, we can set

$$J_p(v) := \begin{cases} \displaystyle\int_{\Omega_1} j_1(v_1(x))\,dx + \int_{\Omega_2} j_2(v_2(x))\,dx & \text{if } j_i(v_i) \in L^1(\Omega_i) \quad \text{for } i = 1,2, \\ +\infty & \text{otherwise} \end{cases} \tag{3.148}$$

and we also define J_V as the restriction of $J_H := J_2$ to V. Since j_1, j_2 are convex, J_p is a convex function on $L^p(\Omega)$ and consequently J_V is convex on V. Moreover, J_p and J_V are proper functions since, for instance, $J_V(0) = J_p(0) = 0$ and they are l.s.c. by the results of Subsec. 1.2.3. In analogy with [10], we also define, for $v_i \in L^p(\Omega_i)$,

$$J_{i,p}(v_i) := \begin{cases} \displaystyle\int_{\Omega_i} j_i(v_i(x))\,dx & \text{if } j_i(v_i) \in L^1(\Omega_i) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.149}$$

which are as well convex, lower semicontinuous and proper functions on $L^p(\Omega_i)$. In the course of this subsection, $\partial_{p,q}J_p$ will denote the subdifferential of J_p as a graph in $L^p(\Omega) \times L^q(\Omega)$ and $\partial_{p,q}J_{i,p}$ will be the subdifferential of $J_{i,p}$ in $L^p(\Omega_i) \times L^q(\Omega_i)$; in particular, for $p = q = 2$, we write, accordingly with [10], $\partial_H J_H$ and $\partial_H J_{i,H}$ instead of $\partial_{2,2}J_2$ and $\partial_{2,2}J_{i,2}$. We also recall that, for any $u_i \in L^q(\Omega_i)$, $v_i \in L^p(\Omega_i)$,

$$\begin{aligned} &v_i \in D(\partial_{p,q}J_{i,p}) \quad \text{and} \quad u_i \in \partial_{p,q}J_{i,p}(v_i) \iff \\ &\qquad v_i(x) \in D(\alpha_i) \quad \text{and} \quad u_i(x) \in \alpha_i(v_i(x)) \quad \text{a.e. in } \Omega_i; \end{aligned} \tag{3.150}$$

moreover, Theorem 1.2.22 entails that

$$J^*_{i,p}(v_i) = \begin{cases} \displaystyle\int_{\Omega_i} j^*_i(v_i(x))\,dx & \text{if } j^*_i(v_i) \in L^1(\Omega_i) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.151}$$

for $v_i \in L^q(\Omega_i)$. We now extend this characterization of convex conjugate functions and of subdifferentials to the transmission case. The following two properties are easy consequences of Theorem 1.2.22, as well. However, we report them as propositions, since they will result useful in a while.

Proposition 3.2.2. *For $v \in L^q(\Omega)$, we have that:*

$$J^*_p(v) := \begin{cases} \displaystyle\sum_{i=1}^{2} \int_{\Omega_i} j^*_i(v_i(x))\,dx = \sum_{i=1}^{2} J^*_{i,p}(v_i) & \textit{if } j^*_i(v_i) \in L^1(\Omega_i) \\ +\infty & \textit{otherwise.} \end{cases} \tag{3.152}$$ ∎

Proposition 3.2.3. *Let $v \in L^p(\Omega)$, $u \in L^q(\Omega)$. Then $v \in D(\partial_{p,q}J_p)$ and $u \in \partial_{p,q}J_p(v)$ if and only if $v_i \in D(\partial_{p,q}J_{i,p})$ and $u_i \in \partial_{p,q}J_{i,p}(v_i)$.* ∎

Our purpose is now that of passing from the (V, V')-setting, where we have proved the results of Subection 3.1.1, to the H-one, which is more suitable for the physical interpretation of these results, and in particular is required in order to recover the constitutive relation (3.126). The following theorem, which is the cornerstone of this procedure, provides an extension of [10, Lemma 2.4] (and of the procedure we have exploited to recover the results of Subsec. 2.2.2 from the abstract approach).

Theorem 3.2.4. *Let $2_* \le p \le 2^*$, q the conjugate exponent to p (i.e. $p^{-1} + q^{-1} = 1$), $v \in V$, $w \in L^q(\Omega)$. Then, $w \in \partial_{V,V'}J_V(v)$ if and only if $w \in \partial_{p,q}J_p(v)$.*

Proof. First of all, it is immediate to see that $\partial_{p,q}J_p(v) \subset \partial_{V,V'}J_V(v)$ (just apply the definition; observe that here we exploit that $p \le 2^*$). To show the converse inclusion, take $z \in L^p(\Omega)$ and $w \in \partial_{V,V'}J_V(v)$ and consider the family of singular perturbation elliptic problems in Ω_i:

$$\begin{cases} z^n_i \in H^1(\Omega_i) \\ \displaystyle\int_{\Omega_i} z^n_i k_i\,dx + \frac{1}{n}\int_{\Omega_i} \nabla z^n_i \cdot \nabla k_i\,dx = \int_{\Omega_i} z_i k_i\,dx \qquad \text{for all } k_i \in H^1(\Omega_i). \end{cases} \tag{3.153}$$

Indeed, since $z_i \in L^p(\Omega_i) \subset V'$ (and here it is necessary that $p \ge 2_*$) we have that $z^n_i \in H^1(\Omega_i)$ and $z^n_i \to z_i$ in $H^1(\Omega_i)'$–strong [26, Appendix]. Moreover, we claim that

$$z^n_i \to z_i \quad \text{in } L^p\text{–strong.} \tag{3.154}$$

To prove this, take $k_i = |z^n_i|^{p-1}\mathrm{sgn}(z^n_i)$ in the equation in (3.153); thanks to the Young inequality, it is easy to obtain the estimate corresponding to $z^n_i \to z_i$ in $L^p(\Omega_i)$–weak, while the strong convergence follows from the uniform convexity of L^p and [9, Prop. 1.4, page 14].

We now choose a nondecreasing cutoff function $h \in \mathcal{D}(\mathbb{R})$ such that $0 \le h \le 1$, $h(y) = 0$ for $y \le 1$ and $h(y) = 1$ for $y \ge 2$ and we set

$$h^n(x) : \Omega \to \mathbb{R}, \qquad h^n(x) := h(nd(x,\Gamma)), \tag{3.155}$$

where $d(x,\Gamma)$ is the Euclidean distance function from Γ.

As before, we denote with z^n the L^2-function whose restriction to Ω_i coincides with z_i^n. It is clear that in general $z^n \notin H^1(\Omega)$; however, since $d(x,\Gamma)$ is a Lipschitz continuous function, $h^n z^n$ is a sequence in $H^1(\Omega)$. We now prove that

$$h^n z^n \to z \quad \text{ in } L^p(\Omega)\text{–weak}. \tag{3.156}$$

Indeed, thanks to (3.154), it is enough to show that $h^n z^n - z^n$ tends to 0 in L^p–weak. Thus, observe that

$$\begin{aligned}\left|\int_\Omega (h^n z^n - z^n)(x)t(x)\,dx\right| &= \left|\int_{E^n} (h^n z^n - z^n)(x)t(x)\,dx\right| \\ &\le \|(h^n z^n - z^n)\|_{L^p(E^n)}\|t\|_{L^q(E^n)},\end{aligned} \tag{3.157}$$

where it is chosen $t(x) \in L^q(\Omega)$ and we have set $E^n := \{x \in \Omega : d(x,\Gamma) \le 2n^{-1}\}$. Now, the first term on the right hand side is clearly bounded thanks to (3.154), while the second tends to 0, as desired.

According to the notation of the previous subsection, we indicate by α_i^ε the Yosida regularization of α_i and by j_i^ε its convex primitive. Moreover, we define $J_{i,p}^\varepsilon$, J_p^ε, $\partial_{p,q} J_p^\varepsilon$, and so on, in the natural ways.

Thanks to the Lipschitz continuity of α_i^ε, we have that $\alpha_i^\varepsilon(z_i^n) \in H^1(\Omega_i) \subset L^q(\Omega_i)$, whence the relation

$$j_i^\varepsilon(z_i^n(x)) - j_i^\varepsilon(z_i(x)) \le \alpha_i^\varepsilon(z_i^n(x))(z_i^n(x) - z_i(x)) \quad \text{ a.e. in } \Omega_i \tag{3.158}$$

can be integrated in Ω_i and, owing to (3.153), with $k_i = \alpha_i^\varepsilon(z_i^n)$, entails

$$J_{i,p}^\varepsilon(z_i^n) - J_{i,p}^\varepsilon(z_i) \le -\frac{1}{n}\int_{\Omega_i} \nabla z_i^n \cdot \nabla \alpha_i^\varepsilon(z_i^n)\,dx = -\frac{1}{n}\int_{\Omega_i} (\alpha_i^\varepsilon)'(z_i^n)|\nabla z_i^n|^2\,dx \le 0. \tag{3.159}$$

On the other hand, it is $|h_i^n(x) z_i^n(x)| \le |z_i^n(x)|$ for every $x \in \Omega_i$, whence, being $\min j_i^\varepsilon = j_i^\varepsilon(0) = 0$, we have

$$J_{i,p}^\varepsilon(h_i^n z_i^n) \le J_{i,p}^\varepsilon(z_i^n) \le J_{i,p}^\varepsilon(z_i) \le J_{i,p}(z_i) \tag{3.160}$$

thanks also to the well-known monotonicity properties of the Yosida approximation. Applying the monotone convergence theorem, we finally get

$$J_{i,p}(h_i^n z_i^n) \le J_{i,p}(z_i) \quad \text{ for all } n \in \mathbf{N}. \tag{3.161}$$

Now, due to the convexity and lower semicontinuity of the functionals $J_{i,p}$ with respect to the L^p-topology and to (3.156), we obtain that

$$\begin{aligned} J_p(z) = \sum_{i=1}^2 J_{i,p}(z_i) &\le \sum_{i=1}^2 \liminf_{n\to\infty} J_{i,p}(h_i^n z_i^n) \le \liminf_{n\to\infty} \sum_{i=1}^2 J_{i,p}(h_i^n z_i^n) \\ &\le \limsup_{n\to\infty} \sum_{i=1}^2 J_{i,p}(h_i^n z_i^n) \le \sum_{i=1}^2 J_{i,p}(z_i) = J_p(z). \end{aligned} \tag{3.162}$$

Since $\sum_{i=1}^{2} J_{i,p}(h_i^n z_i^n) = J_V(h^n z^n)$, this implies that $J_p(z) = \lim_{n\to\infty} J_V(h^n z^n)$.

Finally, from the previous relation and from (3.156) again, we infer

$$(3.163)\quad \int_\Omega w(z-v)\,dx \leftarrow \int_\Omega w(h^n z^n - v)\,dx \le J_V(h^n z^n) - J_V(v) \to J_p(z) - J_p(v).$$

Since this holds for any $z \in L^p(\Omega)$, we deduce that $w \in \partial_{p,q} J_p(v)$, as desired. ■

We now introduce a new convex functional, setting for $w \in V'$:

$$(3.164)\quad J_{*,V'}(w) := \begin{cases} \sum_{i=1}^{2} \int_{\Omega_i} j_i^*(w_i(x))\,dx & \text{if } w \in L^{2_*}(\Omega) \text{ and } j_i^*(w_i) \in L^1(\Omega_i) \\ +\infty & \text{otherwise,} \end{cases}$$

which is nothing else that the $(+\infty)$-extension of $J_{2_*} = J_{2^*}^*$ to the larger space V'. What is not obvious from the definition, being actually a consequence of the coercivity assumption (3.134), is the following

Proposition 3.2.5. *$J_{*,V'}$ is lower semicontinuous with respect to the V'–topology.*

Proof. Let $\{w^n\} \subset D(J_{*,V'})$ such that $w_n \to w$ in V'. It is not restrictive, in view of the proof of the lower semicontinuity, to assume that $J_{*,V'}(w^n) \le C_6$ for some $C_6 > 0$ and for all $n \in \mathbb{N}$. From (3.134), it follows that there exists $R > 0$ such that $j_i^*(r) > m|r|^{2_*}$ for all $|r| \ge R$ and $i = 1, 2$, whence

$$(3.165)\quad \int_{\Omega_i} |w_i^n(x)|^{2_*}\,dx = \int_{\{|w_i^n| \ge R\}} |w_i^n(x)|^{2_*}\,dx + \int_{\{|w_i^n| < R\}} |w_i^n(x)|^{2_*}\,dx$$
$$\le m^{-1} \int_{\{|w_i^n| \ge R\}} j_i^*(w_i^n(x))\,dx + R^{2_*}|\Omega_i| \le m^{-1} C_6 + R^{2_*}|\Omega_i|.$$

Possibly extracting a subsequence, we now deduce that $w_i^n \to w_i$ in L^{2_*}–weak; then,

$$(3.166)\quad J_{*,V'}(w) = \sum_{i=1}^{2} \int_{\Omega_i} j_i^*(w_i)\,dx \le \sum_{i=1}^{2} \liminf_{n\to\infty} \int_{\Omega_i} j_i^*(w_i^n)\,dx$$
$$\le \liminf_{n\to\infty} \sum_{i=1}^{2} \int_{\Omega_i} j_i^*(w_i^n)\,dx = \liminf_{n\to\infty} J_{*,V'}(w^n).\ ■$$

The following simple result is actually the final step in our convex analysis machinery.

Proposition 3.2.6. *It is $J_{*,V'}^* = J_V$ and, owing to the previous proposition and to the Fenchel-Moreau theorem, $J_{*,V'} = J_V^*$.*

Proof. By definition of biconjugate function, recalling also Prop. 3.2.2, for any $v \in V$ we have that

$$J_V(v) = J_{2^*}^{**}(v) = \sup_{z \in L^{2_*}} \left\{ \int_\Omega zv\,dx - \sum_{i=1}^{2} \int_{\Omega_i} j_i^*(z_i(x))\,dx \right\} = J_{*,V'}^*(v)$$

(observe that $J_{*,V'}$ coincides with $J_{2^*}^*$ on its effective domain). ■

3.2.4 Existence and uniqueness of solutions to the transmission problem

Proof of Theorem 3.2.1. We begin by discussing the case of the growth conditions (GC), since it exploits the convex analysis instruments which we have introduced just above; we remark again that the case of (CC) has its natural variational setting in the space H; thus, we shall refer to Subsection 2.2.2 for most of the related resolution procedure.

First of all, we observe that, in order to adapt the transmission problem to the abstract framework, it is possible to proceed essentially as in (3.116–3.119), keeping here into account the more general assumptions on coefficients, of course (some of them were assumed equal to one in the one-substance setting). Under this notation, equations (3.141) and (3.125), together with the boundary and transmission conditions (3.129), (3.131–3.133), and $\partial_{\mathbf{n}}\chi = 0$ on Σ, have their variational formulations in (3.21–3.22); so, Theorem 3.1.2 yields the existence of a solution for such a weak statement and also assures the regularities (3.138–3.140) and the Cauchy conditions (3.24–3.25).

Now, we have to recover (3.126) and (3.142) from the abstract relation (3.23). First of all, observe that, owing for instance to [36, Prop. I.5.1, page 21], we have

$$\langle w(t), \chi(t)\rangle = J_V(\chi(t)) + J_V^*(w(t)) \qquad \text{a.e. in } [0,T], \tag{3.167}$$

while, recalling also estimate (3.26), we infer

$$\|J_V^*(w)\|_{L^2(0,T)} \le \|J_V(\chi)\|_{L^2(0,T)} + \|w\|_{L^2(0,T;V')}\|\chi\|_{L^\infty(0,T;V)} < +\infty. \tag{3.168}$$

Hence, owing to Prop. 3.2.5 and to Prop. 3.2.6, we deduce from the previous relation that $w(t) \in D(J_{*,V'}) \subset L^{2_*}(\Omega)$ for a.e. $t \in [0,T]$; furthermore, applying Theorem 3.2.4 with the choice of $p = 2^*$, we derive that $w(t) \in \partial_{2^*,2_*} J_{2^*}(\chi(t))$ a.e. in $[0,T]$ and, by Prop. 3.2.3, $w_i(t) \in \partial_{2^*,2_*} J_{i,2^*}(\chi_i(t))$ a.e.; relation (3.126) follows now from (3.150).

Anyway, some more effort is still needed in order to recover (3.142). Imitating the proof of Prop. 3.2.5, we deduce that, for some constant $C_7 > 0$ independent of w,

$$\begin{aligned} \|w_i\|^{2\cdot 2_*}_{L^{2\cdot 2_*}(0,T;L^{2_*}(\Omega))} &= \int_0^T \left(\int_{\Omega_i} |w_i(x,t)|^{2_*}\, dx \right)^2 dt \\ &\le \int_0^T \left(R^{2_*}|\Omega_i| + m^{-1} \int_{\Omega_i} j_i^*(w_i(x,t))\, dx \right)^2 dt \\ &\le C_7 \left(1 + \|J_{*,V'}(w)\|^2_{L^2(0,T)} \right) = C_7 \left(1 + \|J_V^*(w)\|^2_{L^2(0,T)} \right) < +\infty, \end{aligned} \tag{3.169}$$

as we see from (3.168). Moreover, proceeding by comparison in the variational equivalent (3.22) of (3.125), we immediately get that

$$B\chi = \operatorname{div}(\nu \nabla \chi) \in L^2(0,T;L^{2_*}(\Omega)), \tag{3.170}$$

whence (3.125) holds in the strong sense and $\partial_{\mathbf{n}}\chi = 0$ and (3.133) follow in the appropriate trace spaces (for the related trace theorems, see for instance [54]).

Moreover, supposing $\theta_0 \in V$ and $g \in H^1(0,T;H^{-1/2}(\partial\Omega))$, we can apply Theorem 3.1.3, which immediately provides (3.144) and, with some more (standard) machinery, also the related Neumann-like conditions for θ (3.129), (3.132).

Finally, if $\nu\nabla\chi_0 \in H(\mathrm{div},\Omega)$ and (3.145) holds, then it is possible to exploit Theorem 3.1.4, which gives (3.146) as well as

$$w \in L^\infty(0,T;V'). \tag{3.171}$$

Now, recalling (3.167), with a procedure similar to (3.168), it is immediate to infer that $J_V^*(w) \in L^\infty(0,T)$, whence also (3.147) follows easily, so concluding the proof of Theorem 3.2.1 in the (GC) setting. ∎

We finally come to the case of the compatibility conditions (CC). Indeed, here we do not refer to the abstract approach developed in Section 3.1, but we choose instead to try and repeat the procedure which we used in Subsec. 2.2.2 for the case of a single domain. Apart from the different notation, the main difference is that the abstract approach is stated in the space V' (as in the case of the Stefan problem (ST)); here, instead, we essentially perform the whole analysis in the space H; this will immediately permit to retrieve the constitutive relation (3.126) in the physical sense, but only provided that we are still able to perform all the apriori estimates as in Subsection 2.2.2. Unfortunately, under this kind of discontinuity of the coefficients, we can still derive the estimates related to the functions θ, χ, but not that of $\alpha(\chi)$ (which, by monotonicity, was almost straightforward in that case).

Then, suppose to have approximated the transmission problem through the Yosida-regularization of both graphs α_1, α_2 (call α_1^ε, α_2^ε the approximated operators) and to have studied this formulation through a Faedo-Galerkin approximation scheme, exactly as in Subsection 2.2.2, actually getting a solution which we name by $(\theta^\varepsilon, \chi^\varepsilon, \xi^\varepsilon)$. Nevertheless, in this procedure it should have been necessary to use some more care in performing the computation, essentially due to the discontinuity of ρ, λ, ℓ (in particular we should have retrieved different regularities for θ and $\rho\theta$, as in the abstract case).

At this point, it is possible to repeat the a priori estimate of that Subsection (no difficulties arise at this point due to the discontinuity of coefficients), obtaining the same kind of bounds for θ^ε and χ^ε as in (2.91–2.92). Now, according to our conventional notation, we can set $\alpha^\varepsilon(\chi^\varepsilon) = \alpha_{i,\varepsilon}(\chi_i^\varepsilon)$ in the domain Ω_i. Observe that it is no longer true that $\alpha^\varepsilon(\chi^\varepsilon)$ belongs to $H^1(\Omega)$; hence, it cannot used as a test function for the phase-field equation. Consequently, we are forced to perform another type of estimate, which is preceeded by a preliminary lemma, extending condition (CC) to the Yosida-approximate graphs.

Lemma 3.2.7. *The compatibility assumption (3.136) holds also for the graphs α_i^ε in the form:*

$$\left.\begin{array}{lcl} |\alpha_1^\varepsilon(x)| & \le & C_{\alpha,1}(1+|\alpha_2^\varepsilon(x)|) \\ |\alpha_2^\varepsilon(x)| & \le & C_{\alpha,2}(1+|\alpha_1^\varepsilon(x)|) \end{array}\right\} \quad \textit{for all } x\in\mathbb{R} \textit{ and } \varepsilon>0. \tag{3.172}$$

Proof. We prove for instance the first relation for $x \ge 0$. For the present, we suppose also $D = D(\alpha_1) = D(\alpha_2)$ open. Let $z \ge 0$; if $y_i \in \alpha_i(z)$, thanks to (3.136), we get

$$z + \varepsilon y_1 \le z + C_{\alpha,1}\varepsilon(1+y_2),$$

whence, taking the inverse functions, we derive

$$J_1^\varepsilon(x) \ge J_2^{C_{\alpha,1}\varepsilon}(x - C_{\alpha,1}\varepsilon) \tag{3.173}$$

for every $x \geq C_{\alpha,1}\varepsilon$. J_i^ε denotes here the *resolvent* of operator α_i; that is, $J_i^\varepsilon = (\mathrm{Id} + \varepsilon\alpha_i)^{-1}$. The monotonicity of the α_i and hypothesis $0 \in \alpha_i(0)$ furthermore, permit to extend easily the validity of the previous relation to every point $x > 0$.

Using the definition of Yosida approximation, we immediately deduce:

$$\alpha_1^\varepsilon(x) = \frac{x - J_1^\varepsilon(x)}{\varepsilon} \leq \frac{x - J_2^{C_{\alpha,1}\varepsilon}(x - C_{\alpha,1}\varepsilon)}{\varepsilon} \tag{3.174}$$

$$= C_{\alpha,1}\alpha_2^{C_{\alpha,1}\varepsilon}(x - C_{\alpha,1}\varepsilon) + C_{\alpha,1} \leq C_{\alpha,1}(1 + \alpha_2^\varepsilon(x)), \tag{3.175}$$

as required, where the last inequality is a consequence of the monotonicity of $\alpha_2^\varepsilon(x)$ with respect both to x and to ε (here we have exploited that $C_{\alpha,1} > 1$).

Finally, we observe that the same procedure can be applied with minor modifications also in the case of D not open. For instance, if $D = [x_-, x_+]$ and $\alpha_i(x_+) = [b_i, +\infty[$, recalling that any maximal monotone graph (in our case α_i) in $\mathbb{R} \times \mathbb{R}$ is constructed as the maximal monotone extension of a monotone (single-valued) function from $\mathbb{R}$ to $\mathbb{R}$ (call it, say, $\overline{\alpha}_i$), we still have that $b_1 \leq C_{\alpha,1}(1 + b_2)$ and $b_2 \leq C_{\alpha,2}(1 + b_1)$; $\overline{\alpha}_i$, in fact, must be continuous on the left in x_+. So, (3.173) can be proved again for all $x \geq 0$ using exactly the same procedure as before. ■

We present in the form of a Lemma the required estimate concerning the functions ξ_i^ε.

