Nonconforming locking-free finite elements for Reissner-Mindlin plates

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Abstract

We prove optimal error estimates in L^2 for a nonconforming finite element for Reissner-Mindlin plates recently introduced in [13]. Moreover, we present numerical experiments using this element and other two nonconforming elements analyzed in [8].

Key words: Finite Element Methods, Reissner-Mindlin plates, Nonconforming Methods, Error Analysis.

Introduction

In recent times there has been a considerable interest in the extension of Discontinuous Galerkin methods to the treatment of elliptic problems for various applications (see, for instance, [3] and the references therein). One of the reason of this increasing popularity is probably the fact that the DG machinery often implies a different approach to the problem, that can sometimes lead, in the end, to new conforming or nonconforming finite elements that would have been more difficult to discover starting with the classical approach. This is surely the case, for instance, of the extension of the Crouzeix-Raviart element

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for Stokes problem or nearly incompressible elasticity problems (see [12]), or the higher order Arnold-Falk elements for Reissner-Mindlin plates (see [4]). The latter paper indicated the basic ingredients for constructing locking-free DG-based finite elements for Reissner-Mindlin plates, that led to the introduction of simple elements of nonconforming type in [8] and [13]. More precisely, the elements of [8] use piecewise linear nonconforming approximation (Crouzeix-Raviart) for both rotations and transversal displacement, with the addition of various types of bubble-functions. In [13] Lovadina proved that the introduction of the bubbles is not needed, and proved optimal estimates, in the broken H^1 norm, with respect to h and to the regularity of the quantities involved.

The aim of the present paper is twofold. On one hand, we prove optimal estimates in L^2, for both rotations and displacement, for the nonconforming P_1 element of [13]. On the other hand, we collect numerical examples to analyze and compare the practical behavior of the elements of [8] and [13].

The paper is organized as follows. In Section 1 we recall the Reissner-Mindlin equations and regularity results, while in Section 2 we briefly recall the elements of [8] and [13]. In Section 3 we prove optimal $O(h^2)$ L^2 error estimates, for both rotations and displacement, for the element of [13]. Section 4 is devoted to numerical results.

Throughout the paper we will use standard notations for Sobolev spaces and norms (cf. [7] and [11], for instance); C will denote a generic constant, which may differ in different occurrences, possibly dependent on the plate geometry and the elastic coefficients, but independent of the meshsize h and the thickness parameter t.

1 The Reissner-Mindlin problem

The Reissner-Mindlin equations for a clamped plate with regular and bounded midplane Ω require to find $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ such that

$$-\operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}) - \boldsymbol{\gamma} = 0 \qquad \text{in } \Omega, \tag{1}$$

$$-\operatorname{div} \boldsymbol{\gamma} = g \qquad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{\gamma} = \lambda t^{-2}(\boldsymbol{\nabla} w - \boldsymbol{\theta}) \qquad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{\theta} = 0, \; w = 0 \qquad \text{on } \partial\Omega. \tag{4}$$

In (1)-(3), $\mathbf{C}$ is the tensor of bending moduli, $\boldsymbol{\theta}$ represents the rotations, w the transversal displacement, $\boldsymbol{\gamma}$ the scaled shear stresses and g a given transversal load. Moreover, ε is the usual symmetric gradient operator, λ is the shear modulus (incorporating also the shear correction factor), and t is the thickness. For simplicity of exposition, we shall assume that both $\mathbf{C}$ and λ are constant in Ω, but the results remain valid in the more general case when $\mathbf{C}$ and λ are smooth functions. We shall need the following regularity result (see, e.g., [5] and [9]).

Proposition 1 *Suppose that Ω is convex and $g \in L^2(\Omega)$. Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ be the solution of* (1)–(4)*. Then the following estimate holds*

$$||\boldsymbol{\theta}||_2 + ||w||_2 + ||\boldsymbol{\gamma}||_{H(\mathrm{div})} + t||\boldsymbol{\gamma}||_1 \le C||g||_0, \tag{5}$$

where

$$||\boldsymbol{\gamma}||^2_{H(\mathrm{div})} = ||\boldsymbol{\gamma}||^2_0 + ||\operatorname{div}\boldsymbol{\gamma}||^2_0.$$

