A locking-free model for Reissner-Mindlin plates: Analysis and isogeometric implementation via NURBS and triangular NURPS

L. Beirão da Veiga *, T.J.R. Hughes †, J. Kiendl ‡
C. Lovadina §, J. Niiranen ¶, A. Reali ∥, H. Speleers *††

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Abstract

We study a reformulated version of Reissner-Mindlin plate theory in which rotation variables are eliminated in favor of transverse shear strains. Upon discretization, this theory has the advantage that the “shear locking” phenomenon is completely precluded, independent of the basis functions used for displacement and shear strains. Any combination works, but due to the appearance of second derivatives in the strain energy expression, smooth basis functions are required. These are provided by Isogeometric Analysis, in particular, NURBS of various degrees and quadratic triangular NURPS. We present a mathematical analysis of the formulation proving convergence and error estimates for all physically interesting quantities, and provide numerical results that corroborate the theory.

*Department of Mathematics, University of Milan, Via Saldini 50, 20133 Milan, Italy. Email: lourenco.beirao@unimi.it
†Institute for Computational Engineering and Sciences, University of Texas at Austin, 201 East 24th Street, Stop C0200 Austin, TX 78712-1229, USA. Email: hughes@ices.utexas.edu
‡Department of Civil Engineering and Architecture, University of Pavia, Via Ferrata 3, 27100, Pavia, Italy. Email: josef.kiendl@unipv.it
§Department of Mathematics, University of Pavia, Via Ferrata 1, 27100 Pavia, Italy. Email: carlo.lovadina@unipv.it
¶Department of Civil and Structural Engineering, Aalto University, PO Box 12100, 00076 AALTO, Finland. Email: jarkko.niiranen@aalto.fi
∥Department of Civil Engineering and Architecture, University of Pavia, Via Ferrata 3, 27100, Pavia, Italy. Email: alessandro.reali@unipv.it
*††Department of Computer Science, University of Leuven, Celestijnenlaan 200A, 3001 Heverlee (Leuven), Belgium. Email: hendrik.speleers@cs.kuleuven.be
‡‡Department of Mathematics, University of Rome “Tor Vergata”, Via della Ricerca Scientifica, 00133 Rome, Italy. Email: speleers@mat.uniroma2.it
1 Introduction

Finite element thin plate bending analysis, based on the Poisson-Kirchhoff theory, began with the MS thesis of Papenfuss at the University of Washington in 1959 [24]. This four-node rectangular element employed $C^1$-continuous interpolation functions, but was deficient in the sense that its basis functions were not complete through quadratic polynomials. Throughout the 1960s the development of quadrilateral and triangular thin plate elements was a focus of finite element research. The problem was surprisingly difficult. Eventually there were technical successes but the elements were complicated, hard to implement, and difficult to use in practice. For an early history, we refer to Felippa [16]. To circumvent the difficulties, in the 1970s attention was redirected to the “thick” plate bending theory of Reissner-Mindlin. In this case only $C^0$ basis functions were required for the displacement and rotations, so the continuity and completeness requirements were easily satisfied with standard isoparametric basis functions, but new difficulties arose associated with “shear locking” in the thin plate limit in which the transverse shear strains must vanish. Nevertheless, through a number of clever ideas, some tricks and some fundamental, successful elements for many applications became available. The simplicity and efficiency of these elements led to immediate incorporation in industrial and commercial structural analysis software programs, and that has been the situation ever since. Research in the subject then became somewhat stagnant for a number of years, but recently things changed.

Isogeometric Analysis was proposed by Hughes, Cottrell & Bazilevs in 2005 [18]. The motivation for its development was to simplify and render more efficient the design-through-analysis process. It is often said that the development of suitable Finite Element Analysis (FEA) models from Computer Aided Design (CAD) files occupies over 80% of overall analysis time [12]. Some design/analysis engineers claim that this is an underestimate and the situation is actually worse perhaps 90%, or even more. Whatever the precise percentage, it is clear that the interface between design and analysis is broken, and it is the stated aim of Isogeometric Analysis to repair it. The way this has been approached in Isogeometric Analysis is to reconstitute analysis within the functions utilized in engineering CAD, such as, for example, NURBS and T-splines, making it possible, at the very least, to perform analysis within the representations provided by design, eliminating redundant data structures and unnecessary geometric approximations. This has been the primary focus of Isogeometric Analysis, but in the process of its development new analysis opportunities have also presented themselves. One emanates from a basic property of the functions utilized in CAD they are smooth, usually at least $C^1$-continuous, more often $C^2$-continuous, and they do not require derivative degrees-of-freedom, one of the paradigmatic deficiencies of the early thin plate bending elements. CAD functions also have other advantageous properties, but we will not go into these here. It turns out that smoothness alone has created new opportunities in plate bending element research.

The first and most obvious to be exploited by Isogeometric Analysis is re-
newed interest in the long abandoned Poisson-Kirchhoff theory, as anticipated in [18]. With convenient, smooth basis functions provided by Isogeometric Analysis, the role of Poisson-Kirchhoff theory is being reconsidered. A prime advantage is that no rotational degrees-of-freedom are required. This immediately reduces the size of equation systems and the computational effort necessary to solve problems. In finite deformation analysis there is a further advantage. This stems from the fact that finite rotations involve a product-group structure, $SO(3)$, a significant complication. It is completely obviated by rotationless thin bending elements.

Another innovation that occurred, due to Long, Bornemann & Cirak [21] and Echter, Oesterle & Bischoff [15] in the context of shells, was again motivated by the existence of smooth basis functions. It was observed that if one could deal with second derivatives appearing in strain energy expressions, then a change of variables from rotations in Reissner-Mindlin plate theory to transverse shear strains would eliminate transverse shear locking, independent of the basis functions employed. Why wasn’t this amazing formulation utilized previously for the development of plate and shell elements? The answer seems to be there simply were not convenient smooth basis functions available within the finite element paradigm. Shear locking has been the fundamental obstacle to the design of effective plate elements based on Reissner-Mindlin theory. It is remarkable that the discovery of a complete and general solution has occurred nearly 50 years after the widespread adoption of the theory as a framework for the derivation of plate and shell elements.

Given the above, the question arises as to what other opportunities might be provided by clever changes of variables? One answer has been presented in the work of Kiendl et al. [19] who have shown that smooth basis functions with only translational displacement degrees-of-freedom can also be employed successfully for “thick” bending elements. The price to pay in the formulation of [19] is that squares of third derivatives appear in the strain energy expression, but these are no problem for $C^2$ continuous Isogeometric Analysis basis functions.

All these new ideas have created a renaissance in the development of methods to solve problems involving thin and thick bending elements. It is clear that these and other related concepts will generate considerable interest in the coming years.

In this paper we undertake the mathematical analysis of a class of methods for Reissner-Mindlin plate theory based on the change of variables introduced in [15, 21]. The dependent variables are then the transverse displacement of the plate and the transverse shear strain vector. The change of variables results in squares of second derivatives of the displacement in the strain energy, which yield to spline discretizations of $C^1$, or higher, continuity. It is apparent that the shear strain vector is not as physically appealing or implementation-convenient as the rotation vector. However, by utilizing weakly enforced rotation boundary conditions, by way of Nitsche’s method [23], these issues are circumvented. Nitsche’s method also provides other analytical benefits in that it alleviates “boundary locking”, a potential problem encountered in the present theory for clamped boundary conditions. In addition to $C^{p-1}$ NURBS dis-
cretizations, we consider quadratic $C^1$ triangular NURPS, that is, non-uniform rational Powell-Sabin splines. We note in passing that Isogeometric Analysis has previously been applied to the standard displacement-rotation forms of the Reissner-Mindlin plate and shell theories [6, 8].

An outline of the remainder of the paper follows: In Section 2 we present the Reissner-Mindlin plate model, first in terms of displacement and rotation variables and then in an equivalent form in terms of displacement and transverse shear strain variables. We introduce the discrete form of the theory and summarize stability and convergence results that apply to displacements, shear strains, rotations, bending moments, and transverse shear force resultants. In Section 3 we describe and perform numerical tests with NURBS spaces. In Section 4 we do likewise with quadratic NURPS, and in Section 5 we present concluding remarks. The technical details of proofs are postponed until Appendix A so as not to interrupt the flow of the main ideas and results in the body of the paper.

2 The model and discretization

We start by considering the classical Reissner-Mindlin model for plates. Let $\Omega$ be a bounded and piecewise regular domain in $\mathbb{R}^2$ representing the midsurface of the plate. We subdivide the boundary $\Gamma$ of $\Omega$ in three disjoint parts (such that each is either void or the union of a finite sum of connected components of positive length),

$$\Gamma = \Gamma_c \cup \Gamma_s \cup \Gamma_f.$$

The plate is assumed to be clamped on $\Gamma_c$, simply supported on $\Gamma_s$ and free on $\Gamma_f$, with $\Gamma_c, \Gamma_s$ defined to preclude rigid body motions. We consider for simplicity only homogeneous boundary conditions. The variational space of admissible solutions is given by

$$\tilde{X} = \{(w, \theta) \in H^1(\Omega) \times [H^1(\Omega)]^2 : w = 0 \text{ on } \Gamma_c \cup \Gamma_s, \theta = 0 \text{ on } \Gamma_c\}.$$

Following the Reissner-Mindlin model, see for instance [4] and [17], the plate bending problem reads as

$$\begin{cases}
\text{Find } (w, \theta) \in \tilde{X}, \text{ such that } \\
a(\theta, \eta) + \mu k t^{-2}(\theta - \nabla w, \eta - \nabla v) = (f, v), \quad \forall (v, \eta) \in \tilde{X},
\end{cases} \quad (1)$$

where $\mu$ is the shear modulus and $k$ is the so-called shear correction factor. In the above model, $t$ represents the plate thickness, $w$ the deflection, $\theta$ the rotation of the normal fibers and $f$ the applied scaled normal load. Moreover, $(\cdot, \cdot)$ stands for the standard scalar product in $L^2(\Omega)$ and the bilinear form $a(\cdot, \cdot)$ is defined by

$$a(\theta, \eta) = (\varepsilon(\theta), \varepsilon(\eta)), \quad (2)$$

with $\varepsilon$ the positive definite tensor of bending moduli and $\varepsilon(\cdot)$ the symmetric gradient operator.
Provided the boundary conditions satisfy the conditions above, the bilinear form appearing in problem (1) is coercive, in the following sense (see Proposition A.1 in Appendix A). There exists a positive constant $\alpha$ depending only on the material constants and the domain $\Omega$ such that

$$a(\eta, \eta) + \mu k t^{-2} (\eta - \nabla v, \eta - \nabla v)$$
$$\geq \alpha \left( ||\eta||^2_{H^1(\Omega)} + t^{-2} ||\eta - \nabla v||^2_{L^2(\Omega)} + ||v||^2_{H^1(\Omega)} \right), \quad \forall (v, \eta) \in \bar{X}. \quad (3)$$

The above result states the coercivity of the bilinear form on the product space $\bar{X}$. It is well known that the discretization of the Reissner-Mindlin model poses difficulties, due to the possibility of the locking phenomenon when the thickness $t \ll \text{diam}(\Omega)$. We will here consider an alternative model that does not suffer from such a drawback. To simplify notation, and without any loss of generality, we will assume $\mu k = 1$ in the following.

### 2.1 An equivalent formulation

The model presented here, and originally introduced in the context of shells in [21, 15], is derived by simply considering the following change of variables:

$$(w, \theta) \leftrightarrow (w, \gamma) \quad \text{with} \quad \theta = \nabla w + \gamma, \quad (4)$$

and changing the Reissner-Mindlin formulation accordingly. The physical interpretation of $\gamma$ is the transverse shear strain.

**Remark 2.1.** In the engineering practice, the shear strain is most frequently defined as $\gamma = -\theta + \nabla w$, instead of $\gamma = \theta - \nabla w$ (see (4)), which is more common in mathematical-oriented papers.

For all sufficiently regular functions $\tau : \Omega \to \mathbb{R}^2$ and $v : \Omega \to \mathbb{R}$, let the “energy” norm be given by

$$|||v, \tau|||^2 = ||\tau + \nabla v||^2_{H^1(\Omega)} + t^{-2} ||\tau||^2_{L^2(\Omega)} + ||v||^2_{H^1(\Omega)}. \quad (5)$$

We then define the variational spaces

$$\tilde{X} = \{\text{closure of } C^\infty(\Omega) \times [C^\infty(\Omega)]^2 \text{ with respect to the norm } |||\cdot, \cdot|||\}.$$  

$$X = \{(v, \tau) \in \tilde{X} : v = 0 \text{ on } \Gamma_e \cup \Gamma_s, \nabla v + \tau = 0 \text{ on } \Gamma_e\}.$$  

It is immediately verified that

$$H^2(\Omega) \times [H^1(\Omega)]^2 \subset \tilde{X} \subset H^1(\Omega) \times [L^2(\Omega)]^2.$$  

Moreover, note that $X$ exactly corresponds to $\tilde{X}$ up to the change of variables (4). Let the variational problem for the hierarchic model be given by

$$\begin{cases}
\text{Find } (w, \gamma) \in X, \text{ such that} \\
a(\nabla w + \gamma, \nabla v + \tau) + t^{-2} (\gamma, \tau) = (f, v), \quad \forall (v, \tau) \in X,
\end{cases} \quad (6)$$
where we recall that we now have \( \mu k = 1 \) for notational simplicity.

Problem (6) is clearly equivalent to (1) up to the above change of variables. In particular, the coercivity property (3) translates immediately into

\[
a(\nabla v + \tau, \nabla v + \tau) + t^{-2}(\tau, \tau) \geq \alpha \|\tau \| \|v\|^2, \quad \forall (v, \tau) \in X,
\]

for the same positive constant \( \alpha \).

### 2.2 Discretization of the model

We introduce a pair of finite dimensional spaces for deflections and shear strains:

\[
W_h \subset H^2(\Omega), \quad \Xi_h \subset [H^1(\Omega)]^2.
\]

We assume that the two spaces above are generated either by some finite element or isogeometric technology, and are therefore associated with a (physical) mesh \( \Omega_h \). In the following, we will indicate by \( K \in \Omega_h \) a typical element of the mesh, and denote by \( h \) its diameter. We denote by \( h \) the maximum mesh size. Moreover, we will indicate by \( E_h \) the set of all (possibly curved) edges of the mesh, by \( e \in E_h \) the generic edge, and by \( h_e \) its length. As usual we assume that the boundary parts \( \Gamma_c, \Gamma_s, \Gamma_f \) are unions of mesh edges. Furthermore, we will assume that the set of elements \( K \) in the family \( \{\Omega_h\}_h \) is uniformly shape regular in the classical sense.

We then consider the discrete space with (partial) boundary conditions

\[
X_h = \{(v_h, \tau_h) \in W_h \times \Xi_h : v_h = 0 \text{ on } \Gamma_c \cup \Gamma_s \};
\]

see also Remark 2.3 below. The rotation boundary condition on \( \Gamma_c \) will be enforced with a penalized formulation in the spirit of Nitsche’s method [23], through the introduction of the following modified bilinear form. Let

\[
a_h(\nabla w_h + \gamma_h, \nabla v_h + \tau_h) = a(\nabla w_h + \gamma_h, \nabla v_h + \tau_h)
- \int_{\Gamma_c} (C\varepsilon(\nabla w_h + \gamma_h)n_e) : (\nabla v_h + \tau_h)
- \int_{\Gamma_c} (C\varepsilon(\nabla v_h + \tau_h)n_e) : (\nabla w_h + \gamma_h)
+ \beta \text{tr}(\Sigma) \sum_{e \in E_h \cap \Gamma_c} c(e)^{-1} \int_e (\nabla w_h + \gamma_h) : (\nabla v_h + \tau_h),
\]

for all \((w_h, \gamma_h), (v_h, \tau_h)\) in \( X_h \) and for \( \beta > 0 \) a stabilization parameter. Above, \( c(e) \) is a characteristic quantity depending on the side \( e \).

**Remark 2.2.** For a shape regular family of meshes, one can choose \( c(e) = h_e \) or \( c(e) = (\text{Area } K_e)^{1/2} \), where \( K_e \) is the element containing \( e \). This latter choice has been employed in the numerical tests of Section 3, while the former has been used in Section 4. However, when a side \( e \) belongs to an element with a large aspect ratio, a different choice could be preferable. For example, significantly
thin elements may be used in the presence of boundary layers, and one may choose \( c(e) = h_e^+ \), where \( h_e^+ \) is the element size in the direction perpendicular to the boundary.

We are now able to present the proposed discretization of the model (6):

\[
\begin{aligned}
\text{Find } (w_h, \gamma_h) \in X_h, \text{ such that } \\
a_h(\nabla w_h + \gamma_h, \nabla v_h + \tau_h) + t^{-2}(\gamma_h, \tau_h) = (f, v_h), \quad \forall (v_h, \tau_h) \in X_h. 
\end{aligned}
\]

(9)

**Remark 2.3.** It is not wise to use a direct discretization of the space \( X \) by enforcing all the boundary conditions on \( W_h \times \Xi_h \). Indeed, the clamped condition on rotations

\[
\nabla w_h + \gamma_h = 0 \text{ on } \Gamma_c, \quad \forall w_h \in W_h, \gamma_h \in \Xi_h,
\]

is very difficult to implement, and can be a source of boundary locking unless the two spaces \( W_h, \Xi_h \) are chosen in a very careful way. To illustrate why a poor approximation might occur on \( \Gamma_c \), let us consider the following example.

Let \( \Omega = (0,1) \times (0,1) \) and \( \Gamma_c = [0,1] \times \{1\} \). Select \( W_h = S_{p-1}^{1,1}(\Omega) \) and \( \Xi_h = S_{p-2,1}^{1,1}(\Omega) \times S_{p-2,1}^{1,1}(\Omega) \), where \( S_{p,q}^{1,r}(\Omega) \) denotes the space of B-splines of degree \( p \) and regularity \( C^r \) with respect to the \( x \) direction, and of degree \( q \) and regularity \( C^* \) with respect to the \( y \) direction.

Imposing \( \nabla w_h + \gamma_h = 0 \text{ on } \Gamma_c \) implies in particular

\[
\frac{\partial w_h}{\partial y}(x,1) = -\gamma_{2,h}(x,1) \quad \forall x \in [0,1],
\]

where \( \gamma_{2,h} \) is the second component of the vector field \( \gamma_h \). Since \( \frac{\partial w_h}{\partial y}(x,1) \in S_{p-1}^{1,1}(0,1) \) and \( \gamma_{2,h}(x,1) \in S_{p-2}^{1,1}(0,1) \), it follows that

\[
\gamma_{2,h}(x,1) \in S_{p-2}^{1,1}(0,1) \cap S_{p-1}^{1,1}(0,1) = S_{p-1}^{1,1}(0,1).
\]

Hence \( \gamma_{2,h}(x,1) \) is necessarily a global polynomial of degree at most \( p - 1 \) on \( \Gamma_c \), regardless of the mesh. This means that on \( \Gamma_c \) \( \gamma_{2,h} \) cannot converge to \( \gamma_2 \), second component of \( \gamma \), as the mesh size tends to zero, in general.