Lemma 3.2.8. *We have that, for some L^2-functions ξ_1, ξ_2, the following convergences are satisfied at least for subsequences*

$$\alpha_1^\varepsilon(\mathcal{X}_1^\varepsilon) \rightharpoonup \xi_1 \quad \text{in } L^2(Q_1)\text{-weak}, \tag{3.176}$$

$$\alpha_2^\varepsilon(\mathcal{X}_2^\varepsilon) \rightharpoonup \xi_2 \quad \text{in } L^2(Q_2)\text{-weak}. \tag{3.177}$$

Proof. Take $v = \alpha_1^\varepsilon(\mathcal{X}^\varepsilon) + \alpha_2^\varepsilon(\mathcal{X}^\varepsilon)$ (that is $v_1 = \alpha_1^\varepsilon(\mathcal{X}_1^\varepsilon) + \alpha_2^\varepsilon(\mathcal{X}_1^\varepsilon)$ and $v_2 = \alpha_1^\varepsilon(\mathcal{X}_2^\varepsilon) + \alpha_2^\varepsilon(\mathcal{X}_2^\varepsilon)$) as a test function in equation (3.125), which is possible thanks to the Lipschitz continuity of the Yosida approximates. Integrating in $[0, t]$ and observing that the monotonicity of the α's and $\alpha_i(0) \ni 0$ entail $\alpha_1^\varepsilon(\mathcal{X}^\varepsilon)\alpha_2^\varepsilon(\mathcal{X}^\varepsilon) \geq 0$ for a.e. $x \in Q$ and that, for instance,

$$\int_{\Omega_1} \nabla\mathcal{X}_1^\varepsilon \cdot \nabla\alpha_2^\varepsilon(\mathcal{X}_1^\varepsilon)\, dx = \int_{\Omega_1} (\alpha_2^\varepsilon)'(\mathcal{X}_1^\varepsilon)\nabla\mathcal{X}_1^\varepsilon \cdot \nabla\mathcal{X}_1^\varepsilon\, dx \geq 0, \tag{3.178}$$

it is a straightforward computation to obtain

$$\begin{aligned} &\mu_1 \left[\int_{\Omega_1} (j_1^\varepsilon(\mathcal{X}_1^\varepsilon(t)) + j_2^\varepsilon(\mathcal{X}_1^\varepsilon(t)))\, dx\right] + \mu_2 \left[\int_{\Omega_2} (j_1^\varepsilon(\mathcal{X}_2^\varepsilon(t)) + j_2^\varepsilon(\mathcal{X}_2^\varepsilon(t)))\, dx\right] \\ &\quad + \int_0^t \int_{\Omega_1} (\alpha_1^\varepsilon(\mathcal{X}_1^\varepsilon))^2\, dx\, dt + \int_0^t \int_{\Omega_2} (\alpha_2^\varepsilon(\mathcal{X}_2^\varepsilon))^2\, dx\, dt \\ &\leq \mu_1 \left[\int_{\Omega_1} (j_1^\varepsilon(\mathcal{X}_{1,0}) + j_2^\varepsilon(\mathcal{X}_{1,0}))\, dx\right] + \mu_2 \left[\int_{\Omega_2} (j_1^\varepsilon(\mathcal{X}_{2,0}) + j_2^\varepsilon(\mathcal{X}_{2,0}))\, dx\right] \\ &\quad + C_\sigma + \sigma \left[\|\alpha_1^\varepsilon(\mathcal{X}_1^\varepsilon)\|^2_{L^2(\Omega_1 \times]0,t[)} + \|\alpha_2^\varepsilon(\mathcal{X}_1^\varepsilon)\|^2_{L^2(\Omega_1 \times]0,t[)}\right] \\ &\quad + \sigma \left[\|\alpha_1^\varepsilon(\mathcal{X}_2^\varepsilon)\|^2_{L^2(\Omega_2 \times]0,t[)} + \|\alpha_2^\varepsilon(\mathcal{X}_2^\varepsilon)\|^2_{L^2(\Omega_2 \times]0,t[)}\right], \end{aligned} \tag{3.179}$$

where $\sigma > 0$ is arbitrary and C_σ has the same role as in (1.24), depending only on the norms of θ^ε and χ^ε in $L^2(Q)$, which have already been estimated, and on the Lipschitz constant of γ. Move now to the left hand side the last two terms, which are negligible for σ sufficiently small, thanks also to condition (3.172); observe also that

$$\begin{aligned} \int_{\Omega_1^+} j_2^\varepsilon(\chi_{1,0}(x))\,dx &= \int_{\Omega_1^+}\int_0^{\chi_{1,0}(x)} \alpha_2^\varepsilon(s)\,ds\,dx \\ &\leq \int_{\Omega_1^+}\int_0^{\chi_{1,0}(x)} C_{\alpha,2}(1+\alpha_1^\varepsilon(s))\,ds\,dx \\ &\leq C_{\alpha,2}\|\chi_{1,0}\|_{L^1(\Omega_1^+)} + C_{\alpha,2}\int_{\Omega_1^+} j_1^\varepsilon(\chi_{1,0}(x))\,dx, \end{aligned} \tag{3.180}$$

where we have set $\Omega_1^+ := \{x \in \Omega_1 : \chi_{1,0}(x) \geq 0\}$, and that analogous relations still hold with the interchange of 1 and 2 as indexes, of $C_{\alpha,1}$ and $C_{\alpha,2}$, or with $\Omega_1\backslash\Omega_1^+$ in place of Ω_1^+; this completes the proof of the lemma. ∎

Now, the proof of Theorem 3.2.1 in the (CC) case is easily concluded, by passing to the limit with respect to $\varepsilon \to 0$ and using as usual the monotonicity argument of Prop. 1.3.8 in the space $L^2(Q)$. ∎

Remark 3.2.9. **1)** *Recalling (3.170) and the second of (3.140), it should be possible to derive the continuity of χ in some better space than $L^2(\Omega)$; for instance, if ν has a $C^1(\Omega)$-regularity and Ω is of C^2 class, then it follows $\chi \in L^2(0,T;W^{2,2_*}(\Omega))$, whence, using some (rather fine) interpolation results [11, Cor. 3.12.3, page 74 and Theorem 6.4.5, page 152], we can deduce*

$$\chi \in C^0([0,T];W^{1,2_*}(\Omega)). \tag{3.181}$$

2) *A comparison of the physical hypotheses of the above discussed applications with the corresponding abstract ones immediately suggests the possibility of choosing more general functions γ (or γ_i, respectively) in the concrete frameworks. For instance, if a suitably regular direction field $\mathbf{x}$ is assigned in Ω, one possibility for the first problem could be $\gamma(v) = \nabla v \cdot \mathbf{x}$, accounting for the presence of a convection phenomenon for the phase field.*

3) *We point out that it should be possible to weaken the hypothesis (3.27) of Theorem 3.1.3, by replacing it with*

$$f \in L^1(0,T;H) + W^{1,1}(0,T;V'). \tag{3.182}$$

This kind of condition would give rise to some technical complications in the derivation of the a priori estimates; in particular, an application of the Gronwall lemma in the form of Prop. 1.1.15 is required, precisely as in Subsection 2.2.2.

4) *It should be interesting to establish if Theorem 3.2.1 remains valid when we substitute the coerciveness assumption (3.134) with the following weaker one*

$$\liminf_{|r|\to\infty} \frac{j_i^*(r)}{|r|} \geq 2m > 0, \qquad \textit{for } i = 1,2, \tag{3.183}$$

which is the equivalent of (2.29) of [10]. Indeed, we do not know if the related procedure of [10] for the deduction of the constitutive relation in the physical sense could be extended to the transmission case.

3.3 Concentrated capacities

In this final section, we present a further physical application of the abstract results developed in the first part of the chapter. In particular, we are going to study a new kind of transmission problem for the phase-field model, where the related statement is actually addressed to a different thermodynamical setting, which we now outline. Maintaining indeed the usual notations for the space and time coordinates, in the new framework we suppose in addition that the domain Ω_2 consists of a very thin layer adjoining to Γ. This layer is still filled with a fluid obeying to the phase-field model, where anyway the thermal and phase conductivities are assumed to be very large at least in the normal direction to Γ. As it has been studied in the case of the weak Stefan problem first by Fasano, Primicerio and Rubinstein [38] and more recently by Magenes [56, 57, 58, 59, 60, 61], and by Savaré and Visintin [76] (who give also a description of the behaviour of the solutions as the conductivity increases), in this setting the heat equation in Ω_2 is very well approximated by an analogous relation defined on the boundary Γ, where indeed a more source term is present, which is left as a trace of the original transmission condition. It is this phenomenon to be named as a *concentrated capacity.*

Since we are dealing instead with the phase-field model, we had to decide whether to study a model presenting a concentrated capacity also for the unknown χ, or to choose a different approximation for the phase diffusion equation on the layer Ω_2. We preferred to deal with a concentrated capacity also for the χ, which corresponds to a blow-up of the coefficient ν_2 of (3.125) (which should now be substituted with a suitably regular tensor field) in the normal direction to Γ, since in this setting the resulting equations really provide an extension of the boundary Stefan problem which has been asymptotically derived and studied in the above quoted papers. Hence, this choice seemed to us to be the most relevant also from the physical point of view; naturally other possible behaviours of the phase-field equation could be object of investigation, as well.

We can now state the precise hypotheses of the concentrated capacity problem, which are very similar indeed to those of the standard transmission case, save for the different space coordinates and analytical setting. First of all, we introduce, as usual, the Hilbert triplet where the variational version of the problem will be stated and we then give some related comments. We choose

$$H := L^2(\Omega_1) \times L^2(\Gamma), \qquad V := \{v \in H^1(\Omega_1) : v_{|\Gamma} \in H^1(\Gamma)\}, \tag{3.184}$$

where V is naturally equipped with the graph norm. As far as notation is concerned, the generic element $v \in H$ will be indicated by the couple (v_1, v_Γ); it is clear that, if $v \in V$ in addition, then v_Γ really turns out to be the trace on Γ of v_1. Only in this case, we shall sometimes identify v with v_1 and see it as a true function on Ω_1 instead that as an ordered couple.

We point out that, in this setting, it could be not completely obvious that (V, H, V') actually form a Hilbert triplet; hence, we give an idea of how to prove this fact. We first notice that the continuity (and compactness) of the inclusion $V \subset H$ is clear; the less trivial point is the density, indeed. Then, to show it, choose $(u_1, u_\Gamma) \in H$ and take first a sequence

$$(w_{1,n}) \subset H_0^1(\Omega_1) \quad \text{such that } \; w_{1,n} \to u_1 \; \text{ in } L^2(\Omega_1). \tag{3.185}$$

Correspondently, pick also a sequence

$$(u_{\Gamma,n}) \subset H^1(\Gamma) \quad \text{such that } \; u_{\Gamma,n} \to u_\Gamma \; \text{ in } L^2(\Gamma). \tag{3.186}$$

Now, since Ω_1 has only the Lipschitz regularity, we have to proceed by localization. Thus, by means of a partition of unity and a (Lipschitz continuous) system of local charts, we reduce the problem to the case of $\Omega_1 = \mathbb{R}^N_+$ and $\Gamma = \mathbb{R}^{N-1} \times \{0\}$. Also, in the new coordinates, we can assume that both the approximating and the approximated functions have a compact support.

Denoting now as $\mathbf{e}_N$ the *inner* normal unit vector to Ω_1 at the generic point $y \in \Gamma$ (i.e., the N-th vector of the canonical basis of $\mathbb{R}^N$), we construct, for $\delta > 0$, the set $\Omega_\delta := \mathbb{R}^{N-1} \times]0, \delta[$, and the sequence (v_n) given by

$$v_n : \Omega_{1/n} \to \mathbb{R}, \qquad v_n(x) := u_{\Gamma,n}(y) \; \text{ for } x = y + t\,\mathbf{e}_N, \quad y \in \Gamma, \quad 0 < t < 1/n. \tag{3.187}$$

It is easy to prove that $(v_n) \subset H^1(\Omega_{1/n})$ and that $(v_n)_{|\Gamma} = u_{\Gamma,n}$.

Now, by regularization of v_n and trivial extension (to the whole Ω_1), we can easily construct a sequence $(z_{1,n}) \subset H^1(\Omega_1)$ satisfying the following:

$$z_{1,n} \to 0 \quad \text{in } L^2(\Omega_1), \tag{3.188}$$

$$(z_{1,n})_{|\Gamma} = u_{\Gamma,n}. \tag{3.189}$$

By virtue of (3.185) and (3.188–3.189), setting $u_{1,n} := w_{1,n} + z_{1,n}$ it is now easy to verify directly that the sequence (u_n) given by $u_n := (u_{1,n}, u_{\Gamma,n})$ fulfils $(u_n) \subset V$ and $u_n \to u$ in H, as desired (naturally, this procedure should be performed by also coming back to the old coordinates).

Although the Sobolev spaces on boundaries have already been briefly introduced in Subsec. 1.1.2, we prefer to specify here some more machinery concerning differential operators on manifolds; for the sake of simplicity, we state the related definitions in the usual $\mathbb{R}^N$-coordinates; we could also proceed by local charts as in [6], but this approach would be perhaps less intuitive. So, let us consider a suitably regular real-valued function f on Γ. Extending it in some smooth way to a small neighbourhood of Γ, and still denoting by f the extension, we define the boundary i-th partial derivative as

$$\delta_i f := \partial_i f - (\nabla f \cdot \mathbf{n})\, n_i. \tag{3.190}$$

The tangential gradient of f is introduced in the following natural way:

$$\nabla_\Gamma f := \sum_{i=1}^N \delta_i f\, \mathbf{e}_i = \nabla f - (\nabla f \cdot \mathbf{n})\, \mathbf{n}, \tag{3.191}$$

where $\mathbf{e}_i$ is the canonical base of $\mathbb{R}^n$, and the Laplace-Beltrami operator is given by:

$$\Delta_\Gamma f := \sum_{i=1}^{N} \delta_i^2 f. \tag{3.192}$$

Finally, for a vector valued function $\mathbf{f}$ on Γ, we set

$$\operatorname{div}_\Gamma \mathbf{f} := \sum_{i=1}^{N} \delta_i f_i. \tag{3.193}$$

It is not difficult to verify that all the above definitions do not depend on the chosen extension (see also [63]).

We have now presented all the machinery which is needed in order to state the mathematical problem. We begin by listing some hypotheses on data. First of all, take α_1, α_Γ as maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$, satisfying as usual $0 \in \alpha_1(0)$, $0 \in \alpha_\Gamma(0)$, and choose initial values θ_0, χ_0 such that

$$\theta_0 \in V, \quad \chi_0 \in V, \quad \text{with } j_1(\chi_{1,0}) \in L^1(\Omega_1),\ j_\Gamma(\chi_{\Gamma,0}) \in L^1(\Gamma), \tag{3.194}$$

with j_1, j_Γ standing for convex primitives of α_1, α_Γ, respectively. Moreover, we assume $f \in L^2(0,T;H)$ as a source term, γ_1, γ_Γ Lipschitz continuous functions, and choose positive parameters ρ_1, ρ_Γ, $\lambda_1, \lambda_\Gamma$, κ_1, κ_Γ, μ_1, μ_Γ, ν_1, ν_Γ, ℓ_1, ℓ_Γ of the usual physical meaning. Next, we suppose that the operators α_1, α_Γ fulfill either one of conditions (CC), (GC). Notice that, as it should already be clear from the mathematical hypotheses assumed on data, we only address the regularity setting of Theorem 3.1.3; indeed, almost no further difficulties arise for the sake of handling different regularity hypotheses. Finally, by simplicity, we require homogeneous Neumann boundary conditions for both θ and χ; thus, the introduction of the following abstract elliptic operators $K, N : V \to V'$, given by

$$\langle Kv, w\rangle := \int_{\Omega_1} \kappa_1 \nabla v_1 \cdot \nabla w_1\, dx + \int_\Gamma \kappa_\Gamma \nabla_\Gamma v_\Gamma \cdot \nabla_\Gamma w_\Gamma\, d\mathcal{H}^{N-1} \quad \text{for } v, w \in V, \tag{3.195}$$

$$\langle Nv, w\rangle := \int_{\Omega_1} \nu_1 \nabla v_1 \cdot \nabla w_1\, dx + \int_\Gamma \nu_\Gamma \nabla_\Gamma v_\Gamma \cdot \nabla_\Gamma w_\Gamma\, d\mathcal{H}^{N-1} \quad \text{for } v, w \in V, \tag{3.196}$$

permits to state here the mathematical formulation of the problem.

Problem 3.3.1. *We look for a triplet of functions (θ, χ, ξ), with*

$$\theta, \chi \in L^\infty(0,T;V) \cap H^1(0,T;H), \tag{3.197}$$

$$\xi \in L^2(0,T;H), \tag{3.198}$$

satisfying the variational equalities

$$\rho \partial_t \theta + \lambda \partial_t \chi + K\theta = f \quad \textit{in } V', \quad \textit{a.e. in }]0,T[, \tag{3.199}$$

$$\mu \partial_t \chi + N\chi + \xi + \gamma(\chi) = \ell\theta \quad \textit{in } V', \quad \textit{a.e. in }]0,T[, \tag{3.200}$$

$$\xi_1 \in \alpha_1(\chi_1) \quad \textit{a.e. in } \Omega_1 \times]0,T[, \tag{3.201}$$

$$\xi_\Gamma \in \alpha_\Gamma(\chi_\Gamma) \quad \textit{a.e. on } \Sigma, \tag{3.202}$$

as well as the initial conditions

$$\theta(0) = \theta_0, \quad \chi(0) = \chi_0, \quad \textit{in } H. \ \blacksquare \tag{3.203}$$

We point out that in the above statement, as in the standard transmission case, we have used a compact notation for some product terms; for instance, the expression $\rho\partial_t\theta$ should be intended as a couple $(\rho_1\partial_t\theta_1, \rho_\Gamma\partial_t\theta_\Gamma) \in H$, and so on.

Remark 3.3.2. *It is worthwhile to try and write the equations of the concentrated capacity system in the strong sense, owing to the regularities (3.197). With this aim, we first choose a function $v \in H_0^1(\Omega_1)$ (corresponding to the couple $(v_1, 0) \in V$) and test by v the equation (3.199). Using standard techniques, we infer the strong relation*

$$\rho_1\partial_t\theta_1 + \lambda_1\partial_t\chi_1 - \kappa_1\Delta\theta_1 = f_1 \qquad \text{a.e. in } \Omega_1\times]0,T[, \tag{3.204}$$

and, by comparison, the further regularity $-\Delta\theta_1 \in L^2(0,T;L^2(\Omega_1))$. The same procedure applied to (3.200) yields analogously

$$\mu_1\partial_t\chi_1 - \nu_1\Delta\chi_1 + \xi_1 - c_1\chi_1 = \ell_1\theta_1 \qquad \text{a.e. in } \Omega_1\times]0,T[\tag{3.205}$$

and $-\Delta\chi_1 \in L^2(0,T;L^2(\Omega_1))$. Furthermore, if we pick $v \in H^1_{0,\Gamma}(\Omega_1)$ instead, one immediately derives the homogeneous Neumann boundary conditions

$$\partial_{\mathbf{n}}\theta = \partial_{\mathbf{n}}\chi = 0 \qquad \text{on } (\partial\Omega_1 \setminus \overline{\Gamma})\times]0,T[. \tag{3.206}$$

Now, let us test (3.199–3.200) (for instance) by a general *test function $v \in V$. On account of (3.204), we would like to derive*

$$\rho_\Gamma\partial_t\theta_\Gamma + \lambda_\Gamma\partial_t\chi_\Gamma - \kappa_\Gamma\Delta_\Gamma\theta_\Gamma = f_\Gamma - \kappa_1\partial_{\mathbf{n}}\theta_1, \tag{3.207}$$

$$\mu_\Gamma\partial_t\chi_\Gamma - \nu_\Gamma\Delta_\Gamma\chi_\Gamma + \xi_\Gamma - c_\Gamma\chi_\Gamma = \ell_\Gamma\theta_\Gamma - \nu_1\partial_{\mathbf{n}}\chi_1, \tag{3.208}$$

say, a.e. on Σ. However, the above expressions need to be furtherly detailed for two reasons: first, due to the mixed boundary conditions for θ_1, χ_1 (homogeneous Neumann – transmission), in general it is not true that the traces $\partial_{\mathbf{n}}\theta_1, \partial_{\mathbf{n}}\chi_1$ belong to $L^2(\Sigma)$ (we can only say that they stay in $L^2(0,T;H^{-1/2}(\Gamma))$, also on account of Prop. 1.1.8); so (3.207–3.208) have to be intended in a correspondingly weaker sense (and in particular not a.e. on Σ). Moreover, the use of the Gauss-Green formula for the Laplace-Beltrami operators can now be done rigorously only as $v_\Gamma \in H_0^1(\Gamma)$ and not for the component v_Γ of the general test function v, since we have not enough regularity to derive the $(N-2)$-dimensional Neumann boundary conditions $\partial_{\mathbf{n}'}\theta_\Gamma = 0$, $\partial_{\mathbf{n}'}\chi_\Gamma = 0$ in a strong sense ($\mathbf{n}'$ denoting here the outer normal unit vector to Γ on its relative $(N-2)$-dimensional boundary Γ'). For this reason, system (3.204–3.208) is not precisely equivalent to (3.199–3.200).

Resolution of Problem 3.3.1. It does not present further difficulties with respect to the case of the standard transmission problem. Indeed, we still have to distinguish between the (GC)-setting, where we have to address the abstract framework of Section 3.1 and the (CC)-one, where it is possible to proceed as in Subsecs. 2.2.2–2.2.3 (see also the paper [77]). ∎

Chapter 4

Convergence results

This final chapter of the dissertation is devoted to the study of some mathematical problems which can be derived as limit statements of the phase-field system under suitable blow-out or, more generally, convergence hypotheses on coefficients; also the convergence behaviour of solution is an object of our study. In general, we shall start from the transmission system (3.124–3.126); then, every investigation will be carried out by keeping essentially fixed the contribution of the equations in Ω_1 and allowing instead the Ω_2-parts to change. Indeed, in any case, the main reason of mathematical interest will lie in the analysis of the behaviour of the compatibility conditions between the Ω_1 and Ω_2-coefficients under the chosen convergence assumptions.

4.1 Singular limits of the transmission problem

In this section, we study the behaviour of Problem (TP) under the blow-out of coefficients μ_2 and ν_2 and under a suitable variation of the graph α_2. While the results of Subsec. 4.1.2 have appeared in the paper [78], the rest of the contents of this Section is new and still unpublished.

4.1.1 Introduction and preliminaries

Here we want to briefly resume in a self-contained way the hypotheses and notation of the transmission problem for the phase field model: with this aim, we first recall that Ω is a smooth bounded domain in $\mathbb{R}^N$, subdivided by the interface Γ into the Lipschitz-regular subdomains Ω_1 and Ω_2, where two different substances are assumed to lie. In all the remainder of the chapter, we shall refer to the transmission problem of Subsec. 3.2.2 as $(\mathrm{TP_n})$; indeed, we are going to consider several perturbation of the coefficients of the phase-field equation in Ω_2; their convergence will take place as the index n tends to ∞.

We recall the hypotheses of $(\mathrm{TP_n})$, which are, by simplicity, related to the intermediate regularity Theorem 3.1.3. Fix, for the present, $f \in L^2(Q)$, and ρ_1, ρ_2, κ_1, κ_2, λ_1, λ_2, μ_1, $\mu_{2,n}$, ν_1, $\nu_{2,n}$, c_1, $c_{2,n}$, ℓ_1, ℓ_2 as positive constants of the usual physical meaning (as it can be controlled in the equations below); the notation is given according to the perturbations of $(\mathrm{TP_n})$ in Ω_2, while the coefficients in Ω_1 in general do not vary with respect to n. Note that, for simplicity, we are supposing here $\gamma_i(r) = -c_i r$. Also, consider two maximal monotone graphs $\alpha_{1,n}$ and $\alpha_{2,n}$. Our conventional notation involving the index $i = 1, 2$ (i.e. $f_i = f_{|\Omega_i}$, for instance) is extended to the case of n-dependent functions this way: for example, μ_n will denote the piecewise constant strictly positive function coinciding with μ_1 (fixed) on Ω_1 and with $\mu_{2,n}$ (possibly varying with n) on Ω_2; the same is intended for ν_n, c_n, and α_n (in the sense of an operator depending also on the space variable), or for the solutions $(\theta_n, \chi_n, \xi_n)$, and so on.

In the rest of the chapter we shall study several problems where some, or all, of the coefficients $\mu_{2,n}, \nu_{2,n}, \alpha_{2,n}, c_{2,n}$ are allowed to vary with n, while the corresponding parameters on Ω_1, save at most the operator $\alpha_{1,n}$, are fixed.

Also, it is possible to allow a variation of the initial datum χ_0 in Ω; i.e., we choose $\chi_{1,0,n} \to \chi_{1,0}$ and $\chi_{2,0,n} \to \chi_{2,0}$; in some cases $\chi_{1,0,n}$ will remain fixed and equal to $\chi_{1,0}$. We can now recall the equations of $(\mathrm{TP_n})$ in a strong form, which is allowed by the regularity of solutions provided by Theorem 3.1.3

$$\rho_1 \partial_t \theta_{1,n} + \lambda_1 \partial_t \chi_{1,n} - \kappa_1 \Delta \theta_{1,n} = f_1 \quad \text{a.e. in } Q_1, \tag{4.1}$$

$$\rho_2 \partial_t \theta_{2,n} + \lambda_2 \partial_t \chi_{2,n} - \kappa_2 \Delta \theta_{2,n} = f_2 \quad \text{a.e. in } Q_2, \tag{4.2}$$

$$\mu_1 \partial_t \chi_{1,n} - \nu_1 \Delta \chi_{1,n} + \xi_{1,n} - c_1 \chi_{1,n} = \ell_1 \theta_{1,n} \quad \text{a.e. in } Q_1, \tag{4.3}$$

$$\mu_{2,n} \partial_t \chi_{2,n} - \nu_{2,n} \Delta \chi_{2,n} + \xi_{2,n} - c_{2,n} \chi_{2,n} = \ell_2 \theta_{2,n} \quad \text{a.e. in } Q_2, \tag{4.4}$$

$$\xi_{1,n} \in \alpha_1(\chi_{1,n}) \quad \text{a.e. in } Q_1; \quad \xi_{2,n} \in \alpha_{2,n}(\chi_{2,n}) \quad \text{a.e. in } Q_2. \tag{4.5}$$

We point out that the above system is complemented with the natural Cauchy conditions of the form

$$\theta_n(0) = \theta_0, \qquad \chi_n(0) = \chi_{0,n}, \tag{4.6}$$

with homogeneous Neumann boundary conditions for χ_n (which, we recall, seem the most appropriate under the thermodynamical viewpoint)

$$\partial_{\mathbf{n}} \chi_n = 0 \quad \text{on } \partial\Omega \times]0, T[, \tag{4.7}$$

and with third type conditions for the temperature, which precisely prescribe the heat flux with respect to the exterior:

$$(\kappa \nabla \theta_n) \cdot \mathbf{n} + p\theta_n = g_n \quad \text{on } \partial\Omega \times]0, T[, \tag{4.8}$$

where it is $p > 0$, and $g \in L^2(0, T; H^{-1/2}(\partial\Omega))$ represents a source term concentrated on the boundary.

