

Interpolation Theory and Shell Problems

Claudio BAIOCCHI

Dip. di Matematica “Guido Castelnuovo”, Università di Roma “La Sapienza”, Italy

Carlo LOVADINA*

Dip. di Ingegneria Meccanica e Strutturale, Università di Trento, Italy

Abstract

The shell problem and its asymptotic are investigated. A connection between the asymptotic behavior of the shell energy and real Interpolation Theory is established. Although only the Koiter shells have been considered, the same procedure can be used for other models, such as Naghdi’s one, for example.

1 Introduction

When a shell problem with thickness ε (cf. [1]) is considered, one is led to solve

$$\begin{aligned} &\text{Find } u_\varepsilon \in \mathcal{U} \text{ such that} \\ &\varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in \mathcal{U}. \end{aligned} \quad (1)$$

Above $a^m(\cdot, \cdot)$ is the membrane bilinear form, $a^b(\cdot, \cdot)$ the bending bilinear form and \mathcal{U} is the admissible displacement space, which also takes into account the kinematical boundary conditions imposed to the structure. The different scaling of the forms involved (the first proportional to ε and the second proportional to ε^3) causes great difficulties in studying the asymptotic behavior of the solution as $\varepsilon \rightarrow 0$. The purpose of this Note is to investigate such behaviors by using the Real Interpolation Theory, anticipating the results proved in a forthcoming paper by the authors (cf. [2]). Referring to problem (1), we consider first the *scaled* problem:

$$\begin{aligned} &\text{Find } u_\varepsilon(\beta) \in \mathcal{U} \text{ such that} \\ &\varepsilon a^m(u_\varepsilon(\beta), v) + \varepsilon^3 a^b(u_\varepsilon(\beta), v) = \varepsilon^\beta \langle f, v \rangle \quad \forall v \in \mathcal{U}, \end{aligned} \quad (2)$$

*Corresponding author. Address:

Carlo Lovadina

Dipartimento di Ingegneria Meccanica e Strutturale, Università di Trento

Via Mesiano 77, I-38050, Italy.

E-mail: lovadina@ing.unitn.it

where β is a real parameter. Then, we will address our attention to the *scaled elastic energy of order β* , namely

$$E(\varepsilon, u_\varepsilon(\beta); \beta) =: \varepsilon^{1-\beta} a^m(u_\varepsilon(\beta), u_\varepsilon(\beta)) + \varepsilon^{3-\beta} a^b(u_\varepsilon(\beta), u_\varepsilon(\beta)) = E_m(\varepsilon, u_\varepsilon(\beta); \beta) + E_b(\varepsilon, u_\varepsilon(\beta); \beta). \quad (3)$$

We will establish a strict connection between the regularity of f (measured by a Real Interpolation technique), and the asymptotic behavior of $E(\varepsilon, u_\varepsilon(\beta); \beta)$. Moreover, we will investigate the features of the function

$$R(\varepsilon) = \frac{E_b(\varepsilon, u_\varepsilon(\beta); \beta)}{E(\varepsilon, u_\varepsilon(\beta); \beta)}, \quad (4)$$

which gives the percentage of the total elastic energy that is stored in the bending part.

2 The shell problem

As it is well known, the Koiter problem for a shell of thickness ε (cf. [1]) reads as follows

$$\begin{aligned} &\text{Find } u_\varepsilon \in \mathcal{U} \text{ such that} \\ &\varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in \mathcal{U}, \end{aligned} \quad (5)$$

where \mathcal{U} is the space of admissible displacements, and $a^m(\cdot, \cdot)$ is related to the elastic membrane energy, while $a^b(\cdot, \cdot)$ to the elastic bending energy. Finally, $f \in \mathcal{U}'$, the topological dual space of \mathcal{U} . In this Note, we will not detail the precise structure of the bilinear forms involved in (5), for which we refer to [1], for instance. We only recall that $a^m(\cdot, \cdot)$ and $a^b(\cdot, \cdot)$ are both continuous on \mathcal{U} , and the sum $a^m(\cdot, \cdot) + a^b(\cdot, \cdot)$ is coercive on \mathcal{U} . It follows that $a^b(\cdot, \cdot)$ is also coercive on the (inextensional displacement) space \mathcal{U}_1 , defined by

$$\mathcal{U}_1 = \{v \in \mathcal{U}, a^m(v, w) = 0, \forall w \in \mathcal{U}\}. \quad (6)$$

Thanks to the continuity of the form $a^m(\cdot, \cdot)$ on \mathcal{U} , \mathcal{U}_1 is a closed subspace of \mathcal{U} . Thus, \mathcal{U}_1 is a Hilbert space. Furthermore, from our assumptions, it easily follows that problem (5) has a unique solution. In the sequel, we will need to consider the space $\mathcal{U}_1^0 \subset \mathcal{U}'$, the polar set of \mathcal{U}_1 , i.e.

$$\mathcal{U}_1^0 = \{f \in \mathcal{U}' : \langle f, v \rangle = 0 \quad \forall v \in \mathcal{U}_1\}. \quad (7)$$

The orthogonal space of \mathcal{U}_1 in \mathcal{U} is given by

$$V := \{u \in \mathcal{U} : a^b(u, v) = 0, \quad \forall v \in \mathcal{U}_1\}, \quad (8)$$

when \mathcal{U} is equipped with the inner product $(a^m(u, v) + a^b(u, v))$. Clearly, V is a Hilbert space, with the norm inherited by \mathcal{U} . Thus, we have

$$\mathcal{U} = \mathcal{U}_1 \oplus V. \quad (9)$$

We also need the following space

$$W = \text{the completion of } V \text{ with the norm } a^m(v, v)^{\frac{1}{2}} := \|v\|_W . \quad (10)$$

Note that on V , $\|v\|_W$ is indeed a norm, not only a seminorm. Notice also that if $f \in \mathcal{U}_1^0$ (cf. (7)), then the Koiter problem (5) can be equivalently formulated as

$$\begin{aligned} &\text{Find } u_\varepsilon \in V \text{ such that} \\ &\varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in V , \end{aligned} \quad (11)$$

where V is defined by (8). As a consequence, in the sequel we will refer to problem (5) if $f \notin \mathcal{U}_1^0$, and to problem (11) if $f \in \mathcal{U}_1^0$. We now introduce the *scaled* problem for each real β and $f \in \mathcal{V}'$

$$\begin{aligned} &\text{Find } u_\varepsilon(\beta) \in \mathcal{V} \text{ such that} \\ &\varepsilon a^m(u_\varepsilon(\beta), v) + \varepsilon^3 a^b(u_\varepsilon(\beta), v) = \varepsilon^\beta \langle f, v \rangle \quad \forall v \in \mathcal{V} , \end{aligned} \quad (12)$$

and we consider the *scaled energy function of order β* defined by

$$E(\varepsilon, u_\varepsilon(\beta); \beta) := \varepsilon^{1-\beta} a^m(u_\varepsilon(\beta), u_\varepsilon(\beta)) + \varepsilon^{3-\beta} a^b(u_\varepsilon(\beta), u_\varepsilon(\beta)) = E_m(\varepsilon, u_\varepsilon(\beta); \beta) + E_b(\varepsilon, u_\varepsilon(\beta); \beta) . \quad (13)$$

Above, the space \mathcal{V} is either \mathcal{U} (if $f \notin \mathcal{U}_1^0$), or V (if $f \in \mathcal{U}_1^0$). We will see in the next section that the behavior of the scaled energy is strictly linked to the regularity of the datum $f \in \mathcal{V}'$.

3 Regularity of f and the scaled energy function

In this section, we collect the results concerning the connection between the regularity of the datum f and the asymptotic behavior of the scaled energy function. We remark that the theorems in the first two subsections are indeed well-known (cf. [3] and [4]).