Instead, as we will prove in the next section, the method proposed above is free of locking for any choice of the discrete spaces \( W_h, \Xi_h \).

### 2.3 Stability and convergence results

In the present section we show the stability and convergence properties of the proposed method. All the proofs can be found in Appendix A.

In what follows, we set \( c(e) = h_e \) in (8), which is a suitable choice for shape regular meshes, see Remark 2.2. We start by introducing the following discrete norm

\[
|||v_h, \tau_h|||^2_h = |||v_h, \tau_h|||^2 + \sum_{e \in \mathcal{E}_h \cap \Gamma_c} h_e^{-1}||\nabla v_h + \tau_h||^2_{L^2(e)}, \tag{10}
\]

for all \((v_h, \tau_h) \in X_h\).

In the theoretical analysis of the method we will make use of the following assumptions on the solution regularity and space approximation properties.

---
A1) We assume that the solution $w \in H^{2+s}(\Omega)$ and $\gamma \in H^{1+s}(\Omega)$ for some $s > 1/2$.

A2) We assume that the following standard inverse estimates hold

$$||v_h||_{H^1(K)} \leq C h^{-1} ||v_h||_{L^2(K)}, \quad ||\tau_h||_{H^1(K)} \leq C h^{-1} ||\tau_h||_{L^2(K)},$$

for all $v_h \in W_h$ and $\tau_h \in \Xi_h$ with $C$ independent of $h$.

A3) We assume the following approximation properties for $X_h$. Let $s = \frac{2}{3}$.

For all $(v, \tau) \in (H^s(\Omega) \times [H^1(\Omega)])^2 \cap X$ there exists $(v_h, \tau_h) \in X_h$ such that

$$||\tau - \tau_h||_{H^j(\Omega)} \leq C h^{1-j} ||\tau||_{H^1(\Omega)}, \quad j = 0, 1,$$
$$||v - v_h||_{H^j(\Omega)} \leq C h^{s-j} ||v||_{H^s(\Omega)}, \quad j = 0, \ldots, s,$$

with $C$ independent of $h$.

Moreover, we will make use of the following natural assumption, in order to avoid rigid body motions:

A4) We assume that $\Gamma_c \cup \Gamma_s$ has positive length and that either

i) $\Gamma_c$ has positive length, or

ii) $\Gamma_s$ is not contained in a straight line.

We now introduce a coercivity lemma stating in particular the invertibility of the linear system associated with (9).

**Lemma 2.1.** Let hypotheses A2 and A4 hold. There exist two positive constants $\beta_0, \alpha'$ such that, for all $\beta \geq \beta_0$, we have

$$a_h(\nabla v_h + \tau_h, \nabla v_{h} + \tau_{h}) + t^{-2}(\tau_h, \tau_h) \geq \alpha' |||\tau_h, v_h|||^2_h, \quad \forall (v_h, \tau_h) \in X_h. \quad (11)$$

The constant $\alpha'$ only depends on the material parameters and the domain $\Omega$, while the constant $\beta_0$ depends only on the shape regularity constant of $\Omega_h$.

Let A1 hold. By an integration by parts, it is immediately verified that the scheme (9) is consistent, in the sense that

$$a_h(\nabla w + \gamma, \nabla v_h + \tau_h) + t^{-2}(\gamma, \tau_h) = (f, v_h), \quad \forall (v_h, \tau_h) \in X_h. \quad (12)$$

where $(w, \gamma)$ is the solution of problem (6) and the left-hand side makes sense due to the regularity assumption A1. Note that condition A1 could be significantly relaxed by interpreting the integrals on $\Gamma_c$ appearing in (8) in the sense of dualities.

By combining the coercivity in Lemma 2.1 with the consistency property (12), the following convergence result follows.
Proposition 2.1. Let $A1$ and $A2$ hold. Let $(w, \gamma)$ be the solution of problem (6) and $(w_h, \gamma_h) \in X_h$ be the solution of problem (9). Then, if $\beta \geq \beta_0$, we have

$$
|||w - w_h, \gamma - \gamma_h|||^2_h
\leq C \inf_{(v_h, \tau_h) \in X_h} \sum_{j=0}^2 \left( \sum_{K \in \Omega_h} h_{K}^{2(j-1)}|w - v_h|_{H^{j+1}(K)}^2 + \sum_{K \in \Omega_h} h_{K}^{2(j-1)}|\gamma - \tau_h|_{H^{j}(K)}^2 \right),
$$

(13)

with $C$ depending only on the material parameters and the domain $\Omega$.

We now state a convergence result concerning bending moments and shear forces, the quantities of interest in engineering applications. To this aim, we first define the scaled bending moments and the scaled shear forces as follows:

$$
M = -C\varepsilon(\nabla w + \gamma), \quad Q = t^{-2}\mu\kappa\gamma = t^{-2}\gamma,
$$

(14)

where $(w, \gamma)$ is the solution to problem (6). The above quantities are known to converge to non-vanishing limits, as $t \to 0$ (see, e.g., [3] and [9]). Once problem (9) has been solved, we can define the scaled discrete bending moments and the scaled discrete shear moments as

$$
M_h = -C\varepsilon(\nabla w_h + \gamma_h), \quad Q_h = t^{-2}\gamma_h.
$$

(15)

We have the following estimates.

Proposition 2.2. Let $A1$ and $A2$ hold. Let $(w, \gamma)$ be the solution of problem (6) and $(w_h, \gamma_h) \in X_h$ be the solution of problem (9). Then, if $\beta \geq \beta_0$, we have

$$
||M - M_h||_{L^2(\Omega)} + t||Q - Q_h||_{L^2(\Omega)} \leq C||w - w_h, \gamma - \gamma_h||_h.
$$

(16)

Moreover, let the mesh family $\{\Omega_h\}_{h}$ be quasi-uniform. Then, we have

$$
h||Q - Q_h||_{L^2(\Omega)} \leq C\left(||w - w_h, \gamma - \gamma_h||_h + h \inf_{s_h \in \Xi_h} ||Q - s_h||_{L^2(\Omega)}\right),
$$

(17)

$$
||Q - Q_h||_{H^{-1}(\Omega)} \leq C\left(||w - w_h, \gamma - \gamma_h||_h + h \inf_{s_h \in \Xi_h} ||Q - s_h||_{L^2(\Omega)}\right).
$$

(18)

In addition, we can formulate the following improved result regarding the error for the rotations in the $L^2$-norm and the error for the deflections in the $H^1$-norm. An analogous result (possibly with a smaller improvement in terms of $h, t$) also holds if the additional hypotheses in the proposition are not satisfied; we do not detail here this more general case.

Proposition 2.3. Let the same assumptions and notation of Proposition 2.1 hold. Moreover, let assumption A3 hold, $\Gamma_c = \Gamma$ and let the domain $\Omega$ be
either regular, or piecewise regular and convex. Then, the following improved approximation result holds

\[ \| \theta - \theta_h \|_{L^2(\Omega)} \leq C(h + t) \| w - w_h, \gamma - \gamma_h \|_h, \]
\[ ||| w - w_h |||_{H^1(\Omega)} \leq C(h + t) ||| w - w_h, \gamma - \gamma_h |||_h + ||| \gamma - \gamma_h |||_{L^2(\Omega)}, \] \hspace{1cm} (19)

where \( \theta_h = \nabla w_h + \gamma_h \) and the constant \( C \) depends only on the material parameters and the domain \( \Omega \).

Note that the last term appearing in (13) is not a source of locking since \( t^{-1} \gamma = tQ \) is a quantity that is known to be uniformly bounded in the correct Sobolev norms (see, e.g., [3] and [9]). The constants appearing in Propositions 2.1 and 2.3 are independent of \( t \), and so the results shown state that the proposed method is locking-free regardless of the discrete spaces adopted. This is a very interesting property that is missing in the standard methods for the Reissner-Mindlin problem. The accuracy of the discrete solution (9) will only depend on the approximation properties of the adopted discrete spaces, and will not be hindered by small values of the plate thickness. We also remark that the norms for the scaled shear forces appearing in the left-hand side of (17) and (18), are indeed the usual norms for which a convergence result can be established (see, e.g., [9] and [10]).

In the following two sections we will present a pair of particular choices for \( W_h, \Xi_h \) within the framework of

- standard tensor-product NURBS-based isogeometric analysis (Section 3);
- triangular NURPS-based isogeometric analysis (Section 4).

For such choices we can apply Propositions 2.1, 2.3 in order to obtain the expected convergence rates in terms of \( h \). For example, in the case of standard tensor-product NURBS, combining Proposition 2.1 with the approximation estimates in [5, 7] we obtain the following convergence result.

\[ h \| w - w_h, \gamma - \gamma_h \|_h \leq Ch^{s-1}( \| w \|_{H^{s+1}(\Omega)} + \| \theta \|_{H^s(\Omega)} + t \| \gamma \|_{H^{s-1}(\Omega)}), \] \hspace{1cm} (21)

\[ h \| M - M_h \|_{L^2(\Omega)} + t \| Q - Q_h \|_{L^2(\Omega)} \leq Ch^{s-1}( \| w \|_{H^{s+1}(\Omega)} + \| \theta \|_{H^s(\Omega)} + t \| \gamma \|_{H^{s-1}(\Omega)}), \] \hspace{1cm} (22)

The constant \( C \) depends only on \( p \), the material parameters and the domain \( \Omega \).
The above corollary can also be combined with Proposition 2.3 to obtain improved error estimates for the $L^2$-norm of the rotations and the $H^1$-norm of the deflections. Also recalling definition (5), one can easily obtain a convergence rate of $(h + t) h^{s-1}$ for such quantities. Finally, note that, like for all high order methods, the regularity requirement in Corollary 2.1 may be too demanding due to the presence of layers in the solution. This situation is typically dealt with by making use of refined meshes, an example of which is shown in the numerical tests.