Finally, there are the transmission conditions at the interface for both variables:

$$\theta_{1,n} = \theta_{2,n} \quad \text{on } \Sigma, \quad \chi_{1,n} = \chi_{2,n} \quad \text{on } \Sigma, \tag{4.9}$$

$$\kappa_1 \partial_{\mathbf{n}} \theta_{1,n} = \kappa_2 \partial_{\mathbf{n}} \theta_{2,n} \quad \text{on } \Sigma, \tag{4.10}$$

$$\nu_1 \partial_{\mathbf{n}} \chi_{1,n} = \nu_{2,n} \partial_{\mathbf{n}} \chi_{2,n} \quad \text{on } \Sigma. \tag{4.11}$$

We recall that, due to the regularity of solutions, joining the contributions of Ω_1 and Ω_2 and accounting for the transmission conditions (4.9–4.11), we could also rewrite the equations of Problem $(\mathrm{TP_n})$ in an equivalent variational form. We finally repeat the regularity assumptions on data and the consequent regularity properties of solutions: setting as usual $V = H^1(\Omega)$ and $H = L^2(\Omega)$ (we point out that Dirichlet boundary conditions, also not homogeneous, could be assumed instead; all the subsequent results should remain valid, with the usual modifications in the variational setting), we assume that $\theta_0 \in V$, $\chi_{0,n} \in V$, and $j_{1,n}(\chi_{1,0,n}) \in L^1(\Omega_1)$, $j_{2,n}(\chi_{2,0,n}) \in L^1(\Omega_2)$, where $j_{1,n}$ ($j_{2,n}$) is the convex primitive of $\alpha_{1,n}$ ($\alpha_{2,n}$, respectively). Moreover, we suppose either of the following (CC), or (GC):

(CC) $D(\alpha_{1,n}) = D(\alpha_{2,n}) =: D_n$; moreover, for every $n \in \mathbb{N}$ there exists a positive constant $C_{\alpha,n}$, such that for every $r \in \mathbb{R}$ and $y_1 \in \alpha_{1,n}(r), y_2 \in \alpha_{2,n}(r)$, it is

$$|y_1| \leq C_{\alpha,n}(1 + |y_2|), \qquad |y_2| \leq C_{\alpha,n}(1 + |y_1|); \tag{4.12}$$

(GC) for some $m_n > 0$ (possibly depending on n), we have that,

$$\liminf_{|r|\to\infty} \frac{j^*_{1,n}(r)}{r^2} \geq m_n, \qquad \liminf_{|r|\to\infty} \frac{j^*_{2,n}(r)}{r^2} \geq m_n \tag{4.13}$$

for every $n \in \mathbb{N}$.

We point out that, for the present, no uniformity in n is required in the above relations; moreover, we remark that, due to the large number of estimates to be performed, in this chapter the counter of the constants $C_1, C_2, \dots$ will be restarted each subsection. Under either (CC) or (GC), the solution $(\theta_n, \chi_n, \xi_n)$ to (4.1–4.11) exists, is unique, and satisfies the following regularity properties:

$$\theta_n \in H^1(0,T;H) \cap C^0([0,T];V); \qquad -\operatorname{div}(\kappa\nabla\theta) \in L^2(0,T;H) \tag{4.14}$$

$$\chi_n \in H^1(0,T;H) \cap C^0([0,T];V); \qquad -\operatorname{div}(\nu\nabla\theta) \in L^2(0,T;H) \tag{4.15}$$

$$\xi_n \in L^2(0,T;H). \tag{4.16}$$

We point out that the expression of condition (GC) reported here is slightly stronger than the corresponding one of Subsection 3.2.2; hence, (4.16) yields some more space regularity than (3.142) (of which it is a particular case); the above statement, anyway, is sufficiently general for the sequel. We also emphasize that, in the following subsections, further hypotheses on coefficients will be given, referring to the specific convergence analysis to be performed time by time.

Here we conclude by presenting some further mathematical instruments which will result useful in the sequel; we start by stating a general lemma concerning maximal monotone operators, which justifies rigorously a formal monotonicity argument that frequently occurs in the derivation of the a priori estimates for this kind of problems.

Lemma 4.1.1. *Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, $V = H^1(\Omega)$ or $V = H^1_0(\Omega)$, $A : V \to V'$ a weakly elliptic operator (recall (1.21)), $u \in V$ such that $Au \in L^2(\Omega)$, $\zeta \in L^2(\Omega)$ such that $\zeta \in \beta(u)$ a.e. in Ω. Then, we have that*

$$\int_\Omega Au\zeta \, dx \geq 0. \tag{4.17}$$

Proof. Denoting by β^ε the Yosida-approximation of β and setting as usual $H := L^2(\Omega)$, we consider the following elliptic problem

$$u^\varepsilon + Au^\varepsilon + \beta^\varepsilon(u^\varepsilon) = u + Au + \zeta \quad \text{in } H. \tag{4.18}$$

It is well-known that such a problem admits a unique solution u^ε, satisfying $u^\varepsilon \in V$ and $Au^\varepsilon \in H$. Since β^ε is Lipschitz continuous, there are no difficulties in deriving the natural apriori estimates for the above problem, leading to the following convergences, which hold, for some $v \in V$, up to the extraction of subsequences (actually uniqueness will guarantee them for the whole sequences):

$$u^\varepsilon \to v \quad \text{in } V\text{–weak (and } H\text{–strong)}, \tag{4.19}$$

$$Au^\varepsilon \to Av \quad \text{in } H\text{–weak}, \tag{4.20}$$

$$\beta^\varepsilon(u^\varepsilon) \to \eta \quad \text{in } H\text{–weak}. \tag{4.21}$$

Then, we see that the limit functions satisfy $v + Av + \eta = u + Au + \zeta$; if we show that $\eta \in \beta(v)$ a.e. in Ω, we can conclude, by monotonicity, that $v = u$ and $\zeta = \eta$. The procedure to obtain this equality exploits, as usual, Prop. 1.3.8. Denoting by $(\cdot,\cdot)$ the scalar product in H, we easily deduce from (4.18)

$$(\beta^\varepsilon(u^\varepsilon), u^\varepsilon) = (u + Au + \zeta - u^\varepsilon - Au^\varepsilon, u^\varepsilon), \tag{4.22}$$

whence $\lim\,(\beta^\varepsilon(u^\varepsilon), u^\varepsilon) = (u + Au + \zeta - v - Av, v) = (\eta, v)$, as desired. We finally verify the required property. At the level $\varepsilon > 0$, we have

$$0 \le \int_\Omega Au^\varepsilon\, \beta^\varepsilon(u^\varepsilon)\, dx = \int_\Omega (u + Au + \zeta - u^\varepsilon - \beta^\varepsilon(u^\varepsilon))\beta^\varepsilon(u^\varepsilon)\, dx, \tag{4.23}$$

whence, passing to the supremum limit and exploiting semicontinuity, we infer

$$0 \le \int_\Omega (u + Au + \zeta - u - \zeta)\zeta\, dx = \int_\Omega Au\, \zeta\, dx, \tag{4.24}$$

as desired. ∎

We now state a series of results which will come out to be crucial in order to manage some nontrivial boundary terms on Γ appearing in the apriori estimates performed in the sequel; for the sake of clarity, also in the statement of these more "theoretical" results we have preferred to keep our usual hypotheses and notations for the domains Ω_i, and the common boundary Γ. So, we introduce the space $H^1_{\Delta,-1/2}(\Omega_i) := \{v \in H^1(\Omega_i) : \Delta v \in H^{-1/2}(\Omega_i)\}$ (where $i = 1$ or $i = 2$, as usual), which, of course, is equipped with the graph norm; also, the constant $c > 0$ in the below statements may vary from line to line, but is naturally independent of the functions appearing on the left hand sides.

Proposition 4.1.2. *Let $u \in H^1_{\Delta,-1/2}(\Omega_i)$; then, we have*

$$\|\partial_{\mathbf{n}} u\|_{H^{-1/2}(\partial\Omega_i)} \le c(\|u\|_{H^1(\Omega_i)} + \|\Delta u\|_{H^{-1/2}(\Omega_i)}). \; ∎ \tag{4.25}$$

We do not give here the proof of the above proposition; notice indeed that it is essentially a consequence of the density of the space $C^\infty(\overline{\Omega})$ in $H^1_{\Delta,-1/2}(\Omega)$ (which can be showed by slightly modifying the proof of the weaker property provided by [42, Lemma 1.5.3.9, pp. 59–60]); in fact, all the terms of the generalized Gauss-Green formula (1.8) make sense also for $p = 2$ and $\mathbf{w} \in L^2(\Omega_1)^N$ with $\operatorname{div} \mathbf{w} \in H^{-1/2}(\Omega_1)$, provided that we intend the integral on the right hand side as a duality.

By virtue of Prop. 1.1.8, we can deduce a consequence of the above property, which will be effectively used in the sequel.

Corollary 4.1.3. *Let $u \in H^1(\Omega_i)$ such that $\Delta u \in L^2(\Omega_i)$ and $\partial_{\mathbf{n}} u = 0$ (in sense of traces) on $\partial\Omega_i \setminus \overline{\Gamma}$; then, we have*

$$\|\partial_{\mathbf{n}} u\|_{H^{-1/2}(\Gamma)} \le c(\|u\|_{H^1(\Omega_i)} + \|\Delta u\|_{H^{-1/2}(\Omega_i)}). \quad \blacksquare \tag{4.26}$$

Also, the above property will be used together with the following standard interpolation result [54, Prop. 1.2.3, page 19 and Th. 1.12.2, pages 71-72]

Lemma 4.1.4. *Let $v \in L^2(\Omega)$ and $K > 0$ such that $\|v\|_{L^2(\Omega)} \le c$ and $\|v\|_{H^{-1}(\Omega)} \le cK$. Then, we have that, $\|v\|_{H^{-\eta}(\Omega)} \le cK^\eta$ for every $0 \le \eta \le 1$, $\eta \ne 1/2$, where the last c is clearly independent of u, K, η.* $\blacksquare$

4.1.2 Limit for $\mu_{2,n} \to 0$

In this subsection, we let the parameter $\mu_{2,n}$ tend to 0 as $n \to \infty$; a formal examination of equations (4.1–4.4) then suggests that the limit problem should couple the standard phase-field model in Ω_1 with a time-stationary (in χ) phase-field model in Ω_2. However, since the time derivative of the χ appears also in the heat equation, a change of unknowns, introducing the enthalpy e, will be needed in order to study the convergence problem. Moreover, some supplementary hypotheses on α_1, α_2 will be assumed and also some coefficients of the problem will be supposed continuous at the interface Γ. The main mathematical interest of this analysis resides in the non-separability of the contributions of the two domains in the weak formulation of the problem, which causes the dependence on μ_2 of some estimates to fall also upon the Ω_1-components of the solutions; this forces the solution of the limit problem to be regarded in a weaker sense, also as the equations in Ω_1 are concerned (even if they are formally invariant with n).

The results reported in this subsection are essentially contained in the paper [78]; other asymptotic analyses of problems similar to this one, but related to the simpler case of a single substance, have been performed in the papers by Damlamian, Kenmochi, and Sato [34] and by Colli, Gilardi, and Grasselli [27] (in the case of the phase field model with memory effects).

First of all, assume that $\nu_{2,n} = \nu_2$, $\alpha_{2,n} = \alpha_2$ and $c_{2,n} = c_2$ for all $n \in \mathbb{N}$, while suppose that $\mu_{2,n} \to 0$ for $n \to \infty$. Also, define (as usual) the enthalpy $e_n := \rho\theta_n + \lambda\chi_n$; in this setting, in order to guarantee regularity in space also to e, we are forced to assume that

$$\rho_1 = \rho_2 =: \rho \quad \text{and} \quad \lambda_1 = \lambda_2 =: \lambda. \tag{4.27}$$

Furthermore, we set $\omega := \rho^{-1}$ and suppose that, for $i = 1, 2$, the (possibly multivalued) operator

$$\delta_i : r \mapsto \alpha_i(r) + (\lambda \ell_i \omega - c_i) r \quad \text{is (maximal) monotone in } \mathbb{R} \times \mathbb{R} \tag{4.28}$$

and denote it by δ_i. Then, accounting for (4.27–4.28), equations (4.1–4.4), together with the transmission conditions (4.9–4.11), can be easily rewritten in the form

$$\partial_t e_n - \operatorname{div}(k \nabla e_n) = f - \operatorname{div}(h \nabla \chi_n) \quad \text{a.e. in } Q, \tag{4.29}$$

$$\mu_n \partial_t \chi_n - \operatorname{div}(\nu \nabla \chi_n) + w_n = \ell \omega \, e_n \quad \text{a.e. in } Q, \tag{4.30}$$

$$w_{1,n} \in \delta_1(\chi_{1,n}) \quad \text{a.e. in } Q_1, \qquad w_{2,n} \in \delta_2(\chi_{2,n}) \quad \text{a.e. in } Q_2, \tag{4.31}$$

where we also defined $k_i := \kappa_i \omega$ and $h_i := \kappa_i \lambda \omega$.

The above system, complemented with the Cauchy and boundary conditions (4.6–4.8) rewritten in terms of the new unknowns and of the (naturally constructed) Cauchy data $e_{0,n}, \chi_{0,n}$, provides the suitable reformulation of Problem $(\mathrm{TP_n})$ for the asymptotic analysis which we now start. First of all, referring to the new unknowns, we recall some regularity hypotheses on the data of the approximating problems and specify some others concerning the limit ones:

$$e_{0,n}, \chi_{0,n} \in V \tag{4.32}$$

$$e_0 \in H \tag{4.33}$$

$$\chi_{1,0} \in L^2(\Omega_1) \tag{4.34}$$

$$f \in L^2(Q). \tag{4.35}$$

As far as the dependence on n is concerned, the minimal convergence-boundedness hypotheses on data are as below:

$$e_{0,n} \to e_0 \quad \text{in } H\text{–strong} \tag{4.36}$$

$$\chi_{1,0,n} \to \chi_{1,0} \quad \text{in } L^2(\Omega_1)\text{–strong} \tag{4.37}$$

$$\mu_{2,n}^{1/2} \chi_{2,0,n} \to 0 \quad \text{in } L^2(\Omega_2)\text{–strong} \tag{4.38}$$

$$\mu_{2,n} \|\theta_{0,n}\|_{H^1(\Omega)} \le M \tag{4.39}$$

$$\mu_{2,n}^{1/2} \|\chi_{0,n}\|_{H^1(\Omega)} \le M \tag{4.40}$$

$$\|\phi_1(\chi_{1,0,n})\|_{L^1(\Omega_1)} \le M \tag{4.41}$$

$$\mu_{2,n} \|\phi_2(\chi_{2,0,n})\|_{L^1(\Omega_2)} \le M. \tag{4.42}$$

In the above hypotheses, ϕ_i denotes the convex primitive of δ_i; moreover, the positive constant M is naturally independent of n.

Setting $V_1 := H^1(\Omega_1)$, $V_2 := H^1(\Omega_2)$, and defining the abstract elliptic operators

$$K : V^2 \to V', \qquad {}_{V'}\langle K(v,u), w \rangle_V := \int_\Omega k(x) \nabla v(x) \cdot \nabla w(x) \, dx + \int_{\partial\Omega} p \omega (v - \lambda u) w \, d\mathcal{H}^{N-1}, \tag{4.43}$$

$$\mathcal{N} : V \to V', \qquad {}_{V'}\langle \mathcal{N} v, w \rangle_V := \int_\Omega \nu(x) \nabla v(x) \cdot \nabla w(x) \, dx, \tag{4.44}$$

$$\mathcal{H} : V \to V', \qquad {}_{V'}\langle \mathcal{H} v, w \rangle_V := \int_\Omega h(x) \nabla v(x) \cdot \nabla w(x) \, dx, \tag{4.45}$$

for $v, u \in V$ and $w \in V'$, we can now state the mathematical formulation of the limit problem, which is the following one:

Problem 4.1.5. *Find*

$$e \in H^1(0,T;V') \cap L^2(0,T;V) \quad \text{and} \tag{4.46}$$

$$\chi \in L^2(0,T;V) \quad \text{with} \quad \chi_1 \in H^1(0,T;V_1') \tag{4.47}$$

such that the following equations hold for a.e. $t \in]0,T[$:

$$\partial_t e + K(e,\chi) = \widetilde{f} + \mathcal{H}\chi \quad \text{in } V', \tag{4.48}$$

$$\mu_1 \partial_t \chi_1 + \mathcal{N}\chi + w = \omega e \quad \text{in } V', \tag{4.49}$$

$$\chi_i \in D(\delta_i) \quad \text{and} \quad w_i \in \delta_i(\chi_i) \quad \text{a.e. in } Q_i. \ \blacksquare \tag{4.50}$$

In the above statement, $\widetilde{f}$ is a generalized source term also keeping into account the contribution of g on the boundary; moreover, $\mu_1 \partial_t \chi_1$ actually stands for its 0-extension to the whole Q. Here is the corresponding convergence result for the solutions of the transmission problem

Theorem 4.1.6. *Under the boundedness-convergence hypotheses* (4.36–4.42), *we have that Problem* 4.1.5 *admits a unique solution (e, χ) which is the limit of solutions of the transmission system (4.29–4.31) in the following sense:*

$$e_n \to e \quad \text{in } L^\infty(0,T;H)\text{–weak}^* \tag{4.51}$$

$$\chi_{1,n} \to \chi_1 \quad \text{in } L^\infty(0,T;L^2(\Omega_1))\text{–weak}^* \tag{4.52}$$

$$e_n \to e \quad \text{in } L^2(0,T;V)\text{–strong} \tag{4.53}$$

$$\chi_n \to \chi \quad \text{in } L^2(0,T;V)\text{–strong} \tag{4.54}$$

$$w_{i,n} \to w_i \quad \text{in } L^2(Q_i)\text{–weak.} \tag{4.55}$$

We also have the following additional convergences for the Ω_2-part:

$$\mu_{2,n}^{1/2} \chi_{2,n} \to 0 \quad \text{in } L^\infty(0,T;V_2)\text{–weak}^* \tag{4.56}$$

$$\mu_{2,n} \chi_{2,n} \to 0 \quad \text{in } H^1(0,T;L^2(\Omega_2))\text{–weak.} \tag{4.57}$$

Moreover,

$$-\nu_2 \Delta \chi_2 + w_2 = \lambda \chi_2 \quad \text{a.e. in } Q_2, \tag{4.58}$$

and $\partial_{\mathbf{n}} \chi_2 = 0$ in the sense of traces on Σ_2. Finally, we have the Cauchy conditions

$$e(0) = e_0 \tag{4.59}$$

$$\chi_1(0) = \chi_{1,0}. \tag{4.60}$$

We now perform the apriori estimates which are needed to obtain the required convergence of solutions. Since the computations are very similar to those of Subsec. 2.2.2, we shall only specify which test functions we choose, without operating the calculations. Some remarks will be given about the delicate points, where the supplementary hypotheses on coefficients become essential.

First of all, we recall a form of the Poincaré inequality that is useful to control some terms in Ω_2:

Lemma 4.1.7. *There exists a purely geometric constant* C_Ω *such that, for any* $v \in H^1(\Omega)$, *we have*

$$\|v_2\|^2_{L^2(\Omega_2)} \leq C_\Omega \left(\|\nabla v_2\|^2_{L^2(\Omega_2)} + \|v_1\|^2_{H^1(\Omega_1)} \right). \quad \blacksquare \tag{4.61}$$

First estimate. We multiply equation (4.29) by e_n and equation (4.3) by $m\chi_n$, where m is a (sufficiently large) positive constant (to be chosen). Notice that the property $e_n \in L^\infty(0,T;V)$, which is essential in order it is an admissible test function, is guaranteed by (4.27). Integrating over Q_t (for $t \leq T$) the obtained relations and summing together, on account of the monotonicity of the δ_i's and of the Young inequality (1.24), it is not difficult to get the following estimates ($K > 0$ depending only on data, and not of n, in particular):

$$\|e_n\|_{L^\infty(0,T;H)} \leq K \tag{4.62}$$

$$\|\chi_n\|_{L^2(0,T;V)} \leq K \tag{4.63}$$

$$\|e_n\|_{L^2(0,T;V)} \leq K \tag{4.64}$$

$$\mu_{i,n}^{1/2} \|\chi_{i,n}\|_{L^\infty(0,T;L^2(\Omega_i))} \leq K. \tag{4.65}$$

The only point concerning the computations which we wish to remark here is the use of the Poincaré inequality which is necessary in the estimation of the norm $\|\chi_{2,n}\|_{L^2(0,T;L^2(\Omega_2))}$ in terms of $\|\nabla\chi_{2,n}\|_{L^2(0,T;L^2(\Omega_2))}$ and the Ω_1-components.

Second estimate. Choose now $\partial_t\chi_n$ as a test function for equation (4.3). Proceeding in the standard way, and exploiting (1.24), it is not difficult to derive the following inequality ($C_2 > 0$ depending again only on data)

$$\begin{aligned}
&\|\mu_n^{1/2}\partial_t\chi_n\|^2_{L^2(Q_t)} + \|\nu^{1/2}\nabla\chi_n(t)\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \int_{\Omega_i} \phi_i(\chi_{i,n}(t))\, dx \\
&\quad \leq \|\nu^{1/2}\nabla\chi_{0,n}\|^2_{L^2(\Omega)} + \sum_{i=1}^{2} \int_{\Omega_i} \phi_i(\chi_{i,0,n})\, dx \\
&\quad + \sum_{i=1}^{2} \frac{\mu_{i,n}}{2} \|\partial_t\chi_{i,n}\|^2_{L^2(0,t;L^2(\Omega_i))} + C_2 \sum_{i=1}^{2} \mu_{i,n}^{-1} \|e_{i,n}\|^2_{L^2(0,t;L^2(\Omega_i))}
\end{aligned}$$

Notice now that we are forced to multiply the whole relation by $\mu_{2,n}$ in order to control by (4.62) the last term on the right hand side; indeed, due to the non-separability of the equations in Ω_1 and Ω_2, the factor μ_2 will fall also on the integrals on Ω_1 on the left hand side, preventing us from finding for them estimates independent of μ_2; so, what we get is

$$\mu_{2,n}^{1/2} \|\chi_{1,n}\|_{H^1(0,T;L^2(\Omega_1))} \leq K, \tag{4.66}$$

$$\mu_{2,n} \|\chi_{2,n}\|_{H^1(0,T;L^2(\Omega_2))} \leq K, \tag{4.67}$$

$$\mu_{2,n}^{1/2} \|\chi_n\|_{L^\infty(0,T;V)} \leq K, \tag{4.68}$$

$$\mu_{2,n} \|\phi_i(\chi_{i,n})\|_{L^\infty(0,T;L^1(\Omega_i))} \leq K. \tag{4.69}$$

Third estimate. Take $\partial_t\theta_n = \omega\,\partial_t(e_n - \lambda\chi_n)$ as a test function for (4.29) and integrate over $[0,t]$. Proceeding as before, it is easy to derive (σ is as in (1.25))

$$\omega\|\partial_t e_n\|^2_{L^2(0,t;H)} + \frac{\min\{\kappa_1,\kappa_2\}}{2}\|\nabla\theta_n(t)\|^2_H \le \frac{\max\{\kappa_1,\kappa_2\}}{2}\|\nabla\theta_{0,n}\|^2_H$$
$$\le C_\sigma\|f\|^2_{L^2(Q_t)} + \sigma\|\partial_t e_n\|^2_{L^2(0,t;H)} + C_\sigma\|\partial_t\chi_n\|^2_{L^2(0,t;H)},$$

whence, multiplying this relation by $\mu^2_{2,n}$, and recalling (4.66–4.67) and hypothesis (4.39), we easily infer

$$\mu_{2,n}\|e_n\|_{H^1(0,T;H)} \le K, \tag{4.70}$$
$$\mu_{2,n}\|e_n\|_{L^\infty(0,T;V)} \le K. \tag{4.71}$$

Proof of Theorem 4.1.6. Estimates (4.62–4.65), (4.67–4.68) immediately provide (4.51–4.52), (4.56–4.57), as well as the the additional weak convergence

$$(e_n,\chi_n)\to(e,\chi) \quad \text{in } L^2(0,T;V)^2\text{–weak}. \tag{4.72}$$

We remark that the dependence on $\mu_{2,n}$ gives rise to a lack of regularity in time which forbids to get, at this level, any strong convergence. Consequently, at present we can say nothing about the validity of relation (4.50). Now, rewriting the system (4.29–4.30) together with the Neumann conditions (4.7–4.8) in the variational form

$$\partial_t e_n + K(e_n,\chi_n) = \widetilde{f} + \mathcal{H}\chi_n \quad \text{in } V', \tag{4.73}$$
$$\mu_n\partial_t\chi_n + \mathcal{N}\chi_n + w_n = \omega e_n \quad \text{in } V' \tag{4.74}$$

and proceeding by comparison in (4.73), it is easy to get

$$\partial_t e_n \to \partial_t e \quad \text{in } L^2(0,T;V')\text{–weak}, \tag{4.75}$$

whence $e\in H^1(0,T;V')$ and (4.48) holds in the required sense. Also, the usual compactness theorems yield

$$e_n\to e \quad \text{in } C^0([0,T];H)\text{–weak}, \tag{4.76}$$
$$e_n\to e \quad \text{in } L^2(0,T;H)\text{–strong}. \tag{4.77}$$

Furthermore, owing to (4.76), we can recover from (4.36) and (4.6) the Cauchy condition (4.59).