□

The classical variational formulation of problem (1)–(4) is

$$\begin{cases} \text{Find } (\boldsymbol{\theta}, w, \boldsymbol{\gamma}) \in \mathbf{H}^1_0(\Omega) \times H^1_0(\Omega) \times \mathbf{L}^2(\Omega) : \\ a(\boldsymbol{\theta}, \boldsymbol{\eta}) + (\boldsymbol{\nabla} v - \boldsymbol{\eta}, \boldsymbol{\gamma}) = (g, v) & (\boldsymbol{\eta}, v) \in \mathbf{H}^1_0(\Omega) \times H^1_0(\Omega), \\ (\boldsymbol{\nabla} w - \boldsymbol{\theta}, \boldsymbol{\tau}) - \lambda^{-1} t^2 (\boldsymbol{\gamma}, \boldsymbol{\tau}) = 0 & \boldsymbol{\tau} \in \mathbf{L}^2(\Omega), \end{cases} \tag{6}$$

where $(\cdot, \cdot)$ is the inner-product in $L^2(\Omega)$ (or in $\mathbf{L}^2(\Omega)$), and

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_\Omega \mathbf{C}\, \varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta})\, \mathrm{d}x. \tag{7}$$

2 Nonconforming elements

We now introduce nonconforming finite element approximations of problem (1)–(4) using the approach detailed in [8]. Let then $\mathcal{T}_h$ be a regular decomposition of Ω into triangular elements T ([11]), and let us set $H^1(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} H^1(T)$. Following [4], we define suitable jump and average operators. Let $\mathcal{E}_h$ denote the set of all the edges in $\mathcal{T}_h$, and $\mathcal{E}^{\mathrm{in}}_h$ the set of internal edges. Let e be an internal edge of $\mathcal{T}_h$, shared by two elements T^+ and T^-, and let φ denote a function in $H^1(\mathcal{T}_h)$, or a vector in $\mathbf{H}^1(\mathcal{T}_h)$, or a tensor in $(H^1(\mathcal{T}_h))^4_s$. We define the average as usual:

$$\{\varphi\} = \frac{\varphi^+ + \varphi^-}{2} \quad \forall e \in \mathcal{E}^{\mathrm{in}}_h. \tag{8}$$

For a scalar function $\varphi \in H^1(\mathcal{T}_h)$ we define its jump as

$$[\varphi] = \varphi^+ \mathbf{n}^+ + \varphi^- \mathbf{n}^- \quad \forall e \in \mathcal{E}^{\mathrm{in}}_h, \tag{9}$$

while the jump of a vector $\boldsymbol{\varphi} \in \mathbf{H}^1(\mathcal{T}_h)$ is given by

$$[\boldsymbol{\varphi}] = (\boldsymbol{\varphi}^+ \otimes \mathbf{n}^+)_S + (\boldsymbol{\varphi}^- \otimes \mathbf{n}^-)_S \quad \forall e \in \mathcal{E}_h^{\text{in}}, \tag{10}$$

where $(\boldsymbol{\varphi}\otimes\mathbf{n})_S$ denotes the symmetric part of the tensor product, and $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) is the outward unit normal to ∂T^+ (resp. to ∂T^-). On the boundary edges we define jumps of scalars as $[\varphi] = \varphi\mathbf{n}$, and jumps of vectors as $[\boldsymbol{\varphi}] = (\boldsymbol{\varphi}\otimes\mathbf{n})_S$, where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. We also define averages of vectors and tensors as $\{\boldsymbol{\varphi}\} = \boldsymbol{\varphi}$.

Following the ideas of [8], we now select finite element spaces $\boldsymbol{\Theta}_h \subset \mathbf{H}^1(\mathcal{T}_h)$, $W_h \subset H^1(\mathcal{T}_h)$, and $\boldsymbol{\Gamma}_h \subset \mathbf{L}^2(\Omega)$, with the property: $\boldsymbol{\nabla}_h W_h \subseteq \boldsymbol{\Gamma}_h$, where $\boldsymbol{\nabla}_h$ denotes the gradient operator element by element. The discrete problem is then

$$\begin{cases} \text{Find } (\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in \boldsymbol{\Theta}_h \times W_h \times \boldsymbol{\Gamma}_h \\ a_h(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + (\boldsymbol{\gamma}_h, \boldsymbol{\nabla}_h v_h - \mathbf{R}_h\boldsymbol{\eta}_h) = (g, v_h) \qquad (\boldsymbