Local solution to Frémond’s full model for irreversible phase transitions

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1. – Introduction

In this work, we aim to analyze the following system of PDE’s:

(1) \[ \theta_t - \theta \chi_t - \Delta \theta = \chi_t^2, \]

(2) \[ \chi_t + \alpha(\chi_t) - \Delta \chi + \beta(\chi) \geq \theta - \theta_c. \]

The system above describes the irreversible phase transition process of a homogeneous substance located inside a bounded container \( \Omega \subset \mathbb{R}^3 \). The evolution of this material is ruled by two state variables, i.e., the absolute temperature \( \theta \) and the phase field \( \chi \). We note that, in (1–2), \( \theta_c > 0 \) is the assigned phase transition temperature and \( \alpha, \beta \subset \mathbb{R} \times \mathbb{R} \) are suitable maximal monotone graphs yielding the desired constraints for the behavior of \( \chi \). More in detail, we assume that \( \alpha = \partial I_{[0,+,\infty)} \), so that \( \chi_t \geq 0 \), and \( \beta = \partial I_{[0,1]} \), that gives \( 0 \leq \chi \leq 1 \). Indeed, \( \chi = 0 \) (\( \chi = 1 \)) is assumed to stand for the pure solid phase (pure liquid, respectively) and \( 0 < \chi < 1 \) denotes the presence of a mixture. Of course, the irreversibility of the process is accounted for by the constraint on \( \chi_t \).

The above model was derived by Frémond et al. [2, 4] starting from the consideration that the microscopic movements of molecules may significantly influence the phase transition process described by the macroscopic variables \( \theta, \chi \). Indeed, the papers quoted above present a detailed derivation of (1–2) starting from physical considerations and in particular give a proof of its thermodynamic consistence via the Clausius-Duhem inequality.

Although much work has been devoted to the analysis of (1–2) and several variants of it [1, 2, 3, 5, 8], up to now the existence of a (global in time) solution is known just in one space dimension [7] (see also [10] for the reversible case, i.e., for \( \alpha \equiv 0 \)). The result of [7] is based on a physically meaningful approximation of (1–2), where a finite maximum speed \( \lambda > 0 \) is imposed to the phase change process by taking \( \alpha = \partial I_{[0,\lambda]} \) in (2). Such a modified system is more accessible from the mathematical
point of view since the above choice for $\alpha$ guarantees the uniform boundedness of $\chi_t$ that is a useful tool for deriving suitable a priori estimates. Actually, a global existence result for the $\lambda$-system is shown in the paper [9]. Starting from this approximation, the authors of [7] prove global existence for the original system (1–2) by letting $\lambda \to +\infty$ and mainly relying on a sharp $\lambda$-independent estimate, originally devised to Dafermos and Hsiao [6] for the study of thermoelasticity, that allows them to get a global $L^2$ boundness for $\theta$ despite of the quadratic growth of the right hand side of (2). Unfortunately, this argument (also exploited in [10]) is strongly dependent on the choice of the one dimensional setting and can not be adapted to our case.

Hence, in this paper, we come back to the $\lambda$-regularized system and derive local-in-time a priori estimates, independent of $\lambda$, for the solution of the approximating problem. The key point is the control of the high power terms resulting from the quadratic nonlinearities in (1); this is reached by performing an accurate choice of the test functions for (2) that permits us to express such terms in the right norms and control them for small times by an extended Gronwall inequality [12].

The rest of the paper is organized as follows: in the next section we present some mathematical preliminaries and detail the precise hypotheses of the problem and the existence theorem. Then, the proof is achieved in Section 3 by a priori estimates and a compactness argument.

2. – Preliminaries and main result

We start by fixing some notations. Let $\Omega \subset \mathbb{R}^3$ be a smooth and bounded domain and $T > 0$ be a final time. Then, set $Q_t := \Omega \times (0, t)$ for all $t \in (0, T]$ and $Q := Q_T$. Letting $n$ stand for the outer normal unit vector to $\partial \Omega$, we set $H := L^2(\Omega)$, $V := H^1(\Omega)$, $W := \{ u \in H^2(\Omega) \text{ such that } \partial_n u = 0 \text{ on } \partial \Omega \}$, endowed with the usual scalar products, and we denote by $\| \cdot \|$ the norm in $H$ and by $\| \cdot \|_E$ the norm of the generic normed space $E$. Finally, we let $V^*$ be the dual of $V$.

Next, we introduce our assumptions on data by requiring that

(3) $\theta_c > 0$ is a prescribed constant,
(4) $\alpha = \partial I_{[0, +\infty)}$, $\alpha_\lambda = \partial I_{[0, \lambda]}$ for $\lambda > 0$,
(5) $\beta = \partial I_{[0, 1]}$,
(6) $\theta_0 \in V \cap L^\infty(\Omega)$, $\theta_0 > 0$ a.e. in $\Omega$,
(7) $\chi_0 \in W^{2,\infty}(\Omega)$, $0 \leq \chi_0 \leq 1$ a.e. in $\Omega$,
(8) there exists $\eta_0 \in L^\infty(\Omega)$ such that $\eta_0 \in \beta(\chi_0)$ a.e. in $\Omega$.

Our main task is the proof of the following existence theorem:

**Theorem 2.1.** There exists a final time $T_0$, with $0 < T_0 \leq T$, and a quadruple $(\theta, \chi, \xi, \eta)$ of functions satisfying

(9) $\theta \in H^1(0, T_0; H) \cap C^0([0, T_0]; V) \cap L^2(0, T_0; W),$
(10) $\chi \in W^{1,\infty}(0, T_0; L^3(\Omega)) \cap L^\infty(0, T_0; W),$
(11) $\xi \in L^\infty(0, T_0; H), \quad \eta \in L^\infty(0, T_0; H),$
(12) $0 \leq \theta$, $0 \leq \chi \leq 1$, and $0 \leq \chi_t$ a.e. in $Q_{T_0}$.
and such that the following relations hold a.e. in $Q_{T_0}$:

\begin{align}
\theta_t + \theta \chi_t - \Delta \theta &= \chi_t^2, \\
\chi_t + \xi - \Delta \chi + \eta &= \theta - \theta_c, \\
\xi &\in \alpha(\chi_t) \quad \text{and} \quad \eta \in \beta(\chi).
\end{align}

Moreover, the following initial conditions hold:

\begin{align}
\theta(x,0) &= \theta_0(x) \quad \text{and} \quad \chi(x,0) = \chi_0(x) \quad \text{a.e. in } \Omega.
\end{align}

**Remark.** We note that the regularity properties of the solution stated in (9–10) are not optimal and one might easily improve them by deriving new a priori estimates and using bootstrap arguments. However, we prefer not to insist on this point and focus our attention just on the existence of a solution.

Let us now recall the main theorem of [9]. Actually, we state it in the slightly modified version proved in [7, Thm. 3.1]. We note that, although this result was originally presented in the one dimensional setting, by looking at the proof one can easily convince that – for fixed $\lambda > 0$ – it also holds in three space dimensions:

**Theorem 2.2.** There exists a quadruple $(\theta_\lambda, \chi_\lambda, \xi_\lambda, \eta_\lambda)$ of functions satisfying

\begin{align}
\theta_\lambda &\in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W), \\
\chi_\lambda &\in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W), \\
\xi_\lambda &\in L^\infty(0, T; H), \quad \eta_\lambda \in L^\infty(0, T; H), \\
0 \leq \theta_\lambda, \quad 0 \leq \chi_\lambda \leq 1, \quad \text{and} \quad 0 \leq \chi_M \leq \lambda \quad \text{a.e. in } Q
\end{align}

and such that the following relations hold a.e. in $Q$:

\begin{align}
\theta_M + \theta_\lambda \chi_M - \Delta \theta_\lambda &= \chi_M^2, \\
\chi_M + \xi_\lambda - \Delta \chi_\lambda + \eta_\lambda &= \theta_\lambda - \theta_c, \\
\xi_\lambda &\in \alpha(\chi_M) \quad \text{and} \quad \eta_\lambda \in \beta(\chi_\lambda).
\end{align}

Moreover, the following initial conditions hold:

\begin{align}
\theta_\lambda(x,0) &= \theta_0(x) \quad \text{and} \quad \chi_\lambda(x,0) = \chi_0(x) \quad \text{a.e. in } \Omega.
\end{align}

### 3. – Proof of Theorem 2.1

We consider a family $(\theta_\lambda, \chi_\lambda, \xi_\lambda, \eta_\lambda)$ of solutions to the $\lambda$-approximated problem with the regularities stated in Theorem 2.2 and derive some a priori estimates, independent of $\lambda$, in order to pass to the limit as $\lambda \to +\infty$. We point out that some estimates turn out to be formal in this setting; however, they could be made rigorous by performing them, e.g., in the framework of a time discretization argument (cf. [7, Section 3]). Furthermore, we note that in the following the symbol $c$ will stand for possibly different positive constants, that are assumed to depend only on $\Omega, T, \theta_c, \theta_0, \chi_0, \eta_0$. In particular, $c$ is not allowed to depend on $\lambda$. 

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**First estimate.** Multiply (21) by $1$ and (22) by $\chi_M$, sum the results and integrate in time over $(0, t)$ for $t \leq T$. Owing to the nonnegativity of $\theta_\lambda$ and on the constraint $0 \leq \chi_\lambda \leq 1$, and proceeding, e.g., as in [7, Subsec. 4.1], we get

$$
\|\theta_\lambda\|_{L^\infty(0,T;L^1(\Omega))} + \|\chi_\lambda\|_{L^2(0,T;V)} \leq c. \tag{25}
$$

**Key estimate.** Differentiate formally (22) with respect to time. Of course, this procedure can be made rigorous at a discrete level (or by using a difference quotients argument). It follows

$$
\chi_M t + \xi_M - \Delta \chi_M + \eta_M = \theta_M. \tag{26}
$$

Then, multiply (21) by $\theta_t$ and (26) by $\chi_M^2$, sum the results and integrate in time over $(0, t)$ for $t \leq T$. Noting that by the monotonicity of $\beta$ and the nonnegativity of $\chi_M$ it is $\eta_M \chi_M^2 \geq 0$ a.e. in $Q_t$, easy computations yield

$$
\|\theta_M\|_{L^2(Q_t)}^2 + \frac{1}{2} \|\nabla \theta(t)\|^2 + \frac{1}{3} \|\chi_M(t)\|^3_{L^3(\Omega)} + \frac{8}{9} \|\nabla (\chi_M^{3/2})\|_{L^2(Q_t)}^2 \\
\leq \frac{1}{2} \|\nabla \theta_0\|^2 + \frac{1}{3} \|\chi_M(0)\|^3_{L^3(\Omega)} + 2 \int_0^t \int_\Omega \theta_M \chi_M^2 \\
- \int_0^t \int_\Omega \theta \theta_M \chi_M - \int_0^t \int_\Omega \xi_M \chi_M^2,
$$

so that we have to control the five terms on the right hand side. Of course, the first one is bounded by (6). For the second, we compute formally (22) for $t = 0$ and note that

$$
\chi_M(0) \in (id + \alpha_L)^{-1}(\Delta x_0 - \eta_0 + \theta_0 - \theta_c) \quad \text{a.e. in } \Omega.
$$

Hence, noting that $(id + \alpha_L)^{-1}$ is a contraction, by (6–8) we derive that the $L^3$ norm of $\chi_M(0)$ is uniformly bounded in $\lambda$.

Then, we have to work with the three integral terms. By the elementary Young inequality and Sobolev’s embedding theorem, the first integral term gives

$$
2 \int_0^t \int_\Omega \theta_M \chi_M^2 \leq \frac{1}{2} \|\theta_M\|_{L^2(Q_t)}^2 + 2 \int_0^t \int_\Omega \chi_M^4
$$

and

$$
2 \int_0^t \int_\Omega \chi_M^4 \leq 2 \int_0^t \|\chi_M^{3/2}\|_V \|\chi_M^{5/2}\|_V^2 \\
\leq \frac{1}{3} \int_0^t \|\chi_M^{3/2}\|_V^2 + c \int_0^t \|\chi_M^{5/2}\|_{L^6(\Omega)}^2 \\
\leq \frac{1}{3} \|\chi_M\|_{L^3(Q_t)}^3 + \frac{1}{3} \|\nabla (\chi_M^{3/2})\|_{L^2(Q_t)}^2 + c \int_0^t \|\chi_M\|_{L^3(\Omega)}^5.
$$

As for the second integral on the right hand side of (27), we get

$$
\left| \int_0^t \int_\Omega \theta \theta_M \chi_M \right| \leq \frac{1}{4} \|\theta_M\|_{L^2(Q_t)}^2 + 2 \int_0^t \int_\Omega \chi_M^4 + \frac{1}{8} \int_0^t \int_\Omega \theta_c^4,
$$

whence the second term can be treated as in (28), while the third one yields

$$
\frac{1}{8} \int_0^t \int_\Omega \theta_c^4 \leq c \int_0^t \|\theta_c\|_V^4 \leq c \int_0^t \|\theta_c\|^4 + c \int_0^t \|\nabla \theta_c\|^4.
$$
Thus, using a three dimensional Gagliardo-Nirenberg inequality [11, p. 125] and recalling the first of (25),
\[
c \int_0^t \| \theta_\lambda \|^4 \leq c \int_0^t \| \theta_\lambda \|^{8/5}_{L^2(\Omega)} \| \nabla \theta_\lambda \|^{12/5} + c \int_0^t \| \theta_\lambda \|^{4}_{L^2(\Omega)} \\
\leq c \int_0^t \| \theta_\lambda \|^{8/5}_{L^\infty(0,T;L^1(\Omega))} \| \nabla \theta_\lambda \|^{12/5} + c \leq c \int_0^t \| \nabla \theta_\lambda \|^{4} + c.
\]
Finally, we work on the latter term in (27) and note that, a.e., in Q, it is
\[
\lambda^2 \mu \in (\alpha_\lambda^{-1})^2(\xi_\lambda),
\]
where \((\alpha_\lambda^{-1})^2\) is easily seen to be a maximal monotone graph in \(\mathbb{R} \times \mathbb{R}\). Integrating by parts and proceeding similarly as in [7, Subsec. 4.4], we derive
\[
- \int_0^t \int_\Omega \xi_{\lambda} \mu^2 = - \lambda^2 \int_\Omega \xi_{\lambda}^+ (t) + \lambda^2 \int_\Omega \xi_{\lambda}^+(0),
\]
with \((\cdot)^+\) standing for the positive part function. Now, the first integral on the right hand side above is clearly nonpositive, while the latter one can be controlled this way. First, note that, by formally computing (22) for \(t = 0\), we have
\[
\xi_\lambda(0) = \alpha_\lambda(\xi_\lambda(0)) = \alpha_\lambda[(\text{id} + \alpha_\lambda)^{-1}(\Delta \chi_0 - \eta_0 + \theta_0 - \theta_\varepsilon)].
\]
Then, by (6–8) it follows that the term inside the square brackets is bounded in \(L^\infty(\Omega)\) independently of \(\lambda\), so that for sufficiently large \(\lambda\), \(\xi_\lambda(0)\) is nonpositive and its positive part is zero.

Now, taking all the above considerations into account, it is not difficult to deduce from (27) that, for \(\lambda\) large enough, it is
\[
\frac{1}{4} \| \theta_\lambda \|^2_{L^2(Q_T)} + \frac{1}{2} \| \nabla \theta_\lambda(t) \|^2 + \frac{1}{3} \| \lambda_\lambda(t) \|^2_{L^2(\Omega)} + \frac{2}{9} \| \nabla (\lambda_\lambda^{3/2}) \|^2_{L^2(Q_T)} \\
\leq c + \frac{2}{3} \int_0^t \| \lambda_\lambda \|^2_{L^2(\Omega)} + c \int_0^t \| \lambda_\lambda \|^2_{L^2(\Omega)} + c \int_0^t \| \nabla \theta_\lambda \|^4.
\]
Then, from the relation above one sees that the extended Gronwall lemma in the form of, e.g., [12, Thm. 7.1, p. 33] applies to the function
\[
t \mapsto \| \lambda_\lambda(t) \|^3_{L^3(\Omega)} + \| \nabla \theta_\lambda(t) \|^2
\]
and this yields a finite time \(T_0 > 0\), possibly with \(T_0 < T\), such that the following bounds hold independently of \(\lambda\):
\[
\tag{30}
\| \theta_\lambda \|_{L^2(0,T_0;H)} + \| \theta_\lambda \|_{L^\infty(0,T_0;V)} \leq c,
\]
\[
\tag{31}
\| \lambda_\lambda \|_{L^\infty(0,T_0;L^3(\Omega))} + \| \nabla (\lambda_\lambda^{3/2}) \|_{L^2(0,T_0;H)} \leq c.
\]

**Third estimate.** Multiply (22) by \(-\Delta \chi_\lambda + \eta_\lambda\)_t and integrate over \((0,t),\) as before. Owing to the first bound in (30), one can proceed exactly as in [7, Subsec. 4.6] to get the bounds
\[
\tag{32}
\| \chi_\lambda \|_{L^\infty(0,T_0;W)} + \| \xi_\lambda \|_{L^\infty(0,T_0;H)} + \| \eta_\lambda \|_{L^\infty(0,T_0;H)} \leq c.
\]
Fourth estimate. It remains to achieve the complete parabolic regularity (9) for $\theta$. With this aim, note that (21) can be rewritten as

(33) \[ \theta_\lambda - \Delta \theta_\lambda = -\theta_\lambda \chi_\lambda + \chi^2_\lambda, \]

and we just need a $\lambda$-uniform bound of the right hand side in $L^2(0, T_0; H)$. As for the first term, this is an immediate consequence of the second of (30), the first of (31), and the Sobolev embedding $V \subset L^6(\Omega)$. The latter term is controlled upon noticing that (31) yields a bound of $\chi^2_\lambda$ in

\[ L^\infty(0, T_0; L^{3/2}(\Omega)) \cap L^{3/2}(0, T_0; L^{9/2}) \subset L^4(0, T_0; H), \]

where of course the inclusion, given by elementary interpolation, is continuous.

Passage to the limit. This last step of the procedure can be performed exactly as in [7, Sec. 5], since we have the same bounds on the approximating functions, albeit they are local in time in our setting. This concludes the proof of Theorem 2.1.

REFERENCES