Let us now subtract (4.48) from (4.73); recalling (4.72) and testing the resulting relation by $e_n - e$, with the aid of (1.24) and of the convergence hypotheses on the initial data, after easy computations (exploiting (4.36) and (4.77)), we obtain

$$\|e_n(T)-e(T)\|^2_H + \|\nabla(e_n-e)\|^2_{L^2(0,T;H)} \le R_{1,n} + \lambda^2\|\nabla(\chi_n-\chi)\|^2_{L^2(0,T;H)}, \tag{4.78}$$

where $R_{1,n}$ is a numerical sequence tending to 0 an n tends to ∞.

We now see that a similar procedure works also for the phase field equation; first, observe that, thanks to the regularity of Ω, there exists a "reflection-like" operator $R: V_1\to V$, that is a linear and continuous operator such that

$$(Rv_1)_{|\Omega_1} = v_1 \quad \text{for all } v_1\in V_1. \tag{4.79}$$

So, if we choose $w_1 \in \mathcal{D}(0,T;V_1)$ and set $w := Rw_1$, test (4.30) with w and integrate over Q, we get

$$\mu_1 \int_0^T \int_{\Omega_1} \partial_t \chi_{1,n} w_1 \, dx \, ds \leq C_3 \|w\|_{L^2(0,T;V)} \leq C_4 \|w_1\|_{L^2(0,T;V_1)}, \tag{4.80}$$

where C_3 depends only on the norms of w, $\nabla\chi$ and e in $L^2(Q)$ and on the norm of $\mu_{2,n}\partial_t\chi_{2,n}$ in $L^2(Q_2)$, which are all bounded, and C_4 is C_3 times the norm of the operator R. We conclude, like before, that

$$\partial_t \chi_{1,n} \to \partial_t \chi_1 \quad \text{in } L^2(0,T;V_1')\text{–weak} \tag{4.81}$$

and, recalling (4.72), thanks again to the Aubin theorem,

$$\chi_{1,n} \to \chi_1 \quad \text{in } L^2(0,T;L^2(\Omega_1))\text{–strong}, \tag{4.82}$$

$$\chi_{1,n} \to \chi_1 \quad \text{in } C^0([0,T];L^2(\Omega_1))\text{–weak}. \tag{4.83}$$

This is enough to pass to the limit in (4.30) and get back (4.49) in the specified sense.

Now, testing again equation (4.30) with $v \in L^2(0,T;V)$, integrating over $[0,T]$, adding and subtracting some terms, we get

$$\begin{aligned}
&{}_{L^2(0,T;V_1')}\langle \mu_1 \partial_t(\chi_{1,n} - \chi_1), v_1 \rangle_{L^2(0,T;V_1)} + \mu_{2,n} \int_0^T \int_{\Omega_2} \partial_t \chi_{2,n} v_2 \, dx \, ds \\
&\quad + \sum_{i=1}^2 \nu_i \int_0^T \int_{\Omega_i} \nabla(\chi_{i,n} - \chi_i) \cdot \nabla v_i \, dx \, ds + \sum_{i=1}^2 \int_0^T \int_{\Omega_i} w_{i,n} v_i \, dx \, ds \\
&= -{}_{L^2(0,T;V_1')}\langle \mu_1 \partial_t \chi_1, v_1 \rangle_{L^2(0,T;V_1)} - \sum_{i=1}^2 \nu_i \int_0^T \int_{\Omega_i} \nabla\chi_i \cdot \nabla v_i \, dx \, ds \\
&\quad + \int_0^T \int_\Omega \ell\omega(e_n - e) v \, dx \, ds + \int_0^T \int_\Omega \ell\omega e v \, dx \, ds,
\end{aligned} \tag{4.84}$$

In particular, for $v = \chi_n - \chi$, thanks also to (4.83), we get

$$\begin{aligned}
&\frac{\mu_1}{2} \|\chi_{1,n}(T) - \chi_1(T)\|^2_{L^2(\Omega_1)} + \frac{\mu_2}{2} \|\chi_{2,n}(T)\|^2_{L^2(\Omega_2)} + \sum_{i=1}^2 \nu_i \|\nabla(\chi_{i,n} - \chi_i)\|^2_{L^2(0,T;L^2(\Omega_i))} \\
&= -{}_{L^2(0,T;V_1')}\langle \mu_1 \partial_t \chi_1, \chi_{1,n} - \chi_1 \rangle_{L^2(0,T;V_1)} - \sum_{i=1}^2 \int_0^T \int_{\Omega_i} w_{i,n}(\chi_{i,n} - \chi_i) \, dx \, ds \\
&\quad - \sum_{i=1}^2 \nu_i \int_0^T \int_{\Omega_i} \nabla\chi_i \cdot \nabla(\chi_{i,n} - \chi_i) \, dx \, ds \\
&\quad + \int_0^T \int_\Omega \ell\omega(e_n - e)(\chi_n - \chi) \, dx \, ds + \int_0^T \int_\Omega \ell\omega e(\chi_n - \chi) \, dx \, ds \\
&\quad + \frac{\mu_1}{2} \|\chi_{1,0,n} - \chi_{1,0}\|^2_{L^2(\Omega_1)} + \frac{\mu_2}{2} \|\chi_{2,0,n}\|^2_{L^2(\Omega_2)} + \mu_2 \int_0^T \int_{\Omega_2} \partial_t \chi_{2,n} \chi_2 \, dx \, ds.
\end{aligned} \tag{4.85}$$

Adding now the preceding relation to m times (4.78) (with m to be chosen sufficiently small), recalling the convergences (4.57), (4.72), (4.77) and (4.82), and the hypotheses (4.37–4.38), and taking the lim sup as $n \to \infty$ of the resulting equality, by lower semicontinuity of ϕ (indeed of its L^2-extension) it is easy to derive (4.53) and also

$$\|\nabla \chi^\mu - \nabla \chi\|_{L^2(0,T;H)} \to 0,$$

whence we deduce (4.54) by recalling (4.82) and the Poincaré inequality (4.61); furthermore, relation (4.50) is, as before, a consequence of (4.54), (4.55) and Prop. 1.3.8.

Recalling (4.75) and (4.81), we get also the regularities (4.46–4.47); furthermore, reasoning by comparison in (4.30), we get that

$$D\chi_{2,n} \to D\chi_2 \quad \text{in } L^2(Q_2)\text{–weak,}$$

so that (4.58) holds along with the related Neumann boundary condition on Γ_2; this concludes the proof of existence. ∎

Proof of uniqueness. Let us suppose we have a pair of solutions $(\hat{e}, \hat{\chi}, \hat{w})$, $(\check{e}, \check{\chi}, \check{w})$ to the system (4.48–4.50), (4.59–4.60). Set $e := \hat{e} - \check{e}$, $\chi := \hat{\chi} - \check{\chi}$, $w := \hat{w} - \check{w}$. The constants $C_5, C_6, C_7 > 0$ in the following computations only depend on data.

Substitute first $(\hat{e}, \hat{\chi})$ and then $(\check{e}, \check{\chi})$ in equation (4.48), take the difference and test with e, easily obtaining for every $t \in [0, T]$ the inequality

$$\|e(t)\|_H^2 + \|\nabla e\|_{L^2(0,t;H)}^2 \le C_5 \|\nabla \chi\|_{L^2(0,t;H)}^2. \tag{4.86}$$

Applying a similar procedure to equation (4.49) and taking into account the monotonicity of δ_i, we get, for $\sigma > 0$ and for every $t \in [0, T]$,

$$\|\chi_1(t)\|_{L^2(\Omega_1)}^2 + \|\nabla \chi_i\|_{L^2(Q_t)}^2 \le C_6 \left(C_\sigma \|e\|_{L^2(0,t;H)}^2 + \sigma \|\chi\|_{L^2(0,t;H)}^2 \right). \tag{4.87}$$

Now, take the sum of m times (4.86) with (4.87), with $m > 0$, as usual, to be chosen later; exploiting (4.61), we derive

$$\begin{aligned} &\|\chi_1(t)\|_{L^2(\Omega_1)}^2 + \|\nabla \chi\|_{L^2(Q_t)}^2 + m\|e(t)\|_H^2 + m\|\nabla e\|_{L^2(0,t;H)}^2 \\ &\quad \le C_7 (C_\sigma \|e\|_{L^2(0,t;H)}^2 + \sigma C_\Omega \|\nabla \chi\|_{L^2(0,t;H)}^2 \\ &\quad + m\|\nabla \chi\|_{L^2(0,t;H)}^2 + \sigma(1 + C_\Omega) \|\chi_1\|_{L^2(0,t;L^2(\Omega_1))}^2). \end{aligned} \tag{4.88}$$

At this point, if we take m and σ sufficiently small, an application of the Gronwall lemma allows us to complete the proof. ∎

4.1.3 Limit for $\nu_{2,n} \to 0$

We consider again a family of transmission problems, as described in Subsec. 4.1.1, and we wish to study their behaviour as we let the interfacial energy coefficient relative to the substance in Ω_2, i.e. $\nu_{2,n}$, tend to 0, while we keep $\mu_{2,n} = \mu_2$, $\alpha_{2,n} = \alpha_2$, $\alpha_{1,n} = \alpha_1$, $c_{2,n} = c_2$ fixed with respect to n. By formally observing the original equations, we expect to get as a limit a transmission system coupling the phase-field model in Ω_1 with the phase-relaxation one [85] in Ω_2; moreover, due to the vanishing of the phase diffusion term in Ω_2, a variation of the boundary conditions for χ is also expected to occur.

We point out that, in the previous analysis, the convergence result did not depend on the particular kind of conditions ((CC) or (GC)) ensuring the existence of the solution to the approximating system; here, instead, since the spatial diffusion of χ is varying, we are forced to confine the study to the case of (CC); for the growth conditions setting, we think that the problem should be addressed by methods similar to those of the subsequent Section 4.2, to which we refer.

First of all, we list some regularity-convergence assumptions on the initial data which, naturally, complement those described in Subsec. 4.1.1:

$$\chi_{1,0,n} \to \chi_{1,0} \quad \text{in } V\text{–strong,} \tag{4.89}$$

$$\nu_{2,n}^{1/2}\chi_{2,0,n} \to 0 \quad \text{in } V\text{–strong,} \tag{4.90}$$

$$\chi_{2,0,n} \to \chi_{2,0} \quad \text{in } L^2(\Omega_2)\text{–strong,} \tag{4.91}$$

$$\int_{\Omega_i} j_i(\chi_{i,0,n})\,dx \le C \quad \text{for all } n \in \mathbb{N},\ i = 1,2. \tag{4.92}$$

Also, we have to introduce two new elliptic operators $\mathcal{K}$ and $\mathcal{N}_1$; while the first definition is standard, we point out that it is the continuity of the restriction operator $V \to H^1(\Omega_1)$ which permits to construct $\mathcal{N}_1$:

$$\mathcal{K} : V \to V', \qquad {}_{V'}\langle \mathcal{K}v, w\rangle_V := \int_\Omega \kappa(x)\nabla v(x)\cdot\nabla w(x)\,dx + \int_{\partial\Omega} p v w\, d\mathcal{H}^{N-1}, \tag{4.93}$$

$$\mathcal{N}_1 : V \to V', \qquad {}_{V'}\langle \mathcal{N}_1 v, w\rangle_V := \int_{\Omega_1} \nu_1 \nabla v_1(x)\cdot\nabla w_1(x)\,dx. \tag{4.94}$$

The above assumptions are enough to state the precise mathematical formulation of the limit problem and the corresponding convergence theorem:

Problem 4.1.8. *We look for a triplet of functions (θ, χ, ξ), such that the following regularity properties hold*

$$\theta \in H^1(0,T;H) \cap C^0([0,T];V), \qquad \mathcal{K}\theta \in L^2(Q), \tag{4.95}$$

$$\chi \in H^1(0,T;H), \qquad \chi_1 \in C^0([0,T];V_1), \qquad \Delta\chi_1 \in L^2(0,T;L^2(\Omega_1)) \tag{4.96}$$

$$\xi \in L^2(Q) \tag{4.97}$$

and that the equations below are satisfied:

$$\rho\partial_t\theta + \lambda\partial_t\chi - \operatorname{div}(\kappa\nabla\theta) = f \qquad a.e.\ in\ Q, \tag{4.98}$$

$$\mu\partial_t\chi + \mathcal{N}_1\chi + \xi - c\chi = \ell\theta \qquad in\ V' \quad a.e.\ in\ ]0,T[, \tag{4.99}$$

$$\xi_i \in \alpha_i(\chi_i) \qquad a.e.\ in\ Q_i \tag{4.100}$$

together with the boundary condition (4.8) and the Cauchy ones $\theta(0) = \theta_0$, $\chi(0) = \chi_0$.

Theorem 4.1.9. *Problem* 4.1.8 *admits a unique solution; moreover, the following convergences are fulfilled:*

$$\theta_n \to \theta \quad \text{in } L^\infty(0,T;V)\text{–weak}^*, \tag{4.101}$$

$$\theta_n \to \theta \quad \text{in } H^1(0,T;H)\text{–weak}, \tag{4.102}$$

$$\chi_n \to \chi \quad \text{in } H^1(0,T;H)\text{–weak}, \tag{4.103}$$

$$\chi_{2,n} \to \chi_2 \quad \text{in } L^2(0,T;L^2(\Omega_2))\text{–strong}, \tag{4.104}$$

$$\chi_{1,n} \to \chi_1 \quad \text{in } L^\infty(0,T;V_1)\text{–weak}^*, \tag{4.105}$$

$$\nu_{2,n}^{1/2}\chi_{2,n} \to 0 \quad \text{in } L^\infty(0,T;V_2)\text{–weak}^*, \tag{4.106}$$

$$\nu_{2,n}^{1/2}\chi_{2,n} \to 0 \quad \text{in } L^2(0,T;V_2)\text{–strong}. \tag{4.107}$$

Remark 4.1.10. *We point out that, owing to the regularity of solutions, equation (4.99) is equivalent to the following system, splitting the contributions of* Ω_1 *and* Ω_2*:*

$$\mu_1\partial_t\chi_1 - \nu_1\Delta\chi_1 + \xi_1 - c_1\chi_1 = \ell_1\theta_1 \quad a.e. \text{ in } Q_1, \tag{4.108}$$

$$\mu_2\partial_t\chi_2 + \xi_2 - c_2\chi_2 = \ell_2\theta_2 \quad a.e. \text{ in } Q_2, \tag{4.109}$$

complemented with the Neumann condition $\partial_{\mathbf{n}}\chi_1 = 0$ *in sense of traces on* $\partial\Omega_1\times]0,T[$.

We now start by deriving the apriori estimates on which the n-limit procedure is based; since the arguments are rather standard, in the following we shall omit most of the computations, proposing anyway to remark the delicate points.

First estimate. Test equations (4.1–4.2) by $\theta_{1,n}$ and $\theta_{2,n}$ respectively and sum together; integrate over $]0,t[$, for $t \le T$ and exploit the transmission conditions. Then, it is easy to infer, for some constant C_1 only depending on data,

$$\|\theta_n(t)\|_H^2 + \|\nabla\theta_n\|_{L^2(Q_t)}^2 \le C_1\left(\|\partial_t\chi_n\|_{L^2(Q_t)}^2 + \|\theta\|_{L^2(Q_t)}^2 + 1\right). \tag{4.110}$$

Multiplying instead (4.3–4.4) by $\chi_{1,n} + \partial_t\chi_{1,n}$ and by $\chi_{2,n} + \partial_t\chi_{2,n}$, respectively, integrating over Q_t, exploiting the monotonicity of α_i and Young's inequality, it is easy to infer (with C_1 possibly different from above, but still depending only on coefficients)

$$\begin{aligned}
&\|\partial_t\chi_n\|_{L^2(Q_t)}^2 + \|\chi_n(t)\|_H^2 + \nu_1\|\nabla\chi_{1,n}\|_{L^2(0,t;L^2(\Omega_1))}^2 + \nu_{2,n}\|\nabla\chi_{2,n}\|_{L^2(0,t;L^2(\Omega_2))}^2\\
&\qquad + \nu_1\|\nabla\chi_{1,n}(t)\|_{L^2(\Omega_1)}^2 + \nu_{2,n}\|\nabla\chi_{2,n}(t)\|_{L^2(\Omega_2)}^2 + \sum_{i=1}^2\int_{\Omega_i} j_i(\chi_i)(t)\,dx\\
&\le C_1\left(C_\sigma\|\theta_n\|_{L^2(Q_t)}^2 + \sum_{i=1}^2\int_{\Omega_i} j_i(\chi_{i,0,n})\,dx + C_\sigma\|\chi_n\|_{L^2(Q_t)}^2\right)\\
&\qquad + C_1\left(\sigma\|\partial_t\chi_n\|_{L^2(Q_t)}^2 + \sum_{i=1}^2\nu_i\|\nabla\chi_{i,0,n}\|_{L^2(\Omega_i)}^2\right)
\end{aligned} \tag{4.111}$$

Now, summing the previous two relations together and exploiting the standard Gronwall inequality, we can derive, up to the extraction of subsequences, the convergences (4.103) and (4.105–4.106), as well as

$$\theta_n \to \theta \quad \text{in } L^2(0,T;V)\text{–weak}, \tag{4.112}$$

$$\theta_n \to \theta \quad \text{in } L^\infty(0,T;H)\text{–weak}^*. \tag{4.113}$$

Second estimate. Testing (4.1–4.2) with $\partial_t\theta_{1,n}$ and $\partial_t\theta_{2,n}$, respectively, and performing standard computations, on account of (4.103) it is easy to infer the validity of convergences (4.101–4.102).

The procedure used up to now works both in the case of (CC) and of (GC); also, by means of the usual compactness theorems, some strong convergences follow:

$$\theta_n \to \theta \quad \text{in } C^0(0,T;H)\text{–strong,} \tag{4.114}$$

$$\chi_{1,n} \to \chi_1 \quad \text{in } C^0(0,T;L^2(\Omega_1))\text{–strong.} \tag{4.115}$$

Unfortunately, no strong convergence is available yet for $\chi_{2,n}$ and again it will be necessary to derive it through a direct approach. However, we have to manage the terms $\xi_{i,n}$, before, and this is the purpose of the next estimate.

Third estimate. We perform a procedure similar to that of Lemma 3.2.8; however, some more attention need to be paid, since here the approximating operators α_1 and α_2 need not be Lipschitz continuous. Thus, we consider the test function $\widetilde{\xi_n}$ defined as $\widetilde{\xi}_{1,n} := \xi_{1,n} + \alpha_2^0(\chi_{1,n})$ in Q_1 and as $\widetilde{\xi}_{2,n} := \xi_{2,n} + \alpha_1^0(\chi_{2,n})$ in Q_2 (for the definition of α_1^0 and α_2^0, we refer to Subsec. 1.2.2); on account of (CC), it is easy to verify that $\widetilde{\xi_n} \in L^2(Q)$. Hence, multiplying (4.3) by $\widetilde{\xi}_{1,n}$ and (4.4) by $\widetilde{\xi}_{2,n}$, we can apply Lemma 4.1.1 to integrate by parts the term $-\operatorname{div}(\nu_n\nabla\chi_n)\widetilde{\xi_n}$ (note that no troubles arise from the dependence on n of ν_n), so that, following the proof of Lemma 3.2.8, it is easy to obtain, for subsequences,

$$\xi_n \to \xi \quad \text{in } L^2(0,T;H)\text{–weak.} \tag{4.116}$$

By virtue of the monotonicity argument of Prop. 1.3.8, (4.115) and (4.116) are enough to identify $\xi_1 \in \alpha_1(\chi_1)$. As far as ξ_2 is concerned instead, for the present we have no strong convergence for $\chi_{2,n}$ and again such a property has to be checked directly. The below argument, anyway, presents further difficulties with respect to case of the previous subsection; indeed, no help can now be derived from the norm $\|\nabla\chi_{2,n}\|_H$, due to the blow-out $\nu_{2,n} \to 0$.

Strong convergence. First of all, owing to convergences (4.102–4.103) and (4.116) and proceeding by comparison in equations (4.1–4.2), we easily derive

$$\Delta\chi_{1,n} \to \Delta\chi_1 \quad \text{in } L^2(0,T;L^2(\Omega_1))\text{–weak,} \tag{4.117}$$

$$\nu_{2,n}\Delta\chi_{2,n} \to 0 \quad \text{in } L^2(0,T;L^2(\Omega_2))\text{–weak.} \tag{4.118}$$

In particular, this allows to recover relations (4.108–4.109); nothing, anyway, can be said yet about the boundary conditions and the interpretation of ξ_2. So, we proceed with a change of unknowns, by setting now $z_{2,n} := e^{-c_2\mu_2^{-1}t}\chi_{2,n}$. Notice that $z_{2,n}$ has the same regularity as $\chi_{2,n}$. Moreover, computing the time derivative, it is easy to verify that, in the new unknown, the equation (4.4) and the second of conditions (4.5) become

$$\mu_2\partial_t z_{2,n} - \nu_{2,n}\Delta z_{2,n} + \xi_{2,n}e^{-c_2\mu_2^{-1}t} = \ell_2 e^{-c_2\mu_2^{-1}t}\theta_{2,n}, \tag{4.119}$$

$$\xi_{2,n} \in \beta_2(e^{c_2\mu_2^{-1}t}z_{2,n}) \quad \text{almost everywhere.} \tag{4.120}$$

Moreover, $z_{2,n}$ still satisfies homogeneous Neumann boundary conditions on $\partial\Omega_2 \setminus \overline{\Gamma}$ and transmission ones on Γ; in addition, thanks to (4.103), it is easy to check that

$$z_{2,n} \to z_2 := e^{-c_2\mu_2^{-1}t}\chi_2 \quad \text{in } H^1(0,T;L^2(\Omega_2))\text{–weak}. \tag{4.121}$$

Subtracting now the term $\mu_2\partial_t z_2$ from both hand sides of (4.119) and testing the result by $z_{2,n} - z_2$, integrating over $]0,t[$ with $t \le T$, we infer

$$\begin{aligned}
&\frac{\mu_2}{2}\|(z_{2,n} - z_2)(t)\|^2_{L^2(\Omega_2)} - \nu_{2,n}\int_0^t\int_{\Omega_2} \Delta z_{2,n}\, z_{2,n}\, dx\, ds \\
&= \frac{\mu_2}{2}\|\chi_{2,0,n} - \chi_{2,0}\|^2_{L^2(\Omega_2)} - \nu_{2,n}\int_0^t\int_{\Omega_2} \Delta z_{2,n}\, z_2\, dx\, ds \\
&\quad + \ell_2\int_0^t\int_{\Omega_2} e^{-c_2\mu_2^{-1}s}\theta_{2,n}(z_{2,n} - z_2)\, dx\, ds - \mu_2\int_0^t\int_{\Omega_2} \partial_t z_2(z_{2,n} - z_2)\, dx\, ds \\
&\quad + \int_0^t\int_{\Omega_2} \xi_{2,n} e^{-c_2\mu_2^{-1}s}(z_2 - z_{2,n})\, dx\, ds
\end{aligned} \tag{4.122}$$

We start by discussing the term with the Laplacean on the left hand side. Actually, we can integrate it by parts, but on account of the mixed boundary conditions (homogeneous Neumann on $\partial\Omega\setminus\overline{\Gamma}$ and transmission on Γ), we also get a surface term:

$$\begin{aligned}
&- \nu_{2,n}\int_0^t\int_{\Omega_2} \Delta z_{2,n}\, z_{2,n}\, dx\, ds \\
&\quad = \nu_{2,n}\int_0^t\int_{\Omega_2} |\nabla z_{2,n}|^2\, dx\, ds - \nu_{2,n}\int_0^t \langle \partial_{\mathbf{n}} z_{2,n}(s), z_{2,n}(s)\rangle\, ds \\
&\quad = \nu_{2,n}\int_0^t\int_{\Omega_2} |\nabla z_{2,n}|^2\, dx\, ds - \nu_{2,n}\int_0^t e^{-2c_2\mu_2^{-1}s}\langle \partial_{\mathbf{n}}\chi_{2,n}(s), \chi_{2,n}(s)\rangle\, ds.
\end{aligned} \tag{4.123}$$

Notice that the above duality is written in the correct functional spaces $H^{-1/2}(\Gamma)$, $H^{1/2}(\Gamma)$ by virtue of Prop. 1.1.8. Now, the first term on the right hand side of the preceding relation gives a nonnegative contribution; the second one, instead, must be split and estimated by means of Lemma 4.1.3 this way:

$$\begin{aligned}
&\nu_{2,n}\int_0^t \left| e^{-2c_2\mu_2^{-1}s}\langle \partial_{\mathbf{n}}\chi_{2,n}(s), \chi_{2,n}(s)\rangle\right| ds \\
&\qquad \le C_2\nu_{2,n}\|\partial_{\mathbf{n}}\chi_{2,n}\|_{L^2(0,t;H^{-1/2}(\Gamma))}\|\chi_{1,n}\|_{L^2(0,t;H^1(\Omega_1))} \\
&\quad \le C_2\nu_{2,n}\big(\|\chi_{2,n}\|_{L^2(0,t;H^1(\Omega_2))} + \|\Delta\chi_{2,n}\|_{L^2(0,t;H^{-1/2}(\Omega_2))}\big)\|\chi_{1,n}\|_{L^2(0,t;H^1(\Omega_1))}
\end{aligned} \tag{4.124}$$

We now observe that, from (4.106), (4.118) it immediately follows, for some constant $C > 0$ not depending on n,

$$\|\Delta\chi_{2,n}\|_{L^2(0,T;H^{-1}(\Omega_2))} \le C\nu_{2,n}^{-1/2}, \qquad \|\Delta\chi_{2,n}\|_{L^2(0,T;L^2(\Omega_2))} \le C\nu_{2,n}^{-1}, \tag{4.125}$$

whence, using Lemma 4.1.4 with the choice of $\eta = 1/2 - \delta$ with δ arbitrarily small (we cannot take $\delta = 0$ because the exponent $\eta = 1/2$ is not good for the quoted lemma), we have that, for some $C_3(\delta) > 0$ independent of n,

$$\nu_{2,n}\big(\|\chi_{2,n}\|_{L^2(0,t;H^1(\Omega_2))} + \|\Delta\chi_{2,n}\|_{L^2(0,t;H^{-1/2}(\Omega_2))}\big) \le C_3(\delta)\,\nu_{2,n}^{1/4-\delta/2} \tag{4.126}$$

for every $\delta \in]0,1/2[$, where we remark that $C_3(\delta)$ does not even depend on the choice of $t \in [0,T]$. Thus, the contribution of the boundary term in (4.123) is actually negligible as n tends to ∞.

Let us now work with the right hand side of (4.122). Due to (4.91), (4.114), to the continuity and boundedness of the function $e^{-c_2\mu_2^{-1}t}$, and to the weak convergences (4.121) and

$$\nu_{2,n}\Delta z_{2,n} \to 0 \quad \text{in } L^2(0,T;L^2(\Omega_2))\text{–weak} \tag{4.127}$$

(which immediately follows from (4.118)), we are able to treat all the terms save the lattest one, deserving a more careful analysis that we now perform.

First of all, by definition of subdifferential, recalling (4.120), we have

$$\xi_{2,n}(x,s)e^{-c_2\mu_2^{-1}s}(z_2 - z_{2,n})(x,s) \le e^{-2c_2\mu_2^{-1}s}\big(j_2(\chi_2(x,s)) - j_2(\chi_{2,n}(x,s))\big) \tag{4.128}$$

almost everywhere in $\Omega_2\times]0,t[$. Remarking that the evaluation operator

$$\delta_s : H^1(0,t;L^2(\Omega_2)) \to L^2(\Omega_2), \qquad \delta_s : v \mapsto v(s) \tag{4.129}$$

is linear and continuous for every $s \in [0,t]$, using (4.103) and the semicontinuity of j_2, we infer

$$\int_{\Omega_2} j_2(\chi_2(x,s))\,dx \le \liminf_{n\to\infty} \int_{\Omega_2} j_2(\chi_{2,n}(x,s))\,dx \tag{4.130}$$

for every $s \in [0,t]$, whence, on account of the nonnegativity of j_2 and of Fatou's lemma, it easily follows

$$\int_0^t e^{-2c_2\mu_2^{-1}s}\int_{\Omega_2} j_2(\chi_2(x,s))\,dx\,ds \le \liminf_{n\to\infty}\int_0^t e^{-2c_2\mu_2^{-1}s}\int_{\Omega_2} j_2(\chi_{2,n}(x,s))\,dx\,ds, \tag{4.131}$$

whence, comparing with (4.128), we can pass to the supremum limit for $n \to \infty$ in relation (4.122). Choosing now $t = T$ and recalling (4.123), we immediately infer (4.107), while, for general t, we have

$$\chi_{2,n}(t) \to \chi(t) \quad \text{in } H\text{–strong}, \quad \text{for every } t \in [0,T]. \tag{4.132}$$

Now, this relation easily implies (4.104), provided that we recall (4.129) and exploit the dominated convergence theorem. The proof of Theorem 4.1.9 is now complete, since the Neumann condition $\partial_{\mathbf{n}}\chi_1 = 0$ on $\partial\Omega_1\times]0,T[$ follows now by passing to the limit in (4.11) in the suitable topology ($H^{-1/2}(\Gamma)$); accounting for (4.108–4.109) this permits to get back the variational form (4.99) of the limit phase-field equation. ∎

Uniqueness. It can be proved in the standard way, by substituting a couple of solutions in the system (4.98–4.100), taking the difference of the obtained relations and multiplying it by the difference of solutions. We do not report the explicit computations, since they are even simpler than in the previous case; indeed, now we do not have here the complications deriving from the use of Poincaré's inequality.

4.1.4 Limit for $\alpha_n \to \alpha$

Here, we give a convergence theorem for Problem ($\mathrm{TP_n}$), which is related to the case where a variation is allowed for the monotone graph $\alpha_{2,n}$; indeed, unlike the previous two cases, we also allow the graph α_1 to change, since no further difficulties are provided by this choice. As for the rest, the n-setting is the usual one discussed in Subsec. 4.1.1; thus, we start by listing the required convergence hypotheses on data and coefficients; $C_1 > 0$ below is a fixed constant independent of n.

$$\alpha_{i,n}, \alpha_i \quad \text{such that } 0 \in \alpha_{i,n}(0), 0 \in \alpha_i(0), \quad \text{for } i = 1, 2, \tag{4.133}$$

$$j_{i,n}, j_i \quad \text{convex primitives of } \alpha_{i,n}, \alpha_i, \quad \text{respectively,} \tag{4.134}$$

$$\alpha_{i,n} \to \alpha_i \quad \text{in the sense of graphs in } \mathbb{R} \times \mathbb{R}, \tag{4.135}$$

$$j_{i,n}(r) \leq j_i(r) \quad \text{for all } r \in \mathbb{R}, \quad n \in \mathbb{N}, \quad i = 1, 2 \tag{4.136}$$

$$\chi_{i,0,n} \in V, \quad \chi_{i,0,n}(x) \in D(j_{i,n}) \quad \text{a.e. in } \Omega_i, \tag{4.137}$$

$$\int_{\Omega_i} j_{i,n}(\chi_{i,0,n})\, dx \leq C_1 \quad \text{for } i = 1, 2, \quad n \in \mathbb{N}. \tag{4.138}$$

We point out that assumption (4.136) is fulfilled if, for instance, $\alpha_{i,n}$ is the Yosida-approximation of α_i. Also, we require that the graphs $\alpha_{1,n}$, $\alpha_{2,n}$ satisfy either one of conditions (CC), (GC) of Subsec. 4.1.1 for every $n \in \mathbb{N}$. Moreover, we need some uniformity with respect to n. Namely, in the case of (CC), we assume that the constant $C_{\alpha,n}$ in (4.12) does not depend on n; when (GC) are assumed instead, we require that condition also for the limit graphs α_1, α_2.

We can now report the convergence theorem; notice that some distinctions are present in the statement, depending on the occurrence of either (CC) or (GC).

Theorem 4.1.11. *We have that the solution $(\theta_n, \chi_n, \xi_n)$ of Problem (TP_{n}) tends, for $n \to \infty$, to a triplet of functions (θ, χ, ξ) in the following sense:*

$$\theta_n \to \theta \quad \text{in } H^1(0,T;H)\text{–weak and in } L^\infty(0,T;V)\text{–weak}^*, \tag{4.139}$$

$$\chi_n \to \chi \quad \text{in } H^1(0,T;H)\text{–weak and in } L^\infty(0,T;V)\text{–weak}^*, \tag{4.140}$$

$$\chi_n \to \chi \quad \text{in } L^2(0,T;V)\text{–strong.} \tag{4.141}$$

Depending on the conditions assumed on the $\alpha_{i,n}$'s, we also have the following convergences of the nonlinear terms

$$\xi_n \to \xi \quad \text{in } L^2(0,T;V')\text{–weak} \quad \text{in the case of (GC)}, \tag{4.142}$$

$$\xi_n \to \xi \quad \text{in } L^2(0,T;H)\text{–weak} \quad \text{in the case of (CC)}. \tag{4.143}$$

Moreover, the following equations

$$\rho \partial_t \theta + \lambda \partial_t \chi - \operatorname{div}(\kappa \nabla \theta) = f \quad \text{a.e. in } Q; \tag{4.144}$$

$$\mu \partial_t \chi - \operatorname{div}(\nu \nabla \chi) + \xi - c\chi = \ell \theta \quad \text{a.e. in } Q; \tag{4.145}$$

$$\xi_1 \in \alpha_1(\chi_1) \quad \text{a.e. in } Q_1; \quad \xi_2 \in \alpha_2(\chi_2) \quad \text{a.e. in } Q_2. \tag{4.146}$$

are fulfilled, as well as the initial conditions $\theta(0) = \theta_0$, $\chi(0) = \chi_0$ and the Neumann boundary ones $\partial_{\mathbf{n}}\chi = 0$, $(\kappa \nabla \theta) \cdot \mathbf{n} + p\theta = g$ on $\partial\Omega \times]0, T[$.

Proof. The argument is based as usual on some a priori estimates together with a direct proof of a strong convergence property; naturally, since the limit problemt is formally analogous to the original one, the only interesting part of the theorem is the convergence one. Clearly, also the uniqueness of solutions for the limit statement has already been showed.

First estimate. We proceed exactly as in the first estimate of the previous Subsection; the resulting computations are precisely as in (4.110–4.111), with only two differences, which concern the substitutions of $\nu_{2,n}$ with ν_2 and of j_i with $j_{i,n}$ in (4.111); also, we observe that the terms related to the initial values are now bounded by virtue of (4.138). Thus, the convergences (4.140) are easily proved, as well as the following further estimates:

$$\|\theta_n\|_{L^2(0,T;V)} \leq C \quad \text{for every } n \in \mathbb{N}, \tag{4.147}$$

$$\|\theta_n\|_{L^\infty(0,T;H)} \leq C \quad \text{for every } n \in \mathbb{N}, \tag{4.148}$$

$$\|j_{i,n}(\chi_{i,n})\|_{L^\infty(0,T;L^1(\Omega_i))} \leq C \quad \text{for every } n \in \mathbb{N}, \ \ i = 1, 2. \tag{4.149}$$

Second estimate. Proceeding exactly as in the second estimate of the previous subsection, we immediately infer the convergence (4.139).

At this point, the procedure depends on the kind of conditions concerning α_1 and α_2 that are fulfilled. We first proceed in the (simpler, as usual) case of (CC). In this framework, the natural convergence setting is the L^2-one; consequently, we perform a further apriori estimate as in the former case, by multiplying the phase-field equation (4.3–4.4) by the test function $\widetilde{\xi}_n$, which is defined in a similar way as before, i.e., $\widetilde{\xi}_{1,n} := \xi_{1,n} + \alpha^0_{2,n}(\chi_{1,n})$ and $\widetilde{\xi}_{2,n} := \xi_{2,n} + \alpha^0_{1,n}(\chi_{2,n})$. By virtue of the n-uniformity of condition (CC) Lemma 4.1.1 can be exploited and the convergences (4.143) follow. Now, thanks to the graph convergence $\alpha_{i,n} \to \alpha_i$, the monotonicity argument of Prop. 1.3.8 works, since the strong convergence in L^2 for the χ_n is guaranteed by (4.140) and the usual compact embedding theorems.

Growth conditions case. Due to the usual troubles related to condition (GC), we have to come back to the (V, V')-setting and to perform weaker estimates. As in the previous chapter, a weaker solution will correspondently be obtained, and the condition (GC) (only related to the limit graphs) will be exploited to interpret it in the physical sense. The first step is then to establish a strong convergence in V of the phase variable.

Strong convergence. We first introduce the elliptic operator $\mathcal{N}$ as

$$\mathcal{N} : V \to V', \qquad {}_{V'}\langle \mathcal{N}v, w\rangle_V := \int_\Omega \nu(x)\nabla v(x) \cdot \nabla w(x)\, dx, \tag{4.150}$$

which allows to rewrite (4.3–4.4), together with the transmission conditions (4.9) and (4.11), in the following compact form:

$$\mu\partial_t\chi_n + \mathcal{N}\chi_n + \xi_n - c\chi_n = \ell\theta_n. \tag{4.151}$$

Then, subtracting the term $\mathcal{N}\chi$ from both hands sides, testing the resulting relation by $\chi_n - \chi$, and integrating in $[0,T]$, we easily derive

(4.152)

$$\|\nabla(\chi_n - \chi)\|^2_{L^2(0,t;H)} = \int_0^T \int_\Omega (-\mu \partial_t \chi_n + c\chi_n + \ell\theta_n)(\chi_n - \chi)\,dx\,dt - \int_0^T \langle \mathcal{N}\chi, (\chi_n - \chi)\rangle\,dt + \sum_{i=1}^2 \int_0^T \int_{\Omega_i} \xi_{i,n}(\chi_i - \chi_{i,n})\,dx\,dt$$

Notice now that, by virtue of the strong convergence $\chi_n \to \chi$ in $L^2(Q)$, which is a consequence of (4.140), the first term on the right hand side tends to 0 (recall also (4.115)), while the second one tends to 0 thanks to the second weak convergence in (4.140). As for the last term, passing to the supremum limit, and due to the definition of subdifferential, we get

$$(4.153)\qquad \limsup_{n\to\infty} \sum_{i=1}^2 \int_0^T \int_{\Omega_i} \xi_{i,n}(\chi_i - \chi_{i,n})\,dx\,dt$$
$$\leq \sum_{i=1}^2 \limsup_{n\to\infty} \int_0^T \int_{\Omega_i} j_{i,n}(\chi_i)\,dx\,dt - \sum_{i=1}^2 \liminf_{n\to\infty} \int_0^T \int_{\Omega_i} j_{i,n}(\chi_{i,n})\,dx\,dt$$
$$\leq \sum_{i=1}^2 \limsup_{n\to\infty} \int_0^T \int_{\Omega_i} j_{i,n}(\chi_i)\,dx\,dt - \sum_{i=1}^2 \int_0^T \int_{\Omega_i} j_i(\chi_i)\,dx\,dt, \leq 0$$

also owing to the Mosco-convergence $j_{i,n} \to j_i$ and to (4.136). Collecting now (4.152–4.153), the strong convergence (4.141) follows.

End of proof. Since $\mathcal{N}$ is linear and continuous from V to V', proceeding by comparison in equation (4.151), we easily get (4.142), so that, by the monotonicity argument of Prop. 1.3.8 (applied in the duality (V, V')), we are able to identify the limit element ξ. However, we cannot get, as before, the "physical" relation (4.146), but we only obtain

$$\xi \in \partial_{V,V'} J_{|V}(\chi(t)) \qquad \text{a.e. in }]0,T[$$

(the notation is as in Chapter 3). So, to get back (4.146), we have to take advantage of (GC) and apply the procedure of Subsec. 3.2.4; moreover, (4.149) has to be exploited, by observing that, by the semicontinuity properties of M-convergence, it entails

$$\|j_i(\chi_i)\|_{L^\infty(0,T;L^1(\Omega_i))} \leq C \qquad \text{for } i = 1, 2.$$

Now the procedure is exactly the same as in Subsec. 3.2.4, to which we refer indeed for more details.

Remark 4.1.12. *Notice that, in this case, the growth condition was needed uniquely as the limit graphs were concerned. So, the performed procedure continues holding, with minor modifications, even if no conditions are supposed on the approximating operators $\alpha_{i,n}$. Naturally, in this case the solution of ($TP_{\rm n}$) could not be interpreted in the physical sense, but only in the abstract (V, V') framework of Chapter 3.*

4.2 Convergence to the Stefan problem

In this section, we address the question of the convergence of the Caginalp-Fix phase-field model to the standard Stefan problem in its two-phase formulation [32], and, in particular, we study two different physical situations related to this problem.

Our aim is that of performing an asymptotic study of equations (4.1–4.4), as the coefficients $\mu_{2,n}, \nu_{2,n}, c_{2,n}$ decrease simultaneously to 0 and the operators α_n suitably tend to another monotone graph α. We prove that, if the α_n are Lipschitz continuous, and they converge to α in the graph sense, then the above problem converges to a transmission system coupling the Stefan model in Ω_1 and the phase-field one in Ω_2.

Anyway, since this analysis results considerably more complicate than those of the previous Section, we prefer to let it be preceeded by a study of the convergence of the phase-field system to the weak Stefan problem in the simpler case of a the model related to a single substance. We point out that the related convergence theorem, which moves from the setting of Subsec. 2.2.2, provides only a slight extension of the results of [34] concerning a similar problem (where it is assumed $c_n = 0$ and $\alpha_n = \alpha$ for all $n \in \mathbb{N}$, anyway); we chose to present it essentially for the purpose of clarity.

Then, the transmission case is addressed, and we prove a convergence result for this problem under very mild hypotheses on the limit operators (and, in particular, much weaker than the usual (CC) or (GC), which are assumed on the approximating ones). This is possible thanks to the different type of boundary behaviour at the interface observed by the limit statement, where homogeneous Neumann conditions hold for the phase variable on the whole boundary of Ω_1.

In order to perform this program, following the approach of Visintin [85] and Damlamian-Kenmochi-Sato [34], it has been necessary to rewrite the transmission problem in another form equivalent to the original one; in particular, proceeding by compactness methods (as we essentially do in the one-domain case), we were not able to show the convergence of solutions under the most general hypotheses on the limit monotone graphs. We finally point out that the results of this Section are essentially contained in our paper [80]; some further physical details have been provided in [81].

4.2.1 Mathematical problems and main results

Under the usual hypotheses on Ω, T and the natural choices for V, H, we begin by listing the hypotheses of the simpler one-domain problem, which slightly extend those of Subsec. 2.2.2:

One-domain case. First of all, let us suppose

(4.154) $\rho, \lambda, \mu_n, \nu_n, c_n, \ell, p, m$ assigned strictly positive constants,

with μ_n, ν_n, c_n depending on $n \in \mathbb{N}$. We also assume

(4.155) $f_n \in L^2(0,T;H)$

(4.156) $g \in L^2(0,T;H^{-1/2}(\partial\Omega))$

(4.157) $\alpha_n \subset \mathbb{R} \times \mathbb{R}$ maximal monotone graphs such that $0 \in \alpha_n(0)$.

Correspondently, we construct a sequence j_n of convex primitives of α_n.

As hypotheses on the initial data, we take:

$$\theta_{0,n} \in H, \qquad \chi_{0,n} \in V, \tag{4.158}$$

$$\chi_{0,n}(x) \in D(j_n) \quad \text{for a.e. } x \in \Omega, \quad j_n(\chi_{0,n}) \in L^1(\Omega); \tag{4.159}$$

moreover, we define:

$$e_{0,n} := \rho\theta_{0,n} + \lambda\chi_{0,n}. \tag{4.160}$$

We also suppose that there are two disjoint and relatively open subsets Γ_{T} and Γ_{N} of $\partial\Omega$ such that

$$\Gamma_{\mathrm{N}} = \partial\Omega \setminus \overline{\Gamma}_{\mathrm{T}} \quad \text{and} \quad \mathcal{H}^{N-1}(\Gamma_{\mathrm{T}}) > 0. \tag{4.161}$$

This subdivision accounts for the choice of mixed Neumann-third type boundary conditions for the temperature, with the third type ones holding on a set of positive $(N-1)$-dimensional measure in order to guarantee some coercivity. However, such conditions become implicit in the statement of the problem, due to its variational character; indeed, we put on the space V the scalar product (equivalent to the standard one, owing to the second hypothesis of (4.161) and to $p > 0$)

$$(\!(u, v)\!) := \int_\Omega \kappa \nabla u \cdot \nabla v \, dx + \int_{\Gamma_{\mathrm{T}}} p u v \, d\mathcal{H}^{N-1}. \tag{4.162}$$

Moreover, we denote by $\langle \cdot, \cdot \rangle$ the duality between V' and V and by $\mathcal{F} : V \to V'$ the Riesz operator associated to the above scalar product, as in Subsection 1.1.3; also, we indicate by $(\!(\cdot, \cdot)\!)_*$ the dual scalar product on V', so that, for $h \in H$ and $v \in V$, we have:

$$(h, v) = \langle h, v \rangle = (\!(h, \mathcal{F}v)\!)_* = (\!(\mathcal{F}^{-1}h, v)\!). \tag{4.163}$$

Owing to this machinery, it is possible to introduce a generalized source term $F_n \in L^2(0, T; V')$ for the heat equation this way:

$$\langle F_n, v \rangle := \int_\Omega f_n v \, dx + {}_{-1/2,\partial\Omega}\langle g, v \rangle_{1/2,\partial\Omega} \quad \text{for } v \in V. \tag{4.164}$$

We are now ready to state our first problem in its precise variational formulation:

Problem 4.2.1 ($\mathbf{P_n}$). *We look for a triplet of suitably regular functions $(\theta_n, \chi_n, \xi_n) : Q \to \mathbb{R}$ satisfying the following system of nonlinear evolution equations:*

$$\partial_t(\rho\theta_n + \lambda\chi_n) + \mathcal{F}\theta_n = F_n \quad \text{in } V' \quad \text{for a.e. } t \in]0, T[\tag{4.165}$$

$$\mu_n \partial_t \chi_n - \operatorname{div}(\nu_n \nabla \chi_n) + \xi_n - c_n \chi_n = \ell\theta_n \quad \text{in } \Omega \quad \text{for a.e. } t \in]0, T[\tag{4.166}$$

$$\xi_n \in \alpha_n(\chi_n) \quad \text{in } \Omega \quad \text{for a.e. } t \in]0, T[\tag{4.167}$$

$$(\rho\theta_n + \lambda\chi_n)(0) = e_{0,n} \quad \text{in } \Omega \tag{4.168}$$

$$\chi_n(0) = \chi_{0,n} \quad \text{in } \Omega \tag{4.169}$$

$$\partial_{\mathbf{n}}\chi_n = 0 \quad \text{on } \partial\Omega \quad \text{for a.e. } t \in]0, T[. \ \blacksquare \tag{4.170}$$

We recall an existence, uniqueness and regularity theorem for the previous problem, which can be easily deduced from our results of Subsections 2.2.2–2.2.3.

Theorem 4.2.2. *For any fixed $n > 0$, there exists a unique solution $(\theta_n, \chi_n, \xi_n)$ of Problem* $(\mathrm{P_n})$ *such that*

$$\theta_n \in L^2(0,T;V) \cap H^1(0,T;V')(\subset C^0([0,T];H)) \tag{4.171}$$

$$\chi_n \in C^0([0,T];V) \cap H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)) \tag{4.172}$$

$$\xi_n \in L^2(0,T;H). \ \blacksquare \tag{4.173}$$

We are now interested in the study of the asymptotic behaviour of $(\mathrm{P_n})$ for vanishing parameters μ_n, ν_n, c_n and varying data $f_n, \alpha_n, e_{0,n}, \chi_{0,n}$. Here are the precise hypotheses we assume on them ($C_0 > 0$ being a constant independent of n; we restart again the counter of the constants $C_0, C_1, \ldots$):

$$f_n \to f \quad \text{in } L^2(0,T;H)\text{–strong}, \tag{4.174}$$

$$\mu_n, \nu_n, c_n \to 0 \quad \text{for } n \to \infty, \tag{4.175}$$

$$c_n/\mu_n \to 0 \quad \text{for } n \to \infty, \tag{4.176}$$

$$\alpha \subset \mathbb{R} \times \mathbb{R} \quad \text{maximal monotone graph such that } 0 \in \alpha(0), \tag{4.177}$$

$$\alpha_n \ \text{ Lipschitz continuous for any } n \in \mathbb{N}, \tag{4.178}$$

$$\liminf_{|r|\to\infty} \frac{j_n(r)}{|r|^2} > m > 0 \quad \text{for every } n \in \mathbb{N}, \tag{4.179}$$

$$\alpha_n \to \alpha \quad \text{in sense of G-convergence in } \mathbb{R} \times \mathbb{R}, \tag{4.180}$$

$$\mu_n^{1/2}\chi_{0,n} \to 0 \quad \text{and} \quad \nu_n^{1/2}\nabla\chi_{0,n} \to 0 \quad \text{in } H\text{–strong}, \tag{4.181}$$

$$\int_\Omega j_n(\chi_{0,n})\,dx \le C_0 \quad \text{for every } n \in \mathbb{N}, \tag{4.182}$$

$$e_{0,n} \to e_0 \quad \text{in } H\text{–strong, for some fixed } e_0 \in H. \tag{4.183}$$

In view of the asymptotic procedure we are going to perform, it is now convenient to introduce another unknown, that is the enthalpy $e_n := \rho\theta_n + \lambda\chi_n$; indeed, this is a more natural variable for the limit statement, which assumes the following form (notice that for suitable choices of the graph α it reduces to the weak formulation of the Stefan problem; in particular, the existence and uniqueness results are the standard ones discussed in Subsec. 2.2.1).

Problem 4.2.3 (P). *We look for a triplet of functions (e, θ, χ), such that $e = \rho\theta + \lambda\chi$, satisfying the regularities*

$$e \in H^1(0,T;V') \cap L^\infty(0,T;H), \tag{4.184}$$

$$\theta \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{4.185}$$

$$\chi \in L^\infty(0,T;H), \tag{4.186}$$

and such that the following equations hold for almost every $t \in]0,T[$:

$$\partial_t e + \mathcal{F}\theta = F \quad \textit{in } V', \tag{4.187}$$

$$\ell\theta(x) \in \alpha(\chi(x)) \quad \textit{for a.e. } x \in \Omega, \tag{4.188}$$

where F is defined as F_n (4.164), but with f in place of f_n. Moreover, we require the limit Cauchy condition

$$e(0) = e_0. \quad \blacksquare \tag{4.189}$$

Theorem 4.2.4. *There exists a unique solution (e, θ, χ) of Problem (P), which is the limit of the solutions (e_n, θ_n, χ_n) of Problems ($\mathrm{P_n}$) in the following sense:*

$$e_n \to e \quad \text{in } H^1(0,T;V')\text{–weak and in } L^\infty(0,T;H)\text{–weak}^*, \tag{4.190}$$

$$\theta_n \to \theta \quad \text{in } L^2(0,T;V)\text{–weak and in } L^\infty(0,T;H)\text{–weak}^*, \tag{4.191}$$

$$\chi_n \to \chi \quad \text{in } L^\infty(0,T;H)\text{–weak}^*. \tag{4.192}$$

Moreover, the following additional convergences hold:

$$\mu_n^{1/2}\chi_n \to 0 \quad \text{in } H^1(0,T;H)\text{–weak}, \tag{4.193}$$

$$\nu_n^{1/2}\nabla\chi_n \to 0 \quad \text{in } L^\infty(0,T;H)\text{–weak}^*, \tag{4.194}$$

$$\xi_n \to \ell\theta \quad \text{in } L^2(0,T;H)\text{–weak}. \quad \blacksquare \tag{4.195}$$

On account of the framework of Subsection 4.1.1, we now pass to the transmission case; however, for the sake of generality, in the following the weaker regularity setting for θ of the one-domain case is maintained.

Transmission case. Since some differences are present with respect to the setting of Subsec. 4.1.1, we specify again the assumptions on the approximating data; indeed, when possible, we try and use the same notations as in the one-domain case, in order to unify the subsequent computations. Since the blow-out of coefficients occurs now only in the domain Ω_2, we can retain the hypotheses (4.155–4.156), (4.158), (4.161) on data and the constructions (4.162) of the scalar product of V and (4.164) of the abstract source term; instead, the other assumptions on coefficients need to be slightly modified. In particular, instead of (4.157), we suppose that, for $i = 1, 2$,

$$\alpha_{i,n} \subset \mathbb{R} \times \mathbb{R} \quad \text{maximal monotone graphs such that } 0 \in \alpha_{i,n}(0); \tag{4.196}$$

furthermore, (4.159) is substituted by

$$\chi_{i,0,n}(x) \in D(j_{i,n}) \quad \text{for a.e. } x \in \Omega_i, \quad \text{with } j_{i,n}(\chi_{i,0,n}) \in L^1(\Omega_i), \text{ for } i = 1, 2, \tag{4.197}$$

$j_{i,n}$ being the convex primitive of $\alpha_{i,n}$. Finally, in place of (4.154), we require that

$$\rho_1, \lambda_1, \mu_1, \nu_1, c_1, \ell_1, p, \rho_2, \lambda_2, \mu_{2,n}, \nu_{2,n}, c_{2,n}, \ell_2, m_n, m \quad \text{are assigned strictly positive constants,} \tag{4.198}$$

with $\mu_{2,n}, \nu_{2,n}, c_{2,n}, m_n$ possibly depending on $n \in \mathbb{N}$. Finally, we assume as usual that the graphs $\alpha_{i,n}$ verify one of assumptions (CC), (GC). In particular, as it will result clear in the following, it is enough to consider here the case of (GC); we point out that no uniformity is required in that condition with respect to n.

These modifications to the one-domain hypotheses are enough to state:

Problem 4.2.5 ($\mathbf{TP_n}$). *We look for a triplet of functions $(\theta_n, \chi_n, \xi_n) : Q \to \mathbb{R}$, of suitable regularity properties, satisfying (4.165–4.166), (4.168–4.170), as well as the constitutive relation*

$$(4.199) \qquad w_{i,n} \in \alpha_{i,n}(\chi_{i,n}) \quad in\ \Omega_i \quad for\ a.e.\ t \in]0,T[. \ \blacksquare$$

For the solution of ($\mathrm{TP_n}$), we naturally refer to Subsec. 4.1.1, whose hypotheses are trivially satisfied. We now pass to the asymptotic study, and begin by presenting the required convergence assumptions, which are

(4.200) $\mu_{2,n}, \nu_{2,n}, c_{2,n} \to 0 \quad \text{for } n \to \infty,$

(4.201) $c_{2,n}/\mu_{2,n} \to 0 \quad \text{for } n \to \infty,$

(4.202) $\alpha_i \subset \mathbb{R} \times \mathbb{R} \quad \text{maximal monotone graphs such that } 0 \in \alpha_i(0),$

(4.203) $\alpha_{1,n}$ Lipschitz continuous for any $n \in \mathbb{N}$,

(4.204) $\alpha_{2,n}$ Lipschitz continuous of Lipschitz constant L_n for any $n \in \mathbb{N}$,

(4.205) $\lim_{n\to\infty} L_n\, \nu_{2,n}^{1/4-\delta/2} = 0 \quad \text{for some (assigned) } \delta \in]0,1/2[,$

(4.206) $\liminf_{|r|\to\infty} \dfrac{j_{2,n}(r)}{|r|^2} \geq m \quad \text{for every } n \in \mathbb{N},$

(4.207) $\alpha_{i,n} \to \alpha_i \quad \text{in the sense of G-convergence in } \mathbb{R} \times \mathbb{R},$

(4.208) the sequence $j_{i,n}(r)$ is nondecreasing in n for every $r \in \mathbb{R}$, $i = 1, 2$,

Moreover, we retain assumption (4.174) as well as (4.183), while, finally, (4.181–4.182) are modified in the following natural way

(4.209) $\mu_{2,n}^{1/2}\chi_{2,0,n} \to 0 \quad \text{and} \quad \nu_{2,n}^{1/2}\nabla\chi_{2,0,n} \to 0 \quad \text{in } L^2(\Omega_2)\text{–strong},$

(4.210) $\chi_{1,0,n} \to \chi_{0,1} \quad \text{in } H^1(\Omega_1)\text{–strong, for some (assigned) } \chi_{0,1} \in H^1(\Omega_1),$

(4.211) $\int_{\Omega_i} j_{i,n}(\chi_{i,0,n})\, dx \leq C_0 \quad \text{for } i = 1, 2 \text{ and for every } n \in \mathbb{N}.$

Remark 4.2.6. *We observe that, at the level $n < +\infty$, two growth hypotheses on the functionals $j_{i,n}$ are required:* (GC) *(which in this framework is a consequence of (4.203–4.204)) and (only regarding $j_{2,n}$) (4.206); they account for a linear (or sublinear) growth of operator $\alpha_{1,n}$ and a linear one of both $\alpha_{2,n}$ and $(\alpha_{2,n})^{-1}$. The main difference is that (4.206) must hold with uniformity on n (actually, this is a standard assumptions for Stefan-like problems [32], resulting essential in order to obtain a weak convergence for $\chi_{2,n}$); in hypothesis* (GC)*, instead, no uniformity is required on n (maybe it could also be avoided by making an ulterior approximation of Yosida type on the graphs α_n and possibly modifying (4.205); anyway this procedure would bring further and boring technical complications); actually, such a condition is no more necessary at the limit step, due to the different form of the boundary conditions on Γ.*

We are now able to write down our limit problem and the related convergence theorem.

Problem 4.2.7 (TP). *We look for a triplet of functions (e, θ, χ) (with $e = \rho\theta + \lambda\chi$) enjoying the regularity relations (4.184–4.186) and also*

$$(4.212) \qquad \chi_1 \in C^0([0,T]; H^1(\Omega_1)) \cap H^1(0,T; L^2(\Omega_1)), \qquad -\Delta\chi_1 \in L^2(Q_1),$$

and such that equation (4.187) holds, together with the Cauchy condition (4.189) and

(4.213) $\ell_1\theta_1 \in \mu_1\partial_t\chi_1 - \nu_1\Delta\chi_1 + \alpha_1(\chi_1) - c_1\chi_1$ *for a.e.* $(x,t) \in \Omega_1\times]0,T[$,

(4.214) $\ell_2\theta_2(x) \in \alpha_2(\chi_2(x))$ *for a.e.* $(x,t) \in \Omega_2\times]0,T[$,

(4.215) $\chi_1(0) = \chi_{0,1}$ *in* Ω_1,

(4.216) $\partial_{\mathbf{n}}\chi_1 = 0$ *on* $\partial\Omega_1$ *for a.e.* $t \in]0,T[$. ■

Theorem 4.2.8. *There exists a unique solution* (e,θ,χ) *to Problem* (TP). *Moreover, the convergences (4.190–4.192) hold, as well as*

(4.217) $\chi_{1,n} \to \chi_1$ *in* $L^\infty(0,T;H^1(\Omega_1))$*–weak** *and* $H^1(0,T;L^2(\Omega_1))$*–weak,*

(4.218) $\mu_{2,n}^{1/2}\chi_{2,n} \to 0$ *in* $H^1(0,T;L^2(\Omega_2))$*–weak,*

(4.219) $\nu_{2,n}^{1/2}\nabla\chi_{2,n} \to 0$ *in* $L^\infty(0,T;L^2(\Omega_2))$*–weak**. ■

4.2.2 A priori estimates

In this subsection, we present some a priori estimates for the solutions of problems ($\mathrm{P_n}$) and ($\mathrm{TP_n}$); we try to carry on the computations in a form adaptable to both cases, possibly remarking the differences, if any.

First estimate. Rewriting equation (4.165) in terms of e_n and θ_n, we get

$$\partial_t e_n + \mathcal{F}\theta_n = F_n \quad \text{in } V' \quad \text{for a.e. } t \in]0,T[. \tag{4.220}$$

Multiply this relation by $\mathcal{F}^{-1}e_n$, which is in both cases a $H^1(0,T;V)$-function thanks to (4.171–4.172) and (3.139–3.140), respectively; integrating the result between 0 and $t \in]0,T]$, and integrating by parts the enthalpy term, owing also to (4.163) and (4.168), we easily derive:

$$\begin{aligned} &\frac{1}{2}\|e_n(t)\|_{V'}^2 + \int_0^t\int_\Omega \rho|\theta_n|^2\,dx\,ds \\ &\quad = \frac{1}{2}\|e_{0,n}\|_{V'}^2 + \int_0^t \langle F_n, \mathcal{F}^{-1}e_n\rangle\,ds - \int_0^t\int_\Omega \lambda\chi_n\theta_n\,dx\,ds. \end{aligned} \tag{4.221}$$

The second step is to multiply the phase-field equation (4.166) by χ_n (which belongs in both cases to $C^0([0,T];V)\cap H^1(0,T;H)$) and integrate the result in $]0,t[$. Integrating by parts in time the first two terms, recalling the Cauchy and Neumann conditions (4.169–4.170) and the properties (4.157) and (4.196) (observe that it is also used that $\alpha_n(0) = \alpha_{1,n}(0) = \alpha_{2,n}(0) = 0$), we infer

$$\begin{aligned} &\frac{1}{2}\int_\Omega \mu_n|\chi_n(t)|^2\,dx + \int_0^t\int_\Omega \nu_n|\nabla\chi_n|^2\,dx\,ds - \int_0^t\int_\Omega c_n|\chi_n|^2\,dx\,ds \\ &\quad \le \int_0^t\int_\Omega \ell\theta_n\chi_n\,dx\,ds + \frac{1}{2}\int_\Omega \mu_n|\chi_{0,n}|^2\,dx. \end{aligned} \tag{4.222}$$

Notice that in the above computations the "coefficients" (ρ,ℓ,μ) are constant in the one-body problem and depend on x in the other case.

We now want to erase the two mixed-unknowns integral terms on the right hand sides of (4.221) and (4.222). So, multiply (4.222) by $\lambda\ell^{-1}$ in the one-body case and by $\lambda_2\ell_2^{-1}$ in the transmission one (this erases only the Ω_2-parts of the integral terms; in Ω_1 their sum can be split by means of (1.24) and controlled by the $\mu_1\|\chi_{1,n}(t)\|^2_{L^2(\Omega_1)}$-term on the left hand side; indeed, μ_1 does not vanish with n); in both cases, sum the result to (4.221).

Now, owing to the the definition of $\mathcal{F}$, it is possible to split the heat source term by inequality (1.24); recalling also the regularity hypotheses on the initial data (4.158), and the limit ones (4.183) and (4.181) (alternatively (4.209–4.210)), we infer:

$$\begin{aligned}(4.223)\qquad &\|e_n(t)\|^2_{V'} + \|\theta_n\|^2_{L^2(0,t;H)} + \|\mu_n^{1/2}\chi_n(t)\|^2_H + \|\nu_n^{1/2}\nabla\chi_n\|^2_{L^2(0,t;H)} \\ &\quad \le C_1\left(1 + \|c_n^{1/2}\chi_n\|^2_{L^2(0,t;H)} + \|e_n\|^2_{L^2(0,t;V')}\right),\end{aligned}$$

where C_1 is a positive constant independent of n. Recalling hypothesis (4.176) ((4.201) for the transmission problem), an application of the Gronwall lemma allows us to derive some apriori estimates (which will be explicitely reported at the end of the subsection) from the previous calculation.

Second estimate. Multiply now (4.165) by θ_n and integrate in $]0,t[$. Note now that the regularity properties (3.138–3.139) of the solutions of $(\mathrm{TP_n})$ are precisely sufficient to integrate by parts in time the product $\partial_t(\rho\theta_n)\theta_n$, so that we easily deduce:

$$\begin{aligned}(4.224)\qquad &\frac{1}{2}\int_\Omega \rho|\theta_n(t)|^2\,dx + \|\theta_n\|^2_{L^2(0,t;V)} \le \int_0^t\int_\Omega \langle F_n,\theta_n\rangle\,ds \\ &\quad + \frac{1}{2}\int_\Omega \rho|\theta_{0,n}|^2\,dx - \int_0^t\int_\Omega \lambda\partial_t\chi_n\theta_n\,dx\,ds.\end{aligned}$$

Testing instead (4.166) by $\partial_t\chi_n$ and integrating in $]0,t[$, we have:

$$\begin{aligned}(4.225)\qquad &\int_0^t\int_\Omega \mu_n|\partial_t\chi_n|^2\,dx\,ds + \frac{1}{2}\int_\Omega \nu_n|\nabla\chi_n(t)|^2\,dx + J_n(\chi_n(t)) \\ &\quad \le \frac{1}{2}\int_\Omega \nu_n|\nabla\chi_{0,n}|^2\,dx + J_n(\chi_{0,n}) + \int_0^t\int_\Omega \ell\partial_t\chi_n\theta_n\,dx\,ds \\ &\quad + \frac{1}{2}\int_\Omega c_n|\chi_n(t)|^2\,dx - \frac{1}{2}\int_\Omega c_n|\chi_{0,n}|^2\,dx,\end{aligned}$$

where we have set, for any $v \in H$,

$$(4.226)\qquad J_n(v) := \begin{cases} \displaystyle\int_\Omega j_n(v(x))\,dx & \text{if } j_n(v) \in L^1(\Omega) \\ +\infty & \text{otherwise}\end{cases}$$

for the one-body problem and

$$(4.227)\qquad J_n(v) := \begin{cases} \displaystyle\sum_{i=1}^2\int_{\Omega_i} j_{i,n}(v_i(x))\,dx & \text{if } j_{i,n}(v_i) \in L^1(\Omega_i) \ \text{ for } i = 1,2 \\ +\infty & \text{otherwise}\end{cases}$$

for the transmission one (in the sequel we shall also meet the functional J whose definition is analogous to the above one, but with the substitution of j_n with j (convex primitive of α) or that of $j_{1,n}, j_{2,n}$ with j_1, j_2 (convex primitives of α_1, α_2, respectively), depending on the physical situation).

Now we can use as before the trick of multiplying (4.225) by $\lambda\ell^{-1}$ (for the first problem) or $\lambda_2\ell_2^{-1}$ (for the second) and we sum the result to (4.224). So, it is now easy to see that hypotheses (4.176), (4.181–4.182) (for the first problem) or (4.201), (4.209–4.211) (for the transmission one) on the Cauchy data and the apriori estimate corresponding to (4.223) permit to control all the terms on the right hand sides of (4.224) and (4.225) and to conclude.

Third estimate. This is the key estimate, *holding only for the first problem*, which will allow us to pass to the limit in the nonlinear term $\alpha_n(\chi_n)$; this becomes possible as we multiply (4.166) by $\alpha_n(\chi_n)$ (which is $C^0([0,T];V)$ owing to the Lipschitz continuity of α_n) and integrate in $]0,t[$ (unfortunately, this procedure is not possible in the transmission case, since neither (CC) nor (GC) are now uniform in n):

$$\begin{aligned}(4.228)\qquad &\mu_n J_n(\chi_n(t)) + \nu_n \int_0^t\int_\Omega (\alpha_n)'(\chi_n)|\nabla\chi_n|^2\,dx\,ds + \|\alpha_n(\chi_n)\|^2_{L^2(0,t;H)}\\ &\le \mu_n J_n(\chi_{0,n}) + \int_0^t\int_\Omega (c_n\chi_n + \ell\theta_n)\,\alpha_n(\chi_n)\,dx\,ds.\end{aligned}$$

Now, the integral term on the left hand side is nonnegative due to the monotonicity of α_n, while the first term on the right hand side is bounded by (4.182) and the second one can be split using (1.24) and controlled by means of the first two estimates. So, we can finally deduce our

Conclusions. From (4.223) and (4.224–4.225), we immediately derive the following block of estimates, which are valid for both problems; in the following, C_2 is a positive constant depending only on data:

$$(4.229)\qquad \|e_n\|_{L^\infty(0,T;V')} \le C_2$$

$$(4.230)\qquad \|\theta_n\|_{L^2(0,T;V)\cap L^\infty(0,T;H)} \le C_2$$

$$(4.231)\qquad \|J_n(\chi_n)\|_{L^\infty(0,T)} \le C_2.$$

In the one-domain case, we also have

$$(4.232)\qquad \|\mu_n^{1/2}\chi_n\|_{H^1(0,T;H)} \le C_2$$

$$(4.233)\qquad \|\nu_n^{1/2}\nabla\chi_n\|_{L^\infty(0,T;H)} \le C_2,$$

and, as a consequence of (4.228),

$$(4.234)\qquad \|\alpha_n(\chi_n)\|_{L^2(0,T;H)} \le C_2.$$

In the transmission framework, the Ω_1 and Ω_2 components have to be split and separately treated:

$$(4.235)\qquad \|\chi_{1,n}\|_{L^\infty(0,T;H^1(\Omega_1))\cap H^1(0,T;L^2(\Omega_1))} \le C_2$$

$$(4.236)\qquad \|\mu_{2,n}^{1/2}\chi_{2,n}\|_{H^1(0,T;L^2(\Omega_2))} \le C_2$$

$$(4.237)\qquad \|\nu_{2,n}^{1/2}\nabla\chi_{2,n}\|_{L^2(0,T;L^2(\Omega_2))} \le C_2.$$

Also, comparing equation (4.220) with hypothesis (4.174) and estimate (4.230), we see that (4.229) can actually be improved to the bound

$$\|e_n\|_{H^1(0,T;V')} \le C_2, \tag{4.238}$$

which will be essential for the final convergence proofs.

4.2.3 Γ-convergence reformulation of $(\mathrm{TP_n})$

In this section, we rewrite problem $(\mathrm{TP_n})$ in a more abstract form, suitable to the limit procedure we are going to perform, which prepares it to be treated with variational convergence techniques for general monotone operators; for the theoretical background, the material of [5] and [31] reported in Section 1.3 should be sufficient.

Reformulation of $(\mathbf{TP_n})$. First of all, we have to introduce some new functionals on H; so, let us set, for $v \in H$,

$$G_n(v) := \begin{cases} \dfrac{\nu_1}{2}\displaystyle\int_{\Omega_1} |\nabla v_1|^2\,dx + \dfrac{\nu_{2,n}}{2}\displaystyle\int_{\Omega_2} |\nabla v_2|^2\,dx & \text{if } v \in V \\ +\infty & \text{otherwise,} \end{cases} \tag{4.239}$$

$$G(v) := \begin{cases} \dfrac{\nu_1}{2}\displaystyle\int_{\Omega_1} |\nabla v_1|^2\,dx & \text{if } v_1 \in H^1(\Omega_1) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.240}$$

It is immediate to verify that G_n and G are convex, l.s.c. and proper with respect to the $L^2(\Omega)$-topology.

In the following we will be concerned with the restatement of problems $(\mathrm{TP_n})$ and (TP) in terms of the functionals $(J_n + G_n)$ and $(J + G)$ respectively. So, we observe that the domain of $(J_n + G_n)$ consists precisely of the $H^1(\Omega)$-functions v such that $j_{1,n}(v_1) \in L^1(\Omega_1)$ and $j_{2,n}(v_2) \in L^1(\Omega_2)$; analogously, we have that $v \in D(J+G)$ if and only if $v_1 \in H^1(\Omega_1)$, $j_1(v_1) \in L^1(\Omega_1)$ and $j_2(v_2) \in L^1(\Omega_2)$. Furthermore, also (J_n+G_n) and $(J + G)$ are convex, l.s.c. and proper functionals on H; we can characterize their subdifferentials in the following way:

Theorem 4.2.9. (a) *$\partial(J_n+G_n)$ coincides with the operator $\mathcal{B}_n : H \to 2^H$ introduced as follows: we define*

$$D(\mathcal{B}_n) := \{v \in H^1(\Omega) : \operatorname{div}(\nu_n \nabla v) \in H \quad \text{and } \partial_{\mathbf{n}} v = 0 \quad \text{on } \partial\Omega\}; \tag{4.241}$$

moreover, for $v \in D(\mathcal{B}_n)$, $w \in H$, we set

$$w \in \mathcal{B}_n(v) \quad \text{if and only if} \quad \begin{cases} w_1 = -\nu_1 \Delta v_1 + \alpha_{1,n}(v_1) & \text{in } \Omega_1 \\ w_2 = -\nu_{2,n} \Delta v_1 + \alpha_{2,n}(v_2) & \text{in } \Omega_2 \\ \nu_1 \partial_{\mathbf{n}} v_1 = \nu_{2,n} \partial_{\mathbf{n}} v_2 & \text{on } \Gamma \\ \partial_{\mathbf{n}} v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.242}$$

(b) *Analogously, we have that $\partial(J + G) = \mathcal{B}$, where*

$$\begin{aligned} D(\mathcal{B}) := \{&v \in L^2(\Omega) : v_1 \in H^1(\Omega_1),\ \Delta v_1 \in L^2(\Omega_1),\ \partial_{\mathbf{n}} v_1 = 0 \quad \text{on } \partial\Omega_1 \\ &\text{and there exist } \zeta_i \in L^2(\Omega_i) \quad \text{verifying } \zeta_i \in \alpha_i(v_i) \quad \text{a.e. in } \Omega_i \quad \text{for } i = 1, 2\} \end{aligned} \tag{4.243}$$

and, for $v \in D(\mathcal{B})$, $w \in H$, we have put

$$w \in \mathcal{B}(v) \quad \textit{if and only if} \quad \begin{cases} w_1 \in -\nu_1 \Delta v_1 + \alpha_1(v_1) & \textit{in } \Omega_1 \\ w_2 \in \alpha_2(v_2) & \textit{in } \Omega_2 \\ \partial_{\mathbf{n}} v_1 = 0 & \textit{on } \partial\Omega_1. \end{cases} \tag{4.244}$$
■

Proof. First of all, it is easy to see that $\mathcal{B}_n$ ($\mathcal{B}$) is a monotone operator contained (in the sense of inclusion of graphs) into $\partial(J_n + G_n)$ ($\partial(J + G)$, respectively): to see this, for instance in the case of $\mathcal{B}_n$ (the other is similar), just take $(v, w) \in \mathcal{B}_n$ and verify the definition of subdifferential; that is, show that, for any $z \in D(J_n + G_n)$, it is

$$\int_\Omega w(z - v)\, dx \le J_n(z) + G_n(z) - J_n(v) - G_n(v). \tag{4.245}$$

The computations do not present difficulties; observe only that $z \in D(J_n + G_n)$ entails $z \in V$, which allows the use of the Gauss-Green formula.

The second (and more interesting) step is the proof of the maximalities of $\mathcal{B}_n$ and $\mathcal{B}$, which we perform separately in the two cases, beginning with the less difficult:

(b) Denoting with I the identity operator of H, we have to prove that $R(I + \mathcal{B}) = H$; that is, for every $h \in H$, we look for a solution $v \in D(\mathcal{B})$ of the system

$$h_1 \in -\nu_1 \Delta v_1 + \alpha_1(v_1) + v_1 \quad \text{in } \Omega_1 \tag{4.246}$$

$$\partial_{\mathbf{n}} v_1 = 0 \quad \text{on } \partial\Omega_1 \tag{4.247}$$

$$h_2 \in \alpha_2(v_2) + v_2 \quad \text{in } \Omega_2. \tag{4.248}$$

Observing that the equations (4.246) and (4.248) of the system are actually de-coupled, standard approximation techniques for elliptic problems with monotone nonlinearities [17] easily permit to conclude.

(a) As before, taken $h \in H$ and $n \in \mathbb{N}$, we look for a solution of the elliptic system:

$$h_1 = -\nu_1 \Delta v_1 + \alpha_{1,n}(v_1) + v_1 \quad \text{in } \Omega_1 \tag{4.249}$$

$$h_2 = -\nu_{2,n} \Delta v_2 + \alpha_{2,n}(v_2) + v_2 \quad \text{in } \Omega_2 \tag{4.250}$$

$$\nu_1 \partial_{\mathbf{n}} v_1 = \nu_{2,n} \partial_{\mathbf{n}} v_2 \quad \text{on } \Gamma \tag{4.251}$$

$$\partial_{\mathbf{n}} v = 0 \quad \text{on } \partial\Omega. \tag{4.252}$$

In this case, the resolution of the system is not completely trivial; infact it presents the usual troubles concerning space regularity, which have already been managed in Subsec. 3.2.3 in the case of the parabolic transmission system. We briefly sketch here the way of operating in the elliptic setting, referring to that subsection for more details on the procedure.

First of all, denote as $J_{V,n}$ the restriction to V of the functional J_n. Clearly, $J_{V,n}$ is convex, l.s.c., and proper, too. In the framework of the Hilbert triplet (V, H, V'), it is natural to see the subdifferential of $J_{V,n}$ as a maximal monotone operator from V to $2^{V'}$ which we denote as $\partial_{V,V'} J_{V,n}$. Also, it is not difficult to verify that every solution of system (4.249–4.252) is also a solution of the following more general and compact formulation:

$$h = D_n v + q + v \quad \text{in } V', \tag{4.253}$$

$$q \in \partial_{V,V'} J_{V,n}(v), \tag{4.254}$$

where $D_n : V \to V'$ is the operator

$$\langle D_n u, z\rangle := \int_\Omega \nu_n \nabla u \cdot \nabla z \, dx \quad \text{for } u, z \in V. \tag{4.255}$$

Moreover, observe that (4.253) can be seen as an abstract equality in the space V' and solved therein with standard techniques. The crucial point is now to prove that the obtained solution – call it v – is also a solution of (4.249–4.252). and this is precisely the point where the results of Subsec. 3.2.3, and in particular Theorem 3.2.4, are needed. ■

Let us now conjecture what still remains to do: consider problem ($\mathrm{TP_n}$) and set

$$\gamma_n := -\mu_n \partial_t \chi_n + c_n \chi_n + \ell \theta_n. \tag{4.256}$$

Owing to part (a) of the previous theorem, relations (4.166), (4.199) and (4.170) of ($\mathrm{TP_n}$) can be written in the equivalent abstract form (see also Remark 4.2.10 below):

$$\gamma_n \in \partial(J_n + G_n)(\chi_n) \quad \text{in } H \quad \text{a.e. in }]0,T[. \tag{4.257}$$

Analogously, thanks to Theorem 4.2.9 (b), setting in (TP)

$$\gamma := \begin{cases} -\mu_1 \partial_t \chi_1 + c_1 \chi_1 + \ell_1 \theta_1 & \text{in } \Omega_1 \\ \ell_2 \theta_2 & \text{in } \Omega_2, \end{cases} \tag{4.258}$$

we see that (4.213–4.214) and (4.216) can be condensed as

$$\gamma \in \partial(J + G)(\chi) \quad \text{in } H \quad \text{a.e. in }]0,T[. \tag{4.259}$$

Furthermore, we observe that estimates (4.235), (4.236), (4.230) and hypothesis (4.201) imply, for suitable subsequences, that

$$\gamma_n \to \gamma \quad \text{in } L^2(0,T;H)\text{–weak}. \tag{4.260}$$

In the next section, we shall essentially see that Prop. 1.3.8 can be applied to the couple $(\chi_n, \gamma_n) \in \partial(J_n + G_n)$ (indeed some purely technical complications will arise, due to the fact that it will be necessary to work in the space $L^2(0,T;H)$ by extending therein the functionals $(J_n + G_n)$ and $(J + G)$); with this aim, it will be necessary to verify the convergence (4.192) (easy), the semicontinuity property (1.61) (easy too) and the G-convergence $\partial(J_n + G_n) \to \partial(J + G)$ (more difficult).

4.2.4 Conclusion of proofs and final remarks

Proof of Theorem 4.2.4. We first observe that (4.191) and the first of (4.190) are easy consequences of estimates (4.230) and (4.238), while from (4.234) we derive that

$$\xi_n \to \xi \quad \text{in } L^2(0,T;H)\text{–weak,} \tag{4.261}$$

for some function $\xi \in L^2(0,T;H)$. Moreover, (4.193) and (4.194) follow from (4.232) and (4.233) respectively. We point out that, here and in the following, all the convergences in exam hold a priori up to the choice of subsequences; the uniqueness of the solution to the limit formulation permits anyway to extend their validity to the whole original sequences.

Now, recalling assumption (4.179), we can choose an $R > 0$ such that

$$\frac{j_n(r)}{|r|^2} > \frac{m}{2} \quad \text{for every } |r| > R \text{ and } n \in \mathbb{N}. \tag{4.262}$$

At this point, owing also to (4.231), we have, for a.e. $t \in]0,T[$ and every $n \in \mathbb{N}$,

$$\|\chi_n(t)\|_H^2 = \int_{\{|\chi_n(t)| \le R\}} |\chi_n(t)|^2\, dx + \int_{\{|\chi_n(t)| > R\}} |\chi_n(t)|^2\, dx \le R^2|\Omega| + \frac{2}{m} C_2, \tag{4.263}$$

whence we easily derive (4.192) and, recalling (4.191), also the second of (4.190).

At this point, regularities (4.184–4.186) are obtained; moreover the quoted convergences and hypothesis (4.174) permit to pass to the limit in (4.165) and get (4.187); also, from (4.190), (4.168) and (4.183), the Cauchy condition (4.189) follows.

Proceeding now by comparison in equation (4.166), we easily get

$$\operatorname{div}(\nu_n \nabla \chi_n) \to 0 \quad \text{in } L^2(0,T;H)\text{–weak,} \tag{4.264}$$

so that, passing to the limit in (4.166) and recalling (4.261), we derive that $w = \ell\theta$ (in H), whence also (4.195) follows; so, it only remains to show (4.188).

At this point, it is convenient to restate our convergence problem in the space $L^2(0,T;H)$; with this purpose, we give a general definition: given a convex, l.s.c. and proper $\mathbb{R}_\infty$-valued functional Ψ on a, say, Hilbert space X, we introduce (and denote) its extension to $L^2(0,T;X)$, as follows (v is taken in $L^2(0,T;X)$):

$$\Psi_T(v) := \begin{cases} \displaystyle\int_0^T \Psi(v(t))\, ds & \text{if } \Psi(v) \in L^1(0,T) \\ +\infty & \text{otherwise} \end{cases} \tag{4.265}$$

In the sequel, when we speak of the functionals $J_{n,T}$, J_T (and others), we shall always refer to the above definition (in general with $X = H$).

Remark 4.2.10. *We recall that, on account of Theorem 1.2.22, given a couple (u,v) of $L^2(0,T;X)$ functions, the following conditions are equivalent:*

(a) $u(t) \in \partial\Psi(v(t))$ *in X, for almost every $t \in]0,T[$;*

(b) $u \in \partial\Psi_T(v)$ *in $L^2(0,T;X)$.*

This permits for instance to reinterpret conditon (4.167) in the equivalent form $\xi_n \in \partial J_{n,T}(\chi_n)$ (indeed, here we have two consecutive extensions: the first, in space, from j_n to J_n, the second from J_n to $J_{n,T}$; in both cases, Theorem 1.2.22 can be applied).

The following result is common to both our physical situations (the proofs anyway will be different):

Lemma 4.2.11. *$J_{n,T} \to J_T$ in the sense of Mosco.*

Proof. In the transmission framework it is immediate once observed that, owing to (4.208), the family of functionals $J_{n,T}$ is nondecreasing; therefore, Prop. 1.3.9 (a) can be applied.

In the one-domain case, instead, we did not suppose any monotonicity (in n) property for j_n; so, we have to work a little bit more.

Owing to Theorem 1.3.7 ((c) $\Rightarrow$ (a)), it is enough to prove that, for every $\epsilon > 0$ and for every $v \in L^2(0,T;H)$, we have that $(\partial J_{n,T})_\epsilon(v) \to (\partial J_T)_\epsilon(v)$ strongly in $L^2(0,T;H)$. First, notice that assumption (4.180), and Theorem 1.3.7 ((b) $\Rightarrow$ (c)), entail that, for every $r \in \mathbb{R}$, it is

$$(\alpha_n)_\epsilon(r) \to \alpha_\epsilon(r), \tag{4.266}$$

so that, using the characterization of Yosida regularizations of $L^2(0,T;H)$-extended operators given again by Theorem 1.2.22 and owing also to the $(1/\epsilon)$-Lipschitz continuity of $(\alpha_n)_\epsilon$ and α_ϵ, we can apply the dominated convergence theorem to the expression

$$\|(\partial J_{n,T})_\epsilon(v) - (\partial J_T)_\epsilon(v)\|^2_{L^2(0,T;H)} = \int_0^T \int_\Omega |(\alpha_n)_\epsilon(v) - \alpha_\epsilon(v)|^2 \, dx \, dt, \tag{4.267}$$

so concluding the proof of the lemma. ∎

Now, owing to (4.192), (4.195) and (4.180), recalling also Prop. 1.3.8, what remains to do is to verify (1.61) with $X = X' = L^2(0,T;H)$, $y = \xi$, $y_n = \xi_n$, $x = \chi$, $x_n = \chi_n$, $\mathcal{A}_n = \partial J_{n,T}$ and $\mathcal{A} = \partial J_T$. So, first of all, observe that (4.190) and [82, Cor. 4, Section 8] entail the following convergence, which is the *only* strong one available for the solutions of $(\mathrm{P_n})$:

$$e_n \to e \quad \text{in } L^2(0,T;H^{-1/4}(\Omega))\text{–strong}. \tag{4.268}$$

At this point, we are ready to verify (1.61); using equation (4.166), we deduce:

$$\begin{aligned} \int_0^T (\xi_n, \chi_n)\, dt = &-\frac{1}{2}\|\mu_n^{1/2}\chi_n(t)\|^2_H + \frac{1}{2}\|\mu_n^{1/2}\chi_{0,n}\|^2_H - \|\nu_n^{1/2}\nabla\chi_n\|^2_{L^2(0,T;H)} \\ &+ \|c_n^{1/2}\chi_n\|^2_{L^2(0,T;H)} + \int_0^T\int_\Omega \ell\theta_n\left(\chi_n + \frac{\rho}{\lambda}\theta_n\right) dx\, dt - \int_0^T\int_\Omega \frac{\ell\rho}{\lambda}|\theta_n|^2\, dx\, dt. \end{aligned} \tag{4.269}$$

Passing now to the supremum limit in the previous computation, we notice that the first and the third term on the right hand side are negative, while the second and the fourth ones tend to vanish by virtue of (4.181) and (4.175), (4.192) respectively. Finally, relation (4.268), the first of (4.191) and the semicontinuity properties of weak convergence permit to manage also the two last terms on the right hand side, so that:

$$\begin{aligned} \limsup_{n\to\infty} \int_0^T (\xi_n(t), \chi_n(t))\, dt &\le \int_0^T\int_\Omega \ell\theta\frac{e}{\lambda}\, dx\, ds - \int_0^T\int_\Omega \frac{\ell\rho}{\lambda}|\theta|^2\, dx\, ds \\ &= \int_0^T (\ell\theta, \chi)\, ds, \end{aligned} \tag{4.270}$$

as desired. Since the uniqueness of the solution to problem (P) is classical, the proof of Theorem 4.2.4 is now complete. ∎

Proof of Theorem 4.2.8. We begin by two preliminary lemmas, the first one accounting for the limit behaviour of a singular perturbation problem with mixed boundary conditions (see also [25, Appendix]); the second showing the M-convergence of the functionals G_n to G.

We point out that this second result could be avoided by suitably modifying the subsequent proof of Theorem 4.2.14; anyway, this approach, although probably longer, permits to emphasize better the difficulties which our problem inherits from the lack of linearity properties which is characteristic of Γ-convergence.

In the sequel, we shall denote by $W_0 := H^1_{0,\Gamma}(\Omega_2)$ the space of $H^1(\Omega_2)$-functions vanishing on Γ in sense of traces. We recall (see also Subsec. 1.1.2) that, for any $\zeta \in H^{1/2}(\Gamma)$, there exists a function $\psi =: \mathcal{R}\zeta \in H^1(\Omega_2)$ extending ζ to Ω_2; that is, $\psi_{|\Gamma} = \zeta$ in sense of traces and $\|\psi\|_{H^1(\Omega_2)} \le C_3\|\zeta\|_{H^{1/2}(\Gamma)}$ for some $C_3 > 0$ (independent of ζ). Since the extension operator $\mathcal{R}$ is a priori nonunique, we choose here ψ precisely as the solution of the following elliptic problem with mixed boundary conditions

$$\begin{cases} -\Delta\psi = 0 & \text{in } \Omega_2 \\ \psi = \zeta & \text{on } \Gamma \\ \partial_{\mathbf{n}}\psi = 0 & \text{on } \partial\Omega_2\backslash\overline{\Gamma}. \end{cases} \tag{4.271}$$

Moreover, in order to state the next lemma, we have to introduce a new Hilbert triplet, that is $(W_0, L^2(\Omega_2), W_0')$, where, on account of the Poincaré inequality, we can choose for W_0 the scalar product given by

$$(\!(\omega_1,\omega_2)\!)_{W_0} := \int_{\Omega_2} \nabla\omega_1 \cdot \nabla\omega_2\,dx \qquad \text{for } \omega_1,\omega_2 \in W_0, \tag{4.272}$$

while we put on $L^2(\Omega_2)$ the standard one, denoted as usual by $(\cdot,\cdot)$.

We also indicate by $\mathcal{D}$ the inverse Riesz operator from W_0' to W_0; in particular, the dual scalar product on W_0' can be defined as

$$\begin{aligned} (\!(\phi,z)\!)_{W_0'} &:= {}_{W_0'}\langle\phi,\mathcal{D}z\rangle_{W_0} && \text{for all } \phi,z\in W_0' \\ &= \int_{\Omega_2} \phi\,\mathcal{D}z\,dx && \text{if, in addition, } \phi\in L^2(\Omega_2). \end{aligned} \tag{4.273}$$

Finally, we remark that, given $\phi \in L^2(\Omega_2)$ and $\omega \in W_0$ such that $\Delta\omega \in L^2(\Omega_2)$ and $\partial_{\mathbf{n}}\omega = 0$ on $\partial\Omega_2\backslash\overline{\Gamma}$ (this homogeneous Neumann condition is essential), we can write

$$\begin{aligned} (-\Delta\omega,\mathcal{D}\phi) &= \int_{\Omega_2} -\Delta\omega\,\mathcal{D}\phi\,dx = \int_{\Omega_2} \nabla\omega\cdot\nabla(\mathcal{D}\phi)\,dx \\ &= (\!(\omega,\mathcal{D}\phi)\!)_{W_0} = {}_{W_0'}\langle\phi,\omega\rangle_{W_0} = \int_{\Omega_2}\omega\phi\,dx. \end{aligned} \tag{4.274}$$

All this machinery permits to state our first preliminary

Lemma 4.2.12. *Given $\zeta \in L^2(0,T;H^{1/2}(\Gamma))$ and $u_2 \in L^2(0,T;L^2(\Omega_2))$, consider the following singular perturbation problem:*

$$(4.275)\qquad \begin{cases} u_{2,n} - \nu_{2,n}\Delta u_{2,n} = u_2 & \textit{in } \Omega_2\times]0,T[\\ u_{2,n} = \zeta & \textit{on } \Gamma\times]0,T[\\ \partial_{\mathbf{n}} u_{2,n} = 0 & \textit{on } (\partial\Omega_2\backslash\overline{\Gamma})\times]0,T[\end{cases}$$

Then, $u_{2,n} \in L^2(0,T;H^1(\Omega_2))$ for all $n \in \mathbb{N}$ and, as $n \to \infty$, the following relations hold:

$$(4.276)\qquad u_{2,n} \to u_2 \quad \textit{in } L^2(0,T;L^2(\Omega_2))\textit{–strong,}$$

$$(4.277)\qquad \nu_{2,n}^{1/2}\nabla u_{2,n} \to 0 \quad \textit{in } L^2(0,T;L^2(\Omega_2))\textit{–strong,}$$

$$(4.278)\qquad \nu_{2,n}^{-1/2}\|u_{2,n} - u_2\|_{L^2(0,T;W_0')} \quad \textit{is bounded,}$$

$$(4.279)\qquad \nu_{2,n}^{-\epsilon/2}\|u_{2,n} - u_2\|_{L^2(0,T;H^{-\epsilon}(\Omega_2))} \quad \textit{is bounded for every } \epsilon \in]0,1/2[.$$

Proof. For the sake of simplicity, we drop the dependence on t and we prove the lemma in the stationary case, only minor adjustments (i.e. integration in time of the various terms) being required for the time dependent setting.

It is well-known that problem (4.275) can be rewritten as a variational equality in the affine manifold $W_\zeta := W_0 + \mathcal{R}\zeta$ in the following way

$$(4.280)\qquad \begin{cases} u_{2,n} \in W_\zeta \\ \int_{\Omega_2} \left((u_{2,n} - u_2)(v - \mathcal{R}\zeta) + \nu_{2,n}\nabla u_{2,n} \cdot \nabla(v - \mathcal{R}\zeta)\right) dx = 0, \end{cases}$$

where the equation must hold for all $v \in W_\zeta$.

Choosing now $v = u_{2,n}$ in the preceding formula, by standard techniques, we infer:

$$(4.281)\qquad \begin{aligned} &\frac{1}{2}\|u_{2,n}\|^2_{L^2(\Omega_2)} + \frac{\nu_{2,n}}{2}\|\nabla u_{2,n}\|^2_{L^2(\Omega_2)} \\ &\qquad \le \frac{1}{2}\|u_2\|^2_{L^2(\Omega_2)} + \frac{\nu_{2,n}}{2}\|\nabla \mathcal{R}\zeta\|^2_{L^2(\Omega_2)} + (u_{2,n} - u_2, \mathcal{R}\zeta), \end{aligned}$$

whence, as a first step, we easily derive by boundedness the convergences (4.276–4.277) in the weak topologies; at this point, we notice that the last two terms on the right hand side of (4.281) tend to 0; consequently, we have

$$(4.282)\qquad \limsup_{n\to\infty}\left[\|u_{2,n}\|^2_{L^2(\Omega_2)} + \nu_{2,n}\|\nabla u_{2,n}\|^2_{L^2(\Omega_2)}\right] \le \|u_2\|^2_{L^2(\Omega_2)}$$

and, owing for instance to [9, Prop. 1.4, page 14], relations (4.276–4.277) follow now in the strong topology.

In order to obtain (4.278), we multiply the first equation of (4.275) by $\mathcal{D}(u_{2,n} - u_2)$, so that, invoking (4.271) and (4.273–4.274), we infer

$$(4.283)\qquad \begin{aligned} \|u_{2,n} - u_2\|^2_{W_0'} &= \nu_{2,n}\int_{\Omega_2} \Delta(u_{2,n} - \mathcal{R}\zeta)\,\mathcal{D}(u_{2,n} - u_2)\,dx \\ &= -\nu_{2,n}\int_{\Omega_2} (u_{2,n} - \mathcal{R}\zeta)\,(u_{2,n} - u_2)\,dx \\ &\le \nu_{2,n}\|u_{2,n} - \mathcal{R}\zeta\|_{L^2(\Omega_2)}\|u_{2,n} - u_2\|_{L^2(\Omega_2)} \le \nu_{2,n} C_4, \end{aligned}$$

where $C_4 > 0$ is a constant depending only on estimate (4.276).

Finally, we observe that (4.279) is a consequence of (4.276), (4.278) (notice that (4.278) entails *in particular* the boundedness of $\nu_{2,n}^{-1/2}\|u_{2,n} - u_2\|_{L^2(0,T;H^{-1}(\Omega_2))}$) and of Lemma 4.1.4. ∎

We now consider the limit behaviour of functionals $G_{n,T}$ for $n \to \infty$, neglecting, as before, the dependence on time.

Lemma 4.2.13. *$G_{n,T} \to G_T$ in the sense of Mosco.*

Proof. It is given for G_n. Observing that the sequence G_n is nonincreasing, owing to Prop. 1.3.9 (b), we can say that $G_n \to \mathrm{sc}^- \inf G_n$ in the sense of Mosco, where, by $\mathrm{sc}^- \inf G_n$, we mean the lower semicontinous regularization (see [31, Chapter 3] or Subsec. 1.2.1) of the functional $\mathcal{H} := \inf G_n$.

Also, it is immediate to verify that $\mathcal{H}(v)$ for $v \in H$ is given by (compare with (4.240))

$$\mathcal{H}(v) := \begin{cases} \dfrac{\nu_1}{2}\displaystyle\int_{\Omega_1} |\nabla v_1|^2\,dx & \text{if } v \in V = H^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.284}$$

So, what remains to prove is that $G = \mathrm{sc}^- \mathcal{H}$. For this purpose, we use the sequential characterization of relaxed functionals [31, Prop. 3.6, page 29] (see also Prop. 1.2.1), by requiring:

$$G(v) \le \liminf_{n\to\infty} \mathcal{H}(v_n) \quad \text{for every } v_n \to v \text{ in } H\text{–strong;} \tag{4.285}$$

$$\text{for every } u \in D(G), \text{ there exists } u_n \subset D(\mathcal{H}) \text{ such that} \tag{4.286}$$
$$u_n \to u \text{ in } H\text{–strong and } G(u) = \lim_{n\to\infty} \mathcal{H}(u_n).$$

Now, (4.285) can be easily checked by lower semicontinuity of norms with respect to the weak convergence.

With regard to the proof of (4.286), if we define $\zeta := (u_1)_{|\Gamma}$ (in the sense of traces), it is possible to take as u_n the original u_1 in Ω_1 and the function $u_{2,n}$ given by (4.275) in Ω_2 (as for the extension to the time-dependent case, we remark that $\zeta \in L^2(0,T;H^{1/2}(\Gamma))$, as it can be verified by recalling Remark 4.2.10, so that Lemma 4.2.12 can be actually applied). The strong convergence $u_n \to u$ in $L^2(\Omega)$ is now given by (4.276), while we even have $G(u) = \mathcal{H}(u_n)$ for every $n \in \mathbb{N}$, as desired.

We point out that, at this level, other approximating procedures could be used for the construction of u_n, but we chose this one since we are going to repeat it in a moment for the global functional $(J_{n,T} + G_{n,T})$. ∎

Theorem 4.2.14. *$(J_{n,T} + G_{n,T}) \to (J_T + G_T)$ in the sense of Mosco.*

Proof. Thanks to Lemma 4.2.11, to Lemma 4.2.13, and to Prop. 1.3.11 we have that $(J_T + G_T) \le \Gamma_w\text{-}\liminf_{n\to\infty}(J_{n,T} + G_{n,T})$ (and we point out again that this could be proved directly without passing through the M-convergences of $G_{n,T}$ and $J_{n,T}$); so, it remains to show condition (1.60) of Prop. 1.3.5, i.e. that, for every $u \in D(J_T + G_T)$, we can find a sequence $(u_n) \subset L^2(0,T;H)$, with $u_n \in D(J_{n,T} + G_{n,T})$ for every $n \in \mathbb{N}$, such that $u_n \to u$ strongly in $L^2(0,T;H)$ and $\liminf_{n\to\infty}((J_T+G_T)(u)-(J_{n,T}+G_{n,T})(u_n)) \geq 0$.