3.1 The case $f \notin \mathcal{U}_1^0$

This is certainly the easiest situation to deal with. In fact, it is well known that the following result holds true (cf. [3])

Theorem 3.1 *Fix $f \notin \mathcal{U}_1^0$ and consider Problem (12). Then it holds*

- If $\beta > 3$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = 0$.
- If $\beta = 3$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = L_b$, where $L_b = \langle f, u_0 \rangle$ and $u_0 \in \mathcal{U}_1$ is the solution of the bending-type problem

$$\begin{aligned} &\text{Find } u_0 \in \mathcal{U}_1 \text{ such that} \\ &a^b(u_0, v_0) = \langle f, v_0 \rangle \quad \forall v_0 \in \mathcal{U}_1 . \end{aligned} \quad (14)$$

- If $\beta < 3$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = +\infty$.

3.2 The case $f \in \mathcal{U}_1^0$ and $f \in W'$

As already noticed, for this case we refer to problem (11). We have the following (cf. [4])

Theorem 3.2 *Fix $f \in \mathcal{U}_1^0$ and suppose moreover that $f \in W'$, where W' is the dual space of W (cf. 10). Consider Problem (12). Then it holds*

- If $\beta > 1$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = 0$.
- If $\beta = 1$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = L_m$, where $L_m = \langle f, u_0 \rangle$ and $u_0 \in W$ is the solution of the membrane-type problem

$$\begin{aligned} & \text{Find } u_0 \in W \text{ such that} \\ & a^m(u_0, v_0) = \langle f, v_0 \rangle \quad \forall v_0 \in W. \end{aligned} \tag{15}$$

- If $\beta < 1$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = +\infty$.

3.3 The case $f \in \mathcal{U}_1^0$ and $f \notin W'$

These are surely the most subtle cases to treat. Since $f \in \mathcal{U}_1^0$, we will refer to problem (11). For such situations, we are not able to establish results as sharp as the ones detailed in Theorems 3.1 and 3.2. Nonetheless, the behavior of the scaled energy functional can be partially characterized by the regularity of the datum f , regularity which, in turn, can be measured by some Interpolation spaces between W' and V' . For the notation and results in Interpolation theory, we refer to [5] and [6]. We have the theorem (cf. [2])

Theorem 3.3 *Fix $f \in \mathcal{U}_1^0$. Suppose moreover that $f \notin W'$. Consider Problem (12). Then we have the following.*

1. Let $f \in (W', V')_{\theta, 2}$, with $0 < \theta < 1$. Set $\alpha = \inf \{2\theta + 1 : f \in (W', V')_{\theta, 2}, 0 < \theta < 1\}$.

- If $\beta > \alpha$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = 0$.
- If $1 \leq \beta < \alpha$ then $\limsup_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = +\infty$.
- If $\beta < 1$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = +\infty$.

2. Let $f \notin (W', V')_{\theta, 2}$ for any $0 < \theta < 1$.

- If $\beta > 3$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = 0$.
- If $1 \leq \beta < 3$ then $\limsup_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = +\infty$.
- If $\beta < 1$ then $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta); \beta) = +\infty$.

Remark 3.1 Referring to part 1. (resp. to part 2.) of the above theorem, we see that the exponent α (resp. the exponent 3) is critical, since it provides the scaling above which the scaled energy function tends to zero. However, in general, we are not able to study the exact asymptotic behavior of $E(\varepsilon, u_\varepsilon(\alpha); \alpha)$ (resp. $E(\varepsilon, u_\varepsilon(3); 3)$).

4 On the asymptotic ratio between the bending and the total elastic energy

In this section we will see that if one can *properly* choose the exponent in the scaled problem (12), then one can get a precise information on the asymptotic ratio between the bending energy and the total elastic energy. We begin with defining the function $R(\varepsilon)$ as (cf. also (13))

$$R(\varepsilon) = \frac{E_b(\varepsilon, u_\varepsilon(\beta); \beta)}{E(\varepsilon, u_\varepsilon(\beta); \beta)}, \quad (16)$$

and we notice that, by bilinearity, $R(\varepsilon)$ does not depend on β . We have the following theorem, whose proof is detailed in [2].

Theorem 4.1 Consider the problem

$$\begin{aligned} & \text{Find } u_\varepsilon(\beta) \in \mathcal{V} \text{ such that} \\ & \varepsilon a^m(u_\varepsilon(\beta), v) + \varepsilon^3 a^b(u_\varepsilon(\beta), v) = \varepsilon^\beta \langle f, v \rangle \quad \forall v \in \mathcal{V}, \end{aligned} \quad (17)$$

where the space \mathcal{V} is either \mathcal{U} (if $f \notin \mathcal{U}_1^0$), or V (if $f \in \mathcal{U}_1^0$). Suppose that there is an exponent $\beta = \beta^*$ such that there exist

$$+\infty > \lim_{\varepsilon \rightarrow 0} E(\varepsilon, u_\varepsilon(\beta^*); \beta^*) > 0, \quad \lim_{\varepsilon \rightarrow 0} E_b(\varepsilon, u_\varepsilon(\beta^*); \beta^*) \geq 0. \quad (18)$$

Then it holds

$$\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = \frac{\beta^* - 1}{2}. \quad (19)$$

Remark 4.1 We remark that Theorem 4.1 answers in a positive way a question raised by E. Sanchez-Palencia on the relationship between β^* and $R(\varepsilon)$. Moreover, it has been proved (cf. [7] and [8]) that if there exists a β^* satisfying (18), then such an exponent is unique and $1 \leq \beta^* \leq 3$. We also remark that

- In the hypotheses of Theorem 3.1, β^* does exist and $\beta^* = 3$. It follows that $\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 1$, i.e. asymptotically the energy goes entirely in bending.

- In the hypotheses of Theorem 3.2, β^* does exist and $\beta^* = 1$. It follows that $\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 0$, i.e. asymptotically the energy goes entirely in membrane.
- In the hypotheses of Theorem 3.3, we are not able to prove that, in general, β^* exists. However, if it exists, then it holds the following
 - If we are in the framework of part 1. of Theorem 3.3, then $\beta^* = \inf\{2\theta + 1 : f \in (W', V')_{\theta,2}, 0 < \theta < 1\}$ and $\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = \inf\{\theta : f \in (W', V')_{\theta,2}, 0 < \theta < 1\}$.
 - If we are in the framework of part 2. of Theorem 3.3, then $\beta^* = 3$ and $\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 1$.

References

- [1] P.G. Ciarlet, *Introduction to Linear Shell Theory*, Series in Applied Mathematics, Gauthier-Villars (1998).
- [2] C. Baiocchi and C. Lovadina, A shell classification by Interpolation, *to appear*.
- [3] D. Chenais and J.C. Paumier, On the locking phenomenon for a class of elliptic problems, *Numer. Math.* **67** 427–440 (1994).
- [4] E. Sanchez-Palencia, Statique et dynamique des coques minces. II. Cas de flexion pure inhibée - Approximation membranaire, *C. R. Acad. Sci. Paris* **309** série I 531–537 (1989).
- [5] J. Bergh and J. Löfstrom, *Interpolation Spaces: an introduction*, Springer–Verlag (1976).
- [6] J.-L. Lions and J. Peetre, Sur une classe d’espaces d’interpolation, *Pubbl. I.H.E.S.* **19** 5–68 (1964).
- [7] A. Blouza, F. Brezzi and C. Lovadina, A New Classification for Shell Problems, *Pubblicazioni IAN–CNR* n. 1128 (1999).
- [8] A. Blouza, F. Brezzi and C. Lovadina, Sur la classification des coques linéairement élastiques, *C.R. Acad. Sci. Paris*, **328** série I 831–836 (1999).