3 Isogeometric discretization with NURBS

In this section, NURBS-based isogeometric analysis is used to perform numerical validations of the presented theory. We begin with a brief summary of B-splines and NURBS (Non-Uniform Rational B-Splines).

3.1 B-splines and NURBS

B-splines are piecewise polynomials defined by the polynomial degree $p$ and a knot vector $[\xi_1, \xi_2, \ldots, \xi_{n+p+1}]$, where $n$ is the number of basis functions. The knot vector is a set of parametric coordinates $\xi_i$, called knots, which divide the parametric space into intervals called knot spans. A knot can also be repeated, in this case it is called a multiple knot. At a single knot the B-splines are $C^{p-1}$-continuous, and at a multiple knot of multiplicity $k$ the continuity is reduced to $C^{p-k}$.

The B-spline basis functions of degree $p$ are defined by the following recursion formula. For $p = 0$,

$$N_{i,0}(x) = \begin{cases} 1, & \xi_i \leq x < \xi_{i+1}, \\ 0, & \text{otherwise}. \end{cases}$$

For $p \geq 1$,

$$N_{i,p}(x) = \frac{x - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(x) + \frac{\xi_{i+p+1} - x}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(x).$$

A bivariate NURBS function $R_{i,j}^{p,q}(x, y)$ is defined as the weighted tensor-product of the B-spline functions $N_{i,p}$ and $M_{j,q}$ with polynomial degrees $p$ and $q$,

$$R_{i,j}^{p,q}(x, y) = \frac{N_{i,p}(x)M_{j,q}(y)\omega_{i,j}}{\sum_{l=1}^{n} \sum_{r=1}^{m} N_{l,p}(x)M_{r,q}(y)\omega_{l,r}},$$

where $\omega_{i,j}$ are called control weights. Following the isogeometric concept, NURBS are employed to both represent the geometry and to approximate the solution, i.e. the isoparametric concept is invoked. Accordingly, the unknown variables
$w$ and $\gamma$ are approximated by

$$w_h(x, y) = \sum_{i=1}^{n_w} \sum_{j=1}^{m_w} R_{i,j}^{p_x,q_x}(x, y) \hat{w}_{i,j}, \quad \gamma_h(x, y) = \sum_{i=1}^{n_\gamma} \sum_{j=1}^{m_\gamma} R_{i,j}^{p_\gamma,q_\gamma}(x, y) \hat{\gamma}_{i,j},$$

with $n_w$, $m_w$ the numbers of basis functions in the two parametric directions for $w_h$, and $n_\gamma$, $m_\gamma$ the numbers for $\gamma_h$. The test functions $v$ and $\tau$ are discretized accordingly.

Since the rotations $\theta$ are not discretized in this approach, rotational boundary conditions cannot be imposed in a standard way by applying them directly on the respective degrees of freedom at the boundary. Instead, such boundary conditions are enforced by the modified bilinear form introduced in equation (8). Displacement boundary conditions are enforced in a standard way through the displacement degrees of freedom on the boundary.

### 3.2 Numerical tests

In this section, the proposed method is tested on different numerical examples. We always employ the same polynomial order and the highest regularity for all the unknowns, but different choices can be made. In order to demonstrate the locking-free behavior of this method we consider both thick and thin plates. Furthermore, we investigate an example which exhibits boundary layers. As an error measure, an approximated $L^2$-norm error for a variable $u$ is computed as

$$\frac{\|u_{ex} - u_h\|_{L^2}}{\|u_{ex}\|_{L^2}} = \sqrt{\frac{\sum_{(x_i,y_i) \in G} (u_{ex}(x,y) - u_h(x,y))^2}{\sum_{(x_i,y_i) \in G} u_{ex}^2(x,y)}},$$

where $G$ is a $101 \times 101$ uniform grid in the parameter domain $[0,1]^2$ mapped onto the physical domain.

#### 3.2.1 Square plate with clamped boundary conditions

The first example consists of a unit square plate $[0,1]^2$ with an analytical solution as described in [11]. The plate is clamped on all four sides, and subject to a load given by

$$f(x, y) = \frac{E}{12(1 - \nu^2)} \left[12y(y - 1)(5x^2 - 5x + 1)(2y^2(y - 1)^2
+ x(x - 1)(5y^2 - 5y + 1))
+ 12x(x - 1)(5y^2 - 5y + 1)(2x^2(x - 1)^2
+ y(y - 1)(5x^2 - 5x + 1))\right].$$
The analytical solution for the displacement $w$ is given by

$$w(x, y) = \frac{1}{3} x^3 (x - 1)^3 y^3 (y - 1)^3$$

$$- \frac{2t^2}{5(1 - \nu)} \left[ y^3 (y - 1)^3 x (x - 1) (5x^2 - 5x + 1) + x^3 (x - 1)^3 y (y - 1) (5y^2 - 5y + 1) \right].$$

We perform an $h$-refinement study using equal polynomial degrees $p = 2, 3, 4, 5$ for $w_h$ and $\gamma_h$, for the case of a thick plate with $t = 10^{-1}$ and a thin plate with $t = 10^{-3}$. The material parameters are taken to be $E = 10^6$ and $\nu = 0.3$. Figure 1(a) shows the convergence plots for the thick plate, whereas Figure 1(b) those for the thin plate. Dashed lines indicate the reference order of convergence. As can be seen, the convergence rates for all polynomial orders are at least of order $p$.

In addition, we study the convergence for bending moments and shear forces since these are of prime interest in the engineering design of plates. Bending moments $m$ and shear forces $q$ are obtained as

$$m = -t^3 C \varepsilon (\nabla w + \gamma), \quad q = k \mu t \gamma.$$

The exact solution for bending moments and shear forces is

$$m_{xx} = -K_b 2 \left( y^3 (y - 1)^3 (x - x^2) (5x^2 - 5x + 1) + \nu (x^3 (x - 1)^3 (y - y^2) (5y^2 - 5y + 1)) \right),$$

$$m_{yy} = -K_b 2 \left( \nu (y^3 (y - 1)^3 (x - x^2) (5x^2 - 5x + 1)) + x^3 (x - 1)^3 (y - y^2) (5y^2 - 5y + 1) \right),$$

$$m_{xy} = m_{yx} = -K_b (1 - \nu) 3y^2 (y - 1)^2 (2y - 1) x^2 (x - 1)^2 (2x - 1),$$

$$q_x = -K_b 2 \left( y^3 (y - 1)^3 (20y^3 - 30y^2 + 12y) + 3y (y - 1) (5y^2 - 5y + 1) x^2 (x - 1)^2 (2x - 1) \right),$$

$$q_y = -K_b 2 \left( x^3 (x - 1)^3 (20x^3 - 30x^2 + 12x - 1) + 3x (x - 1) (5x^2 - 5x + 1) y^2 (y - 1)^2 (2y - 1) \right),$$

where $K_b = \frac{Et^3}{12(1 - \nu^2)}$ is the plate bending stiffness. For the error measure, the Euclidean norms of $m$ and $q$ are used, $m = \sqrt{\sum_{i=1}^{2} \sum_{j=1}^{2} m_{ij}^2}$ and $q = \sqrt{\sum_{i=1}^{2} q_i^2}$. The
convergence plots for bending moments are presented in Figure 2, and those for shear forces in Figure 3. Recalling that $\mathbf{m} = t^3 \mathbf{M}$ (resp. $\mathbf{m}_h = t^3 \mathbf{M}_h$) and $\mathbf{q} = t^3 \mathbf{Q}$ (resp. $\mathbf{q}_h = t^3 \mathbf{Q}_h$), we notice that the convergence rates for the relative errors displayed in Figures 2 and 3 are in accordance with the theoretical results of Corollary 2.1 (see estimates (22) and (23)). In particular, we remark that Figure 3(b) displays an $O(1)$ convergence rate for the $L^2$-norm of the shear force errors, when $p = 2$, in agreement with estimate (23), for $s = 2$. In other words, there is no convergence.
Figure 3: Square plate with clamped boundary conditions. $L^2$-norm approximation error of shear forces with tensor-product B-splines for (a) $t = 10^{-1}$ and (b) $t = 10^{-3}$.

Figure 4: Quarter of annulus plate. Geometry setup.

3.2.2 Quarter of an annulus with clamped and simply supported boundary conditions

The second test consists of a quarter of an annulus with an inner diameter of 1.0 and outer diameter of 2.5, as shown in Figure 4. The plate thickness is $t = 0.01$ and the material parameters are $E = 10^6$ and $\nu = 0.3$. The plate is loaded with a uniform load $f(x, y) = 1$ and two boundary conditions are considered: (a) all edges are clamped, (b) all edges are simply supported. For both cases, this example exhibits boundary layers. Therefore, we adopt a refinement strategy in order to better capture the boundary layers. Given that the knot vectors range from 0 to 1, we introduce as a first step additional knots at 0.1 and 0.9 in
both directions, see Figure 5. Then, we perform uniform refinement of the given knot spans. In the following, we perform convergence studies with and without the boundary refinement strategy. The refinement is performed such that the total number of degrees of freedom is comparable in both cases. Since analytical solutions are not available for these problems, we use as reference the solutions obtained on a very fine mesh (an “overkill” solution) with $100 \times 100$ quintic elements and compute the $L^2$-norm approximation errors for the displacement.

Figure 6 shows the results for the clamped case, (a) with boundary refinement and (b) with uniform refinement, while in Figure 7 the results for the
Figure 7: Quarter of annulus with simply supported boundary conditions. $L^2$-norm approximation error of displacements with tensor-product NURBS and (a) boundary refinement, (b) uniform refinement.

simply supported case are plotted, (a) with boundary refinement and (b) with uniform refinement. As expected, due to the presence of boundary layers, in both the clamped and the simply supported case, the tests confirm that a suitable boundary refined mesh is needed to achieve optimal orders of convergence. With a uniform refinement, suboptimal results are instead obtained; this is more pronounced in the simply supported case.

4 Isogeometric discretization with NURPS

In this section, we use a triangular NURPS-based isogeometric discretization to perform numerical validations. We first briefly summarize the construction and some properties of quadratic B-splines over a Powell-Sabin (PS) refinement of a triangulation and their rational generalization, the so-called NURPS B-splines. Then, the discretized model is tested on the same examples as described in the previous section.

4.1 Quadratic PS and NURPS B-splines

Let $\mathcal{T}$ be a triangulation of a polygonal (parametric) domain $\hat{\Omega}$ in $\mathbb{R}^2$, and let $V_i = (V_{i}^{x}, V_{i}^{y})$, $i = 1, \ldots, N_V$, be the vertices of $\mathcal{T}$. A PS refinement $\mathcal{T}^*$ of $\mathcal{T}$ is the refined triangulation obtained by subdividing each triangle of $\mathcal{T}$ into six subtriangles as follows (see also Figure 8).

- Select a split point $C_i$ inside each triangle $\tau_i$ of $\mathcal{T}$ and connect it to the three vertices of $\tau_i$ with straight lines.
- For each pair of triangles $\tau_i$ and $\tau_j$ with a common edge, connect the two points $C_i$ and $C_j$. If $\tau_i$ is a boundary triangle, then also connect $C_i$ to
an arbitrary point on each of the boundary edges.

These split points must be chosen so that any constructed line segment \([C_i, C_j]\) intersects the common edge of \(\tau_i\) and \(\tau_j\). Such a choice is always possible; for instance, one can take \(C_i\) as the incenter of \(\tau_i\), i.e. the center of the circle inscribed in \(\tau_i\). Usually, in practice, the barycenter of \(\tau_i\) is also a valid choice, but not always.

The space of \(C^1\) piecewise quadratic polynomials on \(T^\ast\) is called the Powell-Sabin spline space [26] and is denoted by \(S^1_2(T^\ast)\). It is well known that the dimension of \(S^1_2(T^\ast)\) is equal to \(3N_V\). Moreover, any element of \(S^1_2(T^\ast)\) is uniquely specified by its value and its gradient at the vertices of \(T\), and can be locally constructed on each triangle of \(T\) once these values and gradients are given.

Dierckx [13] has developed a B-spline like basis \(\{B_{i,j}, j = 1, 2, 3, i = 1, \ldots, N_V\}\) of the space \(S^1_2(T^\ast)\) such that

\[
B_{i,j}(x, y) \geq 0, \quad \sum_{i=1}^{N_V} \sum_{j=1}^{3} B_{i,j}(x, y) = 1, \quad (x, y) \in \hat{\Omega}.
\] (24)

The functions \(B_{i,j}\) will be referred to as Powell-Sabin (PS) B-splines. The PS B-splines \(B_{i,j}, j = 1, 2, 3,\) are constructed to have their support locally in the molecule \(\hat{\Omega}_i\) of vertex \(V_i\), which is the union of all triangles of \(T\) containing \(V_i\). It suffices to specify their values and gradients at any vertex of \(T\). Due to the structure of the support \(\hat{\Omega}_i\), we have

\[
B_{i,j}(V_k) = 0, \quad \frac{\partial}{\partial x} B_{i,j}(V_k) = 0, \quad \frac{\partial}{\partial y} B_{i,j}(V_k) = 0,
\]

for any vertex \(V_k \neq V_i\). Moreover, we set

\[
B_{i,j}(V_i) = \alpha_{i,j}, \quad \frac{\partial}{\partial x} B_{i,j}(V_i) = \beta_{i,j}, \quad \frac{\partial}{\partial y} B_{i,j}(V_i) = \gamma_{i,j}.
\]
The triplets \((\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j})\) can be specified in a geometric way in order to satisfy (24). To this aim, for each vertex \(V_i, i = 1, \ldots, N_v\), we define three points 
\[
\{Q_{i,j} = (Q_{i,j}^x, Q_{i,j}^y), j = 1, 2, 3\},
\]
such that 
\[
\begin{bmatrix}
\alpha_{i,1} & \alpha_{i,2} & \alpha_{i,3} \\
\beta_{i,1} & \beta_{i,2} & \beta_{i,3} \\
\gamma_{i,1} & \gamma_{i,2} & \gamma_{i,3}
\end{bmatrix}
\begin{bmatrix}
Q_{i,1}^x & Q_{i,1}^y & 1 \\
Q_{i,2}^x & Q_{i,2}^y & 1 \\
Q_{i,3}^x & Q_{i,3}^y & 1
\end{bmatrix}
= 
\begin{bmatrix}
V_i^x & V_i^y & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]

The triangle with vertices \(\{Q_{i,j}, j = 1, 2, 3\}\) will be referred to as the PS triangle associated with the vertex \(V_i\) and will be denoted by \(T_i\). Finally, for each vertex \(V_i\) we define its PS points as the vertex itself and the midpoints of all the edges of the PS refinement \(T^*\) containing \(V_i\), see Figure 9. It has been proved in [13] that the functions \(B_{i,j}, j = 1, 2, 3\), are non-negative if and only if the PS triangle \(T_i\) contains all the PS points associated with the vertex \(V_i\). From a stability point of view, it is preferable to choose PS triangles with a small area.

Being equipped with a B-spline like basis, PS splines admit a straightforward rational extension. A NURPS (Non-Uniform Rational PS) basis function is defined as 
\[
R_{i,j}(x, y) = \frac{B_{i,j}(x, y)\omega_{i,j}}{\sum_{i=1}^{N_V} \sum_{r=1}^{3} B_{i,r}(x, y)\omega_{i,r}},
\]
where \(\omega_{i,j}\) are positive control weights. In our plate context, similar to the discretization with NURBS, the unknown variables \(w\) and \(\gamma\) are approximated by 
\[
w_h = \sum_{i=1}^{N_V} \sum_{j=1}^{3} R_{i,j}(x, y)\hat{w}_{i,j}, \quad \gamma_h = \sum_{i=1}^{N_V} \sum_{j=1}^{3} R_{i,j}(x, y)\hat{\gamma}_{i,j},
\]
where \( N_{V,w} \) is the number of vertices for \( w_h \) and \( N_{V,\gamma} \) is the number of vertices for \( \gamma_h \). The test functions \( v \) and \( \tau \) are discretized accordingly. Displacement boundary conditions are enforced in a standard way through the displacement degrees of freedom on the boundary while rotation boundary conditions are enforced by the modified bilinear form introduced in equation (8).

PS and NURPS B-splines have already been successfully employed to solve partial differential problems [30], in particular in the isogeometric environment [32, 31]. Certain spline spaces of higher degree and smoothness (regularity) have also been defined on triangulations endowed with a PS refinement, and they can be represented in a similar way as in the quadratic case. We refer to [27] for \( C^2 \) quintics and to [28] for a family of splines with arbitrary smoothness. Moreover, the quadratic case has been extended to the multivariate setting in [29].

Unfortunately, they are lacking the same flexibility of any combination of polynomial degree and smoothness in contrast with the tensor-product B-spline case. On the one hand, it is known how to construct stable spline spaces on triangulations with a sufficiently high polynomial degree with respect to the global smoothness (see, e.g., [20]). In particular, one can quite easily do degree-elevation for the above mentioned existing spaces (i.e., raising the polynomial degree and keeping the original smoothness). On the other hand, it is extremely challenging to construct spline spaces on triangulations with a very high smoothness relatively to the degree (like the highest continuity \( C^{p-1} \) for a degree \( p \geq 2 \)). Another interesting point of further investigation is the construction of spline spaces with mixed smoothness.

4.2 Numerical tests

In this section, we solve the same examples as illustrated in Section 3.2, using quadratic PS or NURPS B-splines. In particular, the same error measure as described before is adopted. Despite the fact that PS/NURPS splines can be defined on arbitrary triangulations, we will only consider regular meshes in our examples, in order to be able to make a fair comparison with tensor-product splines. Of course, in real applications one should exploit this feature and use triangulations generated by an adaptive refinement strategy. For results with adaptive PS/NURPS approximations in isogeometric analysis, we refer to [32, 31].

4.2.1 Square plate with clamped boundary conditions

We perform the same test described in Section 3.2.1 using quadratic PS B-splines both for deflections and rotations defined on uniform triangulations. The coarsest triangulation is depicted in Figure 10 (left), and the approximation error for the displacement is shown in Figure 11. The dashed line indicates the reference order of convergence. As can be seen, the convergence rate is of order 2. Figure 12 represents the approximation error for bending moments and shear forces. We remark that shear forces seem to converge like \( O(h) \) in the \( L^2 \)-norm for this case. However, other numerical tests (not reported here) exhibit an
Figure 10: Uniform triangulation and its mapping to a quarter of an annulus.

Figure 11: Square plate with clamped boundary conditions. $L^2$-norm approximation error of displacements for $t = 10^{-1}$ and $t = 10^{-3}$ with triangular PS splines.

$O(1)$ convergence rate, in agreement with the theoretical estimate (23), with $s = 2$.

4.2.2 Quarter of an annulus with clamped and simply supported boundary conditions

We perform the same test described in Section 3.2.2 using NURPS B-splines. As before we consider a clamped and a simply supported case and in both cases we perform boundary refinement and uniform refinement. The coarsest uniform mesh and its image are shown in Figure 10. The images of some of the boundary refined meshes are shown in Figure 13. Since there are no analytical solutions available, we have taken as reference solutions the NURPS approximations on a fine mesh (an overkill solution): we have used a triangulation consisting of
Figure 12: Square plate with clamped boundary conditions. $L^2$-norm approximation error of bending moments (left) and shear forces (right) for $t = 10^{-1}$ and $t = 10^{-3}$ with triangular PS splines.

20000 triangles according to the two refinement schemes. Figure 14 shows the approximation error for the displacement. The dashed line indicates the reference order. As can be seen, boundary refinement yields improved results for both cases. In particular, the following remark holds for the investigated range of degrees of freedom. For the simply supported case, the boundary refinement scheme achieves the correct convergence rate, whereas uniform refinement produces a sub-optimal convergence rate. For the clamped case, both the boundary refinement scheme and the uniform refinement scheme give optimal convergence rates, but the former procedure exhibits a numerical better constant in the error plots.
5 Conclusions

In this paper we mathematically and numerically investigated the reformulated variational formulation of Reissner-Mindlin plate theory in which the rotation variables are eliminated in favor of the transverse shear strains. Boundary conditions on the rotations were enforced weakly by way of Nitsche’s method to make the implementation easier and to overcome possible boundary locking phenomena (see Remark 2.3). A distinct advantage of this theory is that shear locking is precluded for any combination of trial functions for displacement and transverse shear strains. However, second derivatives of the displacement appear in the strain energy expression and these require basis functions of at least $C^1$ continuity. To deal with the smoothness requirements we employed Isogeometric Analysis, specifically various degree NURBS of maximal continuity, and quadratic triangular NURPS. The numerical results corroborated the theoretical error estimates for displacement, bending moments and transvers shear force resultants.

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A Proofs of the theoretical results

In the present section we prove all the theoretical results previously presented in the body of the paper. In the following we will assume the obvious condition that $0 < t < \text{diam} (\Omega)$, where $\text{diam} (\Omega)$ denotes the diameter of $\Omega$. We will need the following results.

- **First Korn’s inequality (see [14]).** There exists a positive constant $C$ such that
  \[ \| \varepsilon(\eta) \|_{L^2(\Omega)}^2 + \| \eta \|_{L^2(\Omega)}^2 \geq C \| \eta \|_{H^1(\Omega)}^2, \quad \forall \eta \in H^1(\Omega)^2. \]  
  (25)

- **Second Korn’s inequality (see [14]).** Suppose that $|\Gamma_c| > 0$. Then, there exists a positive constant $C$ such that
  \[ \| \varepsilon(\eta) \|_{L^2(\Omega)}^2 \geq C \| \eta \|_{H^1(\Omega)}^2, \quad \forall \eta \in H^1(\Omega)^2, \text{ such that } \eta_{|\Gamma_c} = 0. \]  
  (26)

- **Agmon’s inequality (see [1, 2]).** Let $e$ be an edge of an element $K_e$. Then
  \[ \exists C_a(K_e) > 0 \text{ only depending on the shape of } K_e \text{ such that} \]
  \[ \| \varphi \|_{L^2(e)}^2 \leq C_a(K_e)(h_e^{-1}\| \varphi \|_{L^2(K_e)}^2 + h_e\| \varphi \|_{H^1(K_e)}^2), \quad \varphi \in H^1(K_e). \]  
  (27)

Clearly, (27) also holds for vector-valued and tensor-valued functions.

A.1 Coercivity of the continuous problem

**Proposition A.1.** Let assumption $A4$ hold. Then there exists a positive constant $\alpha$ depending only on the material constants and the domain $\Omega$ such that

\[ a(\eta, \eta) + \mu kt^{-2}(\eta - \nabla v, \eta - \nabla v) \]
\[ \geq \alpha \left( \| \eta \|_{H^1(\Omega)}^2 + t^{-2}\| \eta - \nabla v \|_{L^2(\Omega)}^2 + \| v \|_{H^1(\Omega)}^2 \right), \quad \forall (v, \eta) \in \tilde{X}. \]  
(28)

**Proof.** It is easy to see that the hypotheses on $\Gamma_c$ and $\Gamma_s$ are sufficient to prevent rigid body motions. We proceed by considering the two different cases.

i) Let $\Gamma_c$ have positive length. Then, from the positive-definiteness of $C$ and the second Korn’s inequality, we get

\[ a(\eta, \eta) = (C\varepsilon(\eta), \varepsilon(\eta)) \geq C_1 \| \varepsilon(\eta) \|_{L^2(\Omega)}^2 \geq C_2 \| \eta \|_{H^1(\Omega)}^2. \]

Therefore, estimate (28) follows from a little algebra and the Poincaré inequality for $v$.

ii) Let $\Gamma_c$ have zero length. Then $\Gamma_s$ is not contained in a straight line, and, since $\Gamma_c \cup \Gamma_s$ has positive length, it follows that $\Gamma_s$ has positive length. It is enough to prove that one has

\[ a(\eta, \eta) + \| \eta - \nabla v \|_{L^2(\Omega)}^2 \geq C \left( \| \eta \|_{H^1(\Omega)}^2 + \| v \|_{H^1(\Omega)}^2 \right), \quad \forall (v, \eta) \in \tilde{X}. \]  
(29)
By contradiction, suppose that estimate (29) does not hold. Then, there exists a sequence \( \{(v_k, \eta_k)\} \in \tilde{X} \) such that
\[
\begin{cases}
a(\eta_k, \eta_k) + ||\eta_k - \nabla v_k||^2_{L^2(\Omega)} \to 0, & \text{for } k \to +\infty; \\
||\eta_k||^2_{H^1(\Omega)} + ||v_k||^2_{H^1(\Omega)} = 1.
\end{cases}
\] (30)

Up to extracting a subsequence, the second equation of (30) shows that
\[
\eta_k \rightharpoonup \eta_0 \text{ weakly in } H^1(\Omega)^2;
\quad v_k \rightharpoonup v_0 \text{ weakly in } H^1(\Omega).
\] (31)

By Rellich’s Theorem we infer that
\[
\eta_k \to \eta_0 \text{ in } L^2(\Omega)^2;
\quad v_k \to v_0 \text{ in } L^2(\Omega).
\] (32)

Therefore, recalling that \( C \) is positive-definite, from \( a(\eta_k, \eta_k) \to 0 \) (cf. (30)), (32) and (25), we get that \( \{\eta_k\} \) is a Cauchy sequence in \( H^1(\Omega)^2 \). Thus, we have
\[
\eta_k \to \eta_0 \text{ in } H^1(\Omega)^2 \quad \text{and} \quad \varepsilon(\eta_0) = 0.
\] (33)

Moreover, since \( \{\eta_k\} \) is a Cauchy sequence in \( L^2(\Omega)^2 \), from \( ||\eta_k - \nabla v_k||^2_{L^2(\Omega)} \to 0 \) (cf. (30)), we have that also \( \{\nabla v_k\} \) is a Cauchy sequence in \( L^2(\Omega)^2 \). Therefore, from (31) and (32) we obtain that
\[
v_k \to v_0 \text{ in } H^1(\Omega) \quad \text{and} \quad \nabla v_0 = \eta_0.
\]

Hence, from (33) we get \( \varepsilon(\nabla v_0) = 0 \), which implies that \( v_0 \) is an affine function. Since \( v_0 = 0 \) on \( \Gamma_s \) and \( \Gamma_s \) is not contained in a straight line, it follows that \( v_0 = 0 \) in \( \Omega \). Therefore, \( \eta_0 = 0 \) and we have proved that \( (\eta_k, v_k) \to (0, 0) \) in \( H^1(\Omega)^2 \times H^1(\Omega) \), which contradicts the second equation of (30).

\[\square\]

A.2 Stability and convergence analysis

In the present section we give the proofs of the results in Section 2.3. We need the following Korn’s type inequality.

**Lemma A.1.** Suppose that \( \Gamma_c \) has positive length. Then, there exists a positive constant \( C \) such that
\[
||\varepsilon(v)||^2_{L^2(\Omega)} + ||v||^2_{L^2(\Gamma_c)} \geq C||v||^2_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega)^2.
\] (34)

**Proof.** By contradiction. If (34) does not hold, then there exists a sequence \( \{v_k\} \) in \( H^1(\Omega)^2 \) such that
\[
\begin{cases}
||\varepsilon(v_k)||^2_{L^2(\Omega)} + ||v_k||^2_{L^2(\Gamma_c)} \to 0, & \text{for } k \to +\infty; \\
||v_k||^2_{H^1(\Omega)} = 1.
\end{cases}
\] (35)

Up to extracting a subsequence, the second equation of (35) and Rellich’s theorem show that there exists \( v_0 \in H^1(\Omega)^2 \) such that
\[
\begin{cases}
v_k \rightharpoonup v_0 \text{ weakly in } H^1(\Omega)^2; \\
v_k \to v_0 \text{ strongly in } L^2(\Omega)^2.
\end{cases}
\]
(36)

From (35) and (36) we get that \( \{(v_k, \varepsilon(v_k))\} \) is a Cauchy sequence in \( L^2(\Omega)^2 \times L^2(\Omega)^2 \). Using the first Korn inequality (25) we deduce that \( \{v_k\} \) is a Cauchy sequence also in \( H^1(\Omega)^2 \), and thus \( v_k \to v_0 \), strongly in \( H^1(\Omega)^2 \). Therefore, from the first equation of (35) we have

\[
\varepsilon(v_0) = 0 \quad \text{in } \Omega; \quad v_0|_{\Gamma_c} = 0.
\]
(37)

Equation (37) easily implies \( v_0 = 0 \). Therefore, \( v_k \to 0 \), which is in contradiction with \( ||v_k||_{H^1(\Omega)} = 1 \) (cf. (35)).