Coming back again to the stationary case, we now choose u_n exactly as in the last lemma; that is $u_n := u_1$ in Ω_1 and $u_n := u_{2,n}$ in Ω_2, with $u_{2,n}$ given by (4.275) with $\zeta := (u_1)_{|\Gamma}$. So, we have

$$(J+G)(u)-(J_n+G_n)(u_n) = \int_{\Omega_1} j_1(u_1)\,dx - \int_{\Omega_1} j_{1,n}(u_1)\,dx - \frac{\nu_{2,n}}{2}\int_{\Omega_2}|\nabla u_{2,n}|^2\,dx + \int_{\Omega_2} j_2(u_2)\,dx - \int_{\Omega_2} j_{2,n}(u_{2,n})\,dx. \tag{4.287}$$

Now, owing to (4.208) (with $i=1$), we immediately see that

$$\int_{\Omega_1} j_1(u_1)\,dx - \int_{\Omega_1} j_{1,n}(u_1)\,dx \geq 0; \tag{4.288}$$

moreover, we infer from (4.277) that

$$\frac{\nu_{2,n}}{2}\int_{\Omega_2}|\nabla u_{2,n}|^2\,dx \to 0. \tag{4.289}$$

Finally, due again to relation (4.208) (with $i=2$) and recalling (4.275) and the definition of subdifferential, we deduce that

$$\begin{aligned}\int_{\Omega_2} j_2(u_2)\,dx - \int_{\Omega_2} j_{2,n}(u_{2,n})\,dx &\geq \int_{\Omega_2} j_{2,n}(u_2)\,dx - \int_{\Omega_2} j_{2,n}(u_{2,n})\,dx \\ &\geq \int_{\Omega_2} \alpha_{2,n}(u_{2,n})(u_2-u_{2,n})\,dx = -\int_{\Omega_2}\alpha_{2,n}(u_{2,n})(\nu_{2,n}\Delta u_{2,n})\,dx \\ &= \nu_{2,n}\int_{\Omega_2}\alpha'_{2,n}(u_{2,n})|\nabla u_{2,n}|^2\,dx - {}_{-1/2,\Gamma}\langle \nu_{2,n}\partial_{\mathbf{n}}u_{2,n}, \alpha_{2,n}(\zeta)\rangle_{1/2,\Gamma}.\end{aligned} \tag{4.290}$$

Note that in the last passage we have exploited a generalized version of the Gauss-Green formula [41, Coroll. 2.6, page 28] (recall also Theorem 1.1.6 (c)); moreover, observe that in order that the last duality make sense, the homogeneous Neumann condition in (4.275) was essential (cfr. Remark 1.1.9).

Now, the first term on the right hand side of the preceding expression is clearly nonnegative thanks to the monotonicity of $\alpha_{2,n}$; moreover, owing to (4.204), (4.275), (4.279) and to Cor. 4.1.3, we have, for some constants $C_5, C_6, C_7 > 0$ and for δ as in (4.205),

$$\begin{aligned} -{}_{-1/2,\Gamma}\langle \nu_{2,n}\partial_{\mathbf{n}}u_{2,n}, \alpha_{2,n}(\zeta)\rangle_{1/2,\Gamma} &\geq -\|\alpha_{2,n}(\zeta)\|_{1/2,\Gamma}\|\nu_{2,n}\partial_{\mathbf{n}}u_{2,n}\|_{-1/2,\Gamma} \\ &\geq -C_5\|\alpha_{2,n}(u_1)\|_{H^1(\Omega_1)}\left(\|\nu_{2,n}u_{2,n}\|_{H^1(\Omega_2)} + \|\nu_{2,n}\Delta u_{2,n}\|_{H^{-1/2}(\Omega_2)}\right) \\ &\geq -C_6\,L_n\|u_1\|_{H^1(\Omega_1)}\left(\|\nu_{2,n}u_{2,n}\|_{H^1(\Omega_2)} + \|u_{2,n}-u_2\|_{H^{-1/2+\delta}(\Omega_2)}\right) \\ &\geq -C_6\,L_n\,\nu_{2,n}^{1/4-\delta/2}\|u_1\|_{H^1(\Omega_1)}\left(\|\nu_{2,n}^{3/4+\delta/2}u_{2,n}\|_{H^1(\Omega_2)} + C_7\right)\end{aligned} \tag{4.291}$$

(actually C_7 is the constant bounding (4.279) for the chosen exponent $\epsilon = 1/2-\delta$).

Now, since $u_1 \in H^1(\Omega_1)$ is fixed, recalling also (4.276–4.277) and exploiting (4.205), we easily see that the preceding expression tends to 0; collecting now the information of (4.288–4.291), the proof of Theorem 4.2.14 can be easily completed. ∎

Proof of existence. First of all, it is possible to reason as in (4.263) in order to deduce (4.186). Naturally, in this case, hypothesis (4.206) has to be exploited for the Ω_2-components; the boundedness on Ω_1, instead, is guaranteed by (4.235), whence also the convergence (4.217) and the related regularity (4.212) (with L^∞ in place of C^0, anyway) follow. Now, (4.185) is a consequence of (4.230), and, coupled with (4.238) and (4.186), entails (4.184). All this procedure also guarantees (4.190–4.192). Furthermore, (4.218–4.219) are immediate consequences of (4.236–4.237). This allows also to derive (4.189) and (4.215) from (4.168), (4.169), (4.183) and (4.210). Finally, equation (4.187) is obtained from (4.165) as in the one-domain case.

As we already pointed out at the end of last section, the rest of the statement is proved if we are able to apply Prop. 1.3.8 to operators $\partial(J_{n,T} + G_{n,T})$ and $\partial(J_T + G_T)$. Recalling the notation (4.256) and (4.258), we see that the evaluation of the scalar product (in $L^2(0,T;H)$) (γ_n, χ_n) is analogous (and even simpler since here we have no space diffusion terms) to the corresponding procedure (4.269–4.270) we used for the one-domain problem (naturally, here it is necessary to separe the contributions of Ω_1 and Ω_2 and use for the first ones the strong convergence $\chi_{1,n} \to \chi_1$ in $L^2(0,T;L^2(\Omega_1))$, which is a consequence of (4.217)). We only point out one difference: here, the mixed unknowns term is adjusted in the following way (compare with (4.270)):

$$\text{(4.292)}\quad \int_0^T \int_\Omega \ell\theta_n\chi_n\,dx\,ds = \sum_{i=1}^2 \int_0^T \int_{\Omega_i} \ell_i\theta_{i,n}\chi_{i,n}\,dx\,ds = \sum_{i=1}^2 \left[\int_0^T \int_{\Omega_i} \frac{\ell_i}{\lambda_i}\theta_{i,n}e_{i,n}\,dx\,ds - \int_0^T \int_{\Omega_i} \frac{\rho_i\ell_i}{\lambda_i}|\theta_{i,n}|^2\,dx\,ds \right].$$

Now, if we define ζ_n as $\ell_i\theta_{i,n}/\lambda_i$ in the domain Ω_i for $i = 1, 2$ (and a limit ζ is constructed from θ in an analogous way), it is well known that the first of (4.191) entails that $\zeta_n \to \zeta$ weakly in $L^2(0,T;H^{1/4}(\Omega))$; thus, (4.268) permits to pass to the limit in the first term on the right hand side, while the second one can be managed as before through a semicontinuity argument. Finally, that the C^0 in (4.212) is as usual a consequence of Theorem 1.1.16. ∎

Proof of uniqueness. Let us suppose to have a couple of solutions, say $(\hat{e}, \hat{\theta}, \hat{\chi})$ and $(\check{e}, \check{\theta}, \check{\chi})$ to (TP). Define also $e := \hat{e} - \check{e}$, $\theta := \hat{\theta} - \check{\theta}$, $\chi := \hat{\chi} - \check{\chi}$.

Now, write equation (4.187) for the two solutions, take the difference and multiply it by $\mathcal{F}^{-1}e$; integrating in $]0,t[$ and proceeding as in (4.221), we infer

$$\text{(4.293)}\quad \frac{1}{2}\|e(t)\|_{V'}^2 + \|\rho^{1/2}\theta\|_{L^2(0,t;H)}^2 = -\int_0^t \int_\Omega \lambda\chi\theta\,dx\,ds.$$

Writing now (4.213) for the two solutions, taking the difference and testing it with χ_1, working as above, we get

$$\frac{\mu_1}{2}\|\chi_1(t)\|^2_{L^2(\Omega_1)} + \nu_1\|\nabla\chi_1\|^2_{L^2(0,t;L^2(\Omega_1))} \leq c_1\|\chi_1\|^2_{L^2(0,t;L^2(\Omega_1))} + \int_0^t\int_{\Omega_1} \ell_1\theta_1\chi_1\,dx\,ds. \tag{4.294}$$

Taking χ_2 as a test function in (4.214) and proceeding as before, thanks to the monotonicity of α_2, we get $\int_0^t\int_{\Omega_2}\theta_2\chi_2\,dx\,ds \geq 0$, so that, calculating (4.293)+$(\lambda_1\ell_1^{-1})$(4.294) and applying the Gronwall inequality, it is straightforward to conclude. ∎

An application. We finally present in some detail the particular physical situation which motivated this analysis. Take in (TP) $\alpha_1(r) = r^3$ and $\alpha_2(r) = \partial I_{[-1,1]}(r)$, $I_{[-1,1]}(r)$ denoting here the *indicator function* of $[-1,1]$, that is $I_{[-1,1]}(r) = 0$ for $r \in [-1,1]$ and $I_{[-1,1]}(r) = +\infty$ otherwise (we are assuming here that the solid state is represented by $\chi = -1$ instead of $\chi = 0$).

Let also $\alpha_{1,n}$ be the Yosida regularization of α_1 of index n^{-1} and $\alpha_{2,n}$ that of α_2 of index L_n^{-1}, where L_n satisfies (4.205). Now the limit situation (TP) accounts for a heat transmission problem between two fluids, of which one (Ω_1) obeys to the Caginalp-Fix phase field model with a double-well (Ginzburg-Landau) energy potential and the other (Ω_2) to the two-phase Stefan model. In the approximating framework $(\mathrm{TP_n})$, instead, a heat diffusion dynamics of Caginalp-Fix type holds on *both* sides. We eventually remark that the equations of $(\mathrm{TP_n})$ are coupled also by (implicit) phase transmission conditions at the interface Γ, while those of (TP) are only coupled through (4.187), which is a global variational equality in the whole Ω.

Bibliography

[1] R. Adams, "Sobolev Spaces", Academic Press, 1975.

[2] I. Athanasopoulos, L. A. Caffarelli, S. Salsa, *Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems*, Ann. of Math., **143** (1996), 413–434.

[3] I. Athanasopoulos, L. A. Caffarelli, S. Salsa, *Regularity of the free boundary in parabolic phase-transition problems*, Acta Math., **176** (1996), 245–282.

[4] I. Athanasopoulos, L. A. Caffarelli, S. Salsa, *Phase transition problems of parabolic type: flat free boundaries are smooth*, Comm. Pure Appl. Math., **51** (1998), 77–112.

[5] H. Attouch, "Variational Convergence for Functions and Operators", Pitman, London, 1984.

[6] T. Aubin, "Nonlinear Analysis on Manifolds. Monge-Ampere Equations", Springer-Verlag, New York-Berlin, 1982.

[7] C. Baiocchi, *Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert*, Ann. Mat. Pura Appl. IV, **76** (1967), 233–304.

[8] C. Baiocchi, *On some trace-estimates*, in "Boundary Value Problems for Partial Differential Equations and Applications", RMA Res. Notes Appl. Math., **29** (1993), 293–297.

[9] V. Barbu, "Nonlinear Semigroups and Differential Equations in Banach Spaces", Noordhoff, Leyden, 1976.

[10] V. Barbu, P. Colli, G. Gilardi, M. Grasselli, *Existence, uniqueness, and long-time behaviour for a generalized Stefan problem with memory*, submitted.

[11] J. Bergh and J. Löfström, "Interpolation Spaces. An Introduction", Springer-Verlag, Berlin-Heidelberg, 1976.

[12] G. Bonfanti, F. Luterotti, *Regularity and convergence results for a phase-field model with memory*, Math. Methods Appl. Sci., **21** (1998), 1085–1105.

[13] H. Brezis, *Intégrales convexes dans les espaces de Sobolev*, Israel J. Math., **13** (1972), 9–23.

[14] H. Brezis, "Opérateurs Maximaux Monotones et Sémi-groupes de Contractions dans les Espaces de Hilbert", North-Holland Math. Studies **5**, North-Holland, Amsterdam, 1973.

[15] H. Brezis, "Analyse Fonctionnelle", Masson, Paris, 1983.

[16] H. Brezis, M. Crandall, *Uniqueness of solutions of the initial-value problem for* $u_t - \Delta\phi(u) = 0$, J. Maths Pures Appl., **58** (1979), 153–163.

[17] H. Brezis, M. Crandall, A. Pazy, *Perturbations of nonlinear maximal monotone sets*, Comm. Pure Appl. Math., **23** (1970), 123–144.

[18] L. A. Caffarelli, *Some aspects of the one-phase Stefan problem*, Indiana Univ. Math. J., **27** (1978), 73–77.

[19] G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal., **92** (1986), 205–245.

[20] G. Caginalp, P. C. Fife, *Dynamics of layered interfaces arising from phase boundaries*, SIAM J. Appl. Math., **48** (1988), 506–517.

[21] R. W. Carroll, R. E. Showalter, "Singular and Degenerate Cauchy Problems", Academic Press, New York, 1976.

[22] E. Cellurale, "Problema di Stefan in spazi di Nikol'skij con peso", PhD. Thesis, University of Pavia, 1997.

[23] M. Cessenat, "Mathematical Methods in Electromagnetism – Linear Theory and Applications", World Scientific Publishing Co., Series on Advances in Mathematics for Applied Sciences, **41**, Singapore, 1996.

[24] B. D. Coleman, M. E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys., **18** (1967), 199–208.

[25] P. Colli, G. Gilardi, M. Grasselli, *Global smooth solution to the standard phase-field model with memory*, Adv. Differential Equations, **2** (1997), 453–486.

[26] P. Colli, G. Gilardi, M. Grasselli, *Well-posedness of the weak formulation for the phase-field model with memory*, Adv. Differential Equations, **2** (1997), 487–508.

[27] P. Colli, G. Gilardi, M. Grasselli, *Asymptotic analysis of a phase-field model with memory for vanishing time relaxation*, Hiroshima Math. J., **29** (1999), 117–143.

[28] P. Colli, G. Gilardi, M. Grasselli, G. Schimperna, *The conserved phase-field system with memory*, submitted.

[29] P. Colli, G. Gilardi, Ph. Laurençot, A. Novick–Cohen, *Uniqueness and long-time behaviour for the conserved phase-field system with memory*, Discrete Contin. Dynam. Systems, **5** (1999), 375–390.

[30] P. Colli, J. Sprekels, *Stefan problems and the Penrose-Fife phase field model*, Adv. Math. Sci. Appl., **2** (1997), 911–934.

[31] G. Dal Maso, “An Introduction to Γ-convergence”, Birkhäuser, Boston, 1993.

[32] A. Damlamian, *Some results on the multi-phase Stefan problem*, Comm. Partial Differential Equations, **10** (1977), 1017–1044.

[33] A. Damlamian, N. Kenmochi, *Evolution equations generated by subdifferentials in the dual space of $H^1(\Omega)$*, Discrete Contin. Dynam. Systems, **5** (1999), 269–278.

[34] A. Damlamian, N. Kenmochi, N. Sato, *Subdifferential operator approach to a class of nonlinear systems for Stefan problems with phase relaxation*, Nonlinear Anal., **23** (1994), 115–142.

[35] M. C. Delfour, J. P. Zolésio, *Shape analysis via oriented distance functions*, J. Funct. Anal., **86** (1989), 129–201.

[36] I. Ekeland, R. Temam, “Analyse Convexe et Problèmes Variationnels”, Dunod, Gauthier-Villars, Paris, 1974.

[37] L. C. Evans, R. Gariepy, “Measure Theory and Fine Properties of Functions”, Studies in Advanced Mathematics, CRC Press, 1992.

[38] A. Fasano, M. Primicerio, L. Rubinstein, *A model problem for heat conduction with a free boundary in a concentrated capacity*, J. Inst. Maths. Applics., **26** (1980), 327–347.

[39] P. Fernandes, G. Gilardi, *Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions*, Math. Models Methods Appl. Sci., **7** (1997), 957–991.

[40] G. J. Fix, *Phase field models for free boundary problems*, in “Free boundary problems: theory and applications; vol. II”, A. Fasano and M. Primicerio eds., Pitman Res. Notes Math. Ser., **79**, Longman, London, 1983, 580–589.

[41] V. Girault, P. A. Raviart, “Finite Element Methods for the Navier-Stokes Equations”, Springer-Verlag, Berlin-Heidelberg, 1986.

[42] P. Grisvard, “Elliptic Problems in Nonsmooth Domains”, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985.

[43] M. E. Gurtin, A. C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal., **31** (1968), 113–126.

[44] D. D. Joseph, L. Preziosi, *Heat waves*, Rev. Modern Phys., **62** (1990), 375–391.

[45] S. L. Kamenomostskaja, *On the Stefan problem*, Mat. Sb., **53** (1961), 489–514 (Russian).

[46] N. Kenmochi, *Neumann problems for a class of nonlinear degenerate parabolic equations*, Differential Integral Equations, **3** (1990), 253–273.

[47] N. Kenmochi, M. Niezgódka, *Non-linear system for non-isothermal diffusive phase separation*, J. Math. Anal. Appl., **188** (1994), 651–679.

[48] N. Kenmochi, M. Niezgódka, I. Pawlow, *Subdifferential operator approach to the Cahn-Hilliard equation with constraint*, J. Differential Equations, **117** (1995), 320–356.

[49] N. Kenmochi, *Systems of nonlinear PDEs arising from dynamical phase transitions*, in "Phase transitions and hysteresis", lectures given in Montecatini Terme at the 3rd session of C.I.M.E., July 13–21, 1993; Lecture Notes in Maths. 1584 (A. Visintin ed.), Springer-Verlag, Heidelberg, 1994.

[50] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type", Translations of Mathematical Monographs, **23**, Amer. Math. Soc., Providence, 1967.

[51] L. D. Landau, E. M. Lifshitz, "Statistical Physics", Addison-Wesley Publishing, Reading, Massachusetts, 1958.

[52] Ph. Laurençot, *Solutions to a Penrose-Fife model of phase-field type*, J. Math. Anal. Appl., **185** (1994), 262–274.

[53] J. L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires", Dunod, Gauthier-Villars, Paris, 1969.

[54] J. L. Lions, E. Magenes, "Non-homogeneous Boundary Value Problems and Applications", volume I, Springer-Verlag, Berlin, 1972.

[55] S. Luckhaus, *Solutions for the two-phase Stefan problem with the Gibbs-Thomson law for the melting temperature*, European J. Appl. Math., **1** (1990), 101–111.

[56] E. Magenes, *On a Stefan problem on the boundary of a domain*, P.D.E.'s and related subjects (M. Miranda ed.), Longman Scient. Techn., 1992, 209–226.

[57] E. Magenes, *Some new results on a Stefan problem in a concentrated capacity*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur. Rend. Lincei IX, **3** (1992), 23–34.

[58] E. Magenes, *The Stefan problem in a concentrated capacity*, Atti Simp. Int. "Problemi attuali dell'analisi e della fisica matematica" (P. E. Ricci ed.), Dip. di Matematica, Univ. "La Sapienza", Roma, 1993, 155–182.

[59] E. Magenes, *Regularity and approximation properties for the solution of a Stefan problem in a concentrated capacity*, Proc. Int. Workshop "Variational Methods, Nonlinear Analysis and Differential Equations", E.C.I.G., Genova, 1994, 88–106.

[60] E. Magenes, *On a Stefan problem in a concentrated capacity*, P.D.E. and Applic. (P. Marcellini, G. Talenti, E. Vesentini eds.), Marcel Dekker, Inc., 1996, 237–253.

[61] E. Magenes, *Stefan problems in a concentrated capacity*, Adv. Math. Comp. and Appl., Proc. AMCA95, N.C.C. Publ., Novosibirsk, 1996, 82–90.

[62] E. Magenes, C. Verdi, A. Visintin, *Semigroup approach to the Stefan problem with nonlinear flux conditions*, Atti Accad. Naz. Lincei. Cl. Sci. Fis. Mat. Natur. Rend. Lincei VIII, **75** (1983), 24–33.

[63] U. Massari, M. Miranda, "Minimal Surfaces of Codimension One", North-Holland, Amsterdam, 1984.

[64] A. M. Meirmanov, "The Stefan Problem", de Gruyter, Berlin-New York, 1992.

[65] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math., **3** (1969), 510–585.

[66] U. Mosco, *On the continuity of the Young-Fenchel transformation*, J. Math. Anal. Appl., **35** (1971), 518–535.

[67] J. Nečas, "Les Méthodes Directes en Theorie des Equations Elliptiques", Masson, Paris, 1967.

[68] A. Novick–Cohen, *Conserved phase-field equations with memory*, in "Curvature flows and related topics", A. Damlamian, J. Spruck and A. Visintin eds., 179–197, GAKUTO Internat. Ser. Math. Sci. Appl., **5**, Gakkōtosho, Tokyo, 1995.

[69] O. A. Oleinik, *A method of solution of the general Stefan problem*, Soviet Math. Dokl., **1** (1960), 1350–1354.

[70] O. Penrose, P. C. Fife, *Thermodynamically consistent models of phase-field type for the kinetics of phase transitions*, Physica D, **43** (1990), 44–62.

[71] P. I. Plotnikov, V. N. Starovoitov, *The Stefan problem with surface tension as the limit of a phase field model*, Differential Equations, **29** (1993), 395–404.

[72] R. T. Rockafellar, *Integrals which are convex functionals*, Pacific J. Math., **24** (1968), 525–539.

[73] R. T. Rockafellar, *Convex functions, monotone operators and variational inequalities; theory and applications of monotone operators*, A. Ghizzetti ed., Tipografia Oderisi Editrice, Gubbio (Italy), 1969, 34–65.

[74] R. T. Rockafellar, *Integrals which are convex functionals II*, Pacific J. Math., **39** (1971), 439–469.

[75] W. Rudin, "Functional Analysis", McGraw-Hill, New York, 1973.

[76] G. Savaré, A. Visintin, *Variational convergence of nonlinear diffusion equations: applications to concentrated capacity problems with change of phase*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur. Rend. Lincei IX Mat. Appl., **8** (1997), 49–89.

[77] G. Schimperna, *Weak solution to a phase-field transmission problem in a concentrated capacity*, Math. Methods Appl. Sci., **22** (1999), 1235–1254.

[78] G. Schimperna, *Asymptotic behaviour of a transmission problem for the parabolic phase-field model*, Appl. Math., to appear.

[79] G. Schimperna, *Abstract approach to evolution equations of phase-field type and applications*, J. Differential Equations, to appear.

[80] G. Schimperna, *Some convergence results for a class of nonlinear phase-field evolution equations*, submitted.

[81] G. Schimperna, *Convergence of phase-field equations to the Stefan model*, Proceedings of the PCC99 ESF-AMIF Workshop (Warsaw, Poland, 24–27/6/1999), T. A. Kowalewski, F. Stella, J. Banaszek, J. Szmyd eds., IPPT-PAN Reports, **5** (1999), 131–134.

[82] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. IV, **146** (1987), 65–96.

[83] J. Sprekels, S. Zheng, *Global smooth solutions to a thermodynamically consistent model of phase-field type in higher space dimensions*, J. Math. Anal. Appl., **176** (1993), 200–223.

[84] J. Stefan, *Über einige probleme der theorie der wärmeleitung*, Sitzungsberichte der Österreichischen Akademie der Wissenschaften Mathematisch-Naturwissenschaftliche Klasse, Abteilung 2, Mathematik, Astronomie, Physik, Meteorologie und Technik, **98** (1889), 473–484; see also, *ibidem*, 616–634, 965–983, 1418–1442.

[85] A. Visintin, *Stefan problem with phase relaxation*, IMA J. Appl. Math., **34** (1985), 225–245.

[86] A. Visintin, "Models of Phase Transitions", Progress in Nonlinear Differential Equations and their Applications, **28**, Birkhäuser, Boston, 1996.

[87] A. Visintin, *Introduction to the models of phase transitions*, Boll. Un. Mat. Ital. Sez. B Artic. Ric. Mat. VIII, **1** (1998), 1–47.