**Proof of Lemma 2.1.** We distinguish two cases.

i) \( \Gamma_c \) has zero length. In this case we have

\[
a_h(\nabla v_h + \tau_h, \nabla v_h + \tau_h) = a(\nabla v_h + \tau_h, \nabla v_h + \tau_h),
|||v_h, \tau_h|||_h = |||v_h, \tau_h|||,
\]
for every \( (v_h, \tau_h) \in X_h \); see (8) and (10). Therefore, estimate (11) immediately follows from estimate (7), since \( \Gamma_c \) with vanishing length implies \( X_h \subset X \).

ii) \( \Gamma_c \) has positive length. First, for every \( (v_h, \tau_h) \in X_h \) we will show that (cf. (10)):

\[
a_h(\nabla v_h + \tau_h, \nabla v_h + \tau_h)
\geq C \left( |||\nabla v_h + \tau_h|||^2_{H^1(\Omega)} + \sum_{e \in \mathcal{E}_h \cap \Gamma_c} h_e^{-1} |||\nabla v_h + \tau_h|||_{L^2(e)}^2 \right). \tag{38}
\]

For notational simplicity, we set \( \theta_h := \nabla v_h + \tau_h \). Then, recalling (8), we have

\[
a_h(\theta_h, \theta_h) = a(\theta_h, \theta_h) - 2 \int_{\Gamma_c} (C \varepsilon(\theta_h)n_e) \cdot \theta_h + \beta \tr(C) \sum_{e \in \mathcal{E}_h \cap \Gamma_c} h_e^{-1} \int_e |\theta_h|^2
\]

\[
= a(\theta_h, \theta_h) + \frac{\beta}{2} \tr(C) \sum_{e \in \mathcal{E}_h \cap \Gamma_c} h_e^{-1} \int_e |\theta_h|^2
\]

\[-2 \int_{\Gamma_c} (C \varepsilon(\theta_h)n_e) \cdot \theta_h + \frac{\beta}{2} \tr(C) \sum_{e \in \mathcal{E}_h \cap \Gamma_c} h_e^{-1} \int_e |\theta_h|^2.
\]

Applying (34) with \( v = \theta_h \), we obtain

\[
a_h(\theta_h, \theta_h) \geq C_K |||\theta_h|||^2_{H^1(\Omega)}
- 2 \int_{\Gamma_c} (C \varepsilon(\theta_h)n_e) \cdot \theta_h + \frac{\beta}{2} \tr(C) \sum_{e \in \mathcal{E}_h \cap \Gamma_c} h_e^{-1} \int_e |\theta_h|^2, \tag{39}
\]
for a suitable positive constant $C_K$. For each edge $e \in \Gamma_e \cap E_h$, let the symbol $K_e$ denote an element of $\Omega_h$ such that $e \in \partial K$. We now have, by simple algebra and using (27):

$$-2 \int_{\Gamma_e} (C\varepsilon(\theta_h)n_e) \cdot \theta_h = \sum_{e \in E_h \cap \Gamma_e} (-2 \int_{\Gamma_e} (C\varepsilon(\theta_h)n_e) \cdot \theta_h)$$

$$\geq \sum_{e \in E_h \cap \Gamma_e} (-2\|C\varepsilon(\theta_h)n_e\|_{L^2(e)}\|\theta_h\|_{L^2(e)})$$

$$\geq \sum_{e \in E_h \cap \Gamma_e} (-2C\varepsilon\|\varepsilon(\theta_h)\|_{L^2(e)}\|\theta_h\|_{L^2(e)})$$

$$\geq -\sum_{e \in E_h \cap \Gamma_e} C_C \left( \frac{e h_e\|\varepsilon(\theta_h)\|_{L^2(e)}}{\gamma_e h_e} + \frac{1}{\gamma_e h_e} \|\theta_h\|_{L^2(e)}^2 \right)$$

$$\geq -\sum_{e \in E_h \cap \Gamma_e} \left( C_C C_a(K_e) \gamma_e \left( \frac{\|\varepsilon(\theta_h)\|_{H^1(K_e)}}{\|\theta_h\|_{L^2(K_e)}} + h_e^2 \|\varepsilon(\theta_h)\|_{H^1(K_e)}^2 \right) + \frac{C_C}{\gamma_e h_e} \|\theta_h\|_{L^2(e)}^2 \right),$$

for positive constants $\{\gamma_e\}_{e \in \Gamma_e \cap E_h}$ to be chosen. By using the inverse inequality

$$\|\varepsilon(\theta_h)\|_{H^1(K_e)}^2 \leq C_{inv}(K_e) h_e^{-2} \|\varepsilon(\theta_h)\|_{L^2(K_e)}^2,$$

and setting

$$C(K_e) = C_C C_a(K_e) \left( 1 + C_{inv}(K_e) \frac{h_e^2}{h_e^2} \right),$$

from (40) it follows that

$$-2 \int_{\Gamma_e} (C\varepsilon(\theta_h)n_e) \cdot \theta_h \geq -\sum_{e \in E_h \cap \Gamma_e} \left( C(K_e) \gamma_e \|\varepsilon(\theta_h)\|_{L^2(K_e)}^2 + \frac{C_C}{\gamma_e h_e} \|\theta_h\|_{L^2(e)}^2 \right).$$

Therefore, from (39) and (41) we get

$$a_h(\theta_h, \theta_h) \geq C_K \|\theta_h\|_{H^1(\Omega)}^2 - \sum_{e \in E_h \cap \Gamma_e} C(K_e) \gamma_e \|\varepsilon(\theta_h)\|_{L^2(K_e)}^2$$

$$+ \sum_{e \in E_h \cap \Gamma_e} \left( \frac{\beta}{2} \text{tr}(C) - \frac{C_C}{\gamma_e} \right) h_e^{-1} \int_{\Gamma_e} |\theta_h|^2 \geq$$

$$\left( C_K - \sum_{e \in E_h \cap \Gamma_e} C(K_e) \gamma_e \right) \|\theta_h\|_{H^1(\Omega)}^2 + \left( \frac{\beta}{2} \text{tr}(C) - \frac{C_C}{\gamma_e} \right) \sum_{e \in E_h \cap \Gamma_e} h_e^{-1} \int_{\Gamma_e} |\theta_h|^2.$$  (42)

Choosing

$$\gamma_e = \frac{C_K}{2} C(K_e)^{-1} \quad \text{and} \quad \beta_0 = \frac{\gamma C_K + 2C_C}{\gamma \text{tr}(C)} \quad \text{with} \quad \gamma = \min_{e \in E_h \cap \Gamma_e} \gamma_e,$$

from (42) we deduce that, for every $\beta \geq \beta_0$, we have

$$a_h(\theta_h, \theta_h) \geq \frac{C_K}{2} \left( \|\theta_h\|_{H^1(\Omega)}^2 + \sum_{e \in E_h \cap \Gamma_e} h_e^{-1} \int_{\Gamma_e} |\theta_h|^2 \right).$$
Recalling that \( \theta_h := \nabla v_h + \tau_h \), we get that (38) holds. Therefore, (11) follows from (38), (5), (10) and the Poincaré inequality applied to \( v_h \) (recall that \( v_h|_{\Gamma_e} = 0 \) and \( |\Gamma_e| > 0 \)). Finally note that, due to the uniform shape regularity of the elements \( K \) in \( \{\Omega_h\}_h \), it is easy to check that the constant \( \beta_0 \) is uniformly bounded from above independently of the mesh size \( h \). □

**Proof of Proposition 2.1.** In the following, \( C \) will denote a generic positive constant independent of \( h \). Given any pair \((v_h, \tau_h)\) in \( X_h \), we denote by \( w_E = w_h - v_h \), \( \gamma_E = \gamma_h - \tau_h \) and by \( w_A = w - v_h \), \( \gamma_A = \gamma - \tau_h \). By applying first the coercivity Lemma 2.1 and then using the linearity of the bilinear forms and the consistency condition (12), we get

\[
\alpha' \|\gamma_E, w_E\|^2_h \leq a_h(\nabla w_E + \gamma_E, \nabla w_E + \gamma_E) + t^{-2}(\gamma_E, \gamma_E) = a_h(\nabla w_A + \gamma_A, \nabla w_E + \gamma_E) + t^{-2}(\gamma_A, \gamma_E).
\]

By definitions (8) and (2) and standard algebra we get from (43)

\[
\|\gamma_E, w_E\|^2_h \leq C T_A^{1/2} T_E^{1/2} + t^{-2}\|\gamma_A\|_{L^2(\Omega)} \|\gamma_E\|_{L^2(\Omega)},
\]

where the scalar terms are given by

\[
T_A = \|\nabla w_A + \gamma_A\|^2_{H^1(\Omega)} + \sum_{e \in \mathcal{E}_h \cap \Gamma_e} h_e \|\mathcal{C}_e(\nabla w_A + \gamma_A) n_e\|^2_{L^2(e)} + \sum_{e \in \mathcal{E}_h \cap \Gamma_e} h_e^{-1} \|\nabla w_A + \gamma_A\|^2_{L^2(e)},
\]

and

\[
T_E = \|\nabla w_E + \gamma_E\|^2_{H^1(\Omega)} + \sum_{e \in \mathcal{E}_h \cap \Gamma_e} h_e \|\mathcal{C}_e(\nabla w_E + \gamma_E) n_e\|^2_{L^2(e)} + \sum_{e \in \mathcal{E}_h \cap \Gamma_e} h_e^{-1} \|\nabla w_E + \gamma_E\|^2_{L^2(e)}.
\]

Term \( T_A \) can be bounded by using (27), as already done in (40). Without again showing the details, and following the same notation for \( K_e \) introduced in (40), we get

\[
T_A \leq C \left( \|\nabla w_A + \gamma_A\|^2_{H^1(\Omega)} + \sum_{e \in \mathcal{E}_h \cap \Gamma_e} h_e^2 |w_A|^2_{H^1(K_e)} + |w_A|^2_{H^2(K_e)} + h_e^2 |\gamma_A|^2_{H^2(K_e)} + h_e^{-2} |w_A|^2_{H^1(K_e)} + h_e^{-2} |\gamma_A|^2_{L^2(K_e)} \right).
\]

From the above bound, a triangle inequality, and the definition of \( w_A, \gamma_A \), we get

\[
T_A \leq C \sum_{j = 0}^2 \left( \sum_{K \in \Omega_h} h_K^{2(j-1)} |w - v_h|^2_{H^{j+1}(K)} + \sum_{K \in \Omega_h} h_K^{2(j-1)} |\gamma - \tau_h|^2_{H^j(K)} \right),
\]

(45)
We now bound $T_E$. Again, using the Agmon inequality (27) and inverse estimates as done in (40), we get for all $e \in \mathcal{E}_h \cap \Gamma_c$:

$$h_e||\mathcal{C}(\nabla w_E + \gamma_E)\mathbf{n}_e||_{L^2(e)}^2 \leq C\left(h_e^2||\nabla w_E + \gamma_E||_{H^2(K_e)}^2 + ||\nabla w_E + \gamma_E||_{H^1(K_e)}^2\right).$$

Combining the above bound with the definition of $T_E$ and (10) yields

$$T_E + t^{-2}||\gamma_E||_{L^2(\Omega)}^2 \leq C||w_E, \gamma_E||_h^2. \quad (46)$$

Now, recalling (44) and using (46) we easily get

$$||\gamma_E, w_E||_h \leq C(T_A^{1/2} + t^{-1}||\gamma_A||_{L^2(\Omega)}).$$

Finally, recalling that the above inequality holds for all $(v_h, \tau_h) \in X_h$, the bound in (45) concludes the proof. □

**Proof of Proposition 2.2.** We only sketch the proof. Estimate (16) immediately follows from (5), by recalling (14) and (15). We prove (17) only for the case of a simply supported plate. In this case, $a_h(\cdot, \cdot) = a(\cdot, \cdot)$, because $\Gamma_c$ is the empty set. However, we notice that different boundary conditions can be dealt with using a similar technique. Let $s_h$ and $\tau_h$ be given in $\Xi_h$. Using (6) and (9), we get

$$(Q_h - s_h, \tau_h) = (Q_h - Q, \tau_h) + (Q - s_h, \tau_h)$$

Choosing $\psi_h = h^2(Q_h - s_h)$, and using the inverse inequality

$$||Q_h - s_h||_{H^1(\Omega)} \leq Ch^{-1}||Q_h - s_h||_{L^2(\Omega)}.$$

we have

$$h^2||Q_h - s_h||_{L^2(\Omega)}^2 = h^2a(\nabla(w - w_h) + (\gamma - \gamma_h), (Q_h - s_h))$$

$$+ h^2(Q - s_h, Q_h - s_h) \leq C||w - w_h, \gamma - \gamma_h||_h h||Q_h - s_h||_{L^2(\Omega)}$$

$$+ h||Q - s_h||_{L^2(\Omega)} h||Q_h - s_h||_{L^2(\Omega)}.$$

Hence, we obtain

$$h||Q_h - s_h||_{L^2(\Omega)} \leq C||w - w_h, \gamma - \gamma_h||_h + h||Q - s_h||_{L^2(\Omega)}.$$

Therefore, the triangle inequality gives

$$h||Q - Q_h||_{L^2(\Omega)} \leq C||w - w_h, \gamma - \gamma_h||_h + 2h||Q - s_h||_{L^2(\Omega)},$$

from which we infer

$$h||Q - Q_h||_{L^2(\Omega)} \leq C||w - w_h, \gamma - \gamma_h||_h + 2h \inf_{s_h \in \Xi_h} ||Q - s_h||_{L^2(\Omega)}.$$
To prove (18), we use the so-called Pitkäranta-Verfürth trick (see [25] and [33]). We start with noticing that
\[ \|Q - Q_h\|_{H^{-1}(\Omega)} = \sup_{\tau \in H_0^1(\Omega)^2} \frac{(Q - Q_h, \tau)}{\|\tau\|_{H^1(\Omega)}}. \] (47)

Then, fix \( \tau \in H_0^1(\Omega)^2 \) and take \( \tau_I \) as its best approximation in \( \Xi_h \) with respect to the \( H^1 \)-norm. Then, we have
\[ (Q - Q_h, \tau) = (Q - Q_h, \tau - \tau_I) + (Q - Q_h, \tau_I) \leq C h \|Q - Q_h\|_{L^2(\Omega)} \|\psi\|_{H^3(\Omega)} + (Q - Q_h, \tau_I). \] (48)

Using (6) and (9), we get
\[ (Q - Q_h, \tau_I) = a(\nabla (w_h - w) + (\gamma_h - \gamma), \tau_I) \leq C \|w - w_h, \gamma - \gamma_h\|_h \|\tau_I\|_{H^1(\Omega)} \] (49)

Therefore, from (47), (48) and (49) we deduce
\[ \|Q - Q_h\|_{H^{-1}(\Omega)} \leq C (h \|Q - Q_h\|_{L^2(\Omega)} + \|w - w_h, \gamma - \gamma_h\|_h). \] (50)

Estimate (18) now follows from (50) and (17). \( \square \)

**Proof of Proposition 2.3.** We begin with introducing the following auxiliary problem:
\[
\begin{cases}
\text{Find } (\tilde{\omega}, \tilde{\gamma}) \in X, \text{ such that } \\
a(\nabla \tilde{\omega} + \tilde{\gamma}, \nabla \tilde{v} + \tau) + t^{-2}(\tilde{\gamma}, \tau) = (\theta - \theta_h, \nabla \tilde{v} + \tau), \quad \forall (\tilde{v}, \tau) \in X.
\end{cases}
\] (51)

Note that problem (51) is equivalent, up to the usual transformation (4), to a classical Reissner-Mindlin problem with the load acting on the rotations. Therefore, also recalling the hypotheses above and the definition of \( \gamma \), the following regularity result holds [22]
\[ \|\tilde{\omega}_1\|_{H^2(\Omega)} + t^{-1}\|\tilde{\omega}_2\|_{H^2(\Omega)} + t^{-1}\|\tilde{\gamma}\|_{H^2(\Omega)} \leq C \|\theta - \theta_h\|_{L^2(\Omega)}, \] (52)

with \( \tilde{\omega} = \tilde{\omega}_1 + \tilde{\omega}_2 \) denoting a splitting of the deflections.

The consistency of the discrete bilinear form, analogously as in (12), implies that
\[ a_h(\nabla \tilde{\omega} + \tilde{\gamma}, \nabla \tilde{v}_h + \tau_h) + t^{-2}(\tilde{\gamma}, \tau_h) = (\theta - \theta_h, \nabla \tilde{v}_h + \tau_h), \] (53)

for all \((v_h, \tau_h) \in X_h\). Note that, whenever both entries are in \( X \), the bilinear form \( a_h(\cdot, \cdot) \) is equal to the original form \( a(\cdot, \cdot) \) since all the additional terms vanish. Therefore, from (51) we also have
\[ a_h(\nabla \tilde{\omega} + \tilde{\gamma}, \nabla v + \tau) + t^{-2}(\tilde{\gamma}, \tau) = (\theta - \theta_h, \nabla v + \tau), \quad \forall (v, \tau) \in X. \] (54)
Combining (53) and (54) we can take \((w - w_h, \gamma - \gamma_h)\) as a test function, and obtain
\[
|||\theta - \theta_h|||_{L^2(\Omega)}^2 = a_h(\nabla \tilde{w} + \tilde{\gamma}, \nabla (w - w_h) + \gamma - \gamma_h) + t^{-2}(\tilde{\gamma}, \gamma - \gamma_h) 
\] (55)
where we recall the usual notation \(\theta = \nabla w + \gamma\) and \(\theta_h = \nabla w_h + \gamma_h\). The remaining steps are standard in Aubin-Nitsche duality arguments and therefore only described briefly. By the symmetry of the bilinear form and using (9), (12), from (55) we get
\[
|||\theta - \theta_h|||_{L^2(\Omega)}^2 = a_h(\nabla (\tilde{w} - \tilde{w}_I) + \tilde{\gamma} - \tilde{\gamma}_I, \nabla (w - w_h) + \gamma - \gamma_h) + t^{-2}(\tilde{\gamma} - \tilde{\gamma}_I, \gamma - \gamma_h), 
\] (56)
where \((\tilde{w}_I, \tilde{\gamma}_I) \in X_h^I\) is an approximant of \((\tilde{w}, \tilde{\gamma})\).

Due to the \(||| \cdot |||_h\)-norm continuity of the bilinear form in (56), all we need to show in order to prove (19) is that
\[
|||\tilde{w} - \tilde{w}_I, \gamma - \gamma_I|||_h \leq C (h + t) |||\theta - \theta_h|||_{L^2(\Omega)}, 
\] (57)
for some uniform constant \(C\). The bound in (57) can be shown by making use of the regularity estimate (52) and the approximation properties A3 of the space \(X_h^I\); we do not show the details.

Finally, estimate (20) follows easily by the triangle inequality
\[
|||w - w_h|||_{H^1(\Omega)} \leq |||\theta - \theta_h|||_{L^2(\Omega)} + |||\gamma - \gamma_h|||_{L^2(\Omega)} 
\]
and using (19).

\[\square\]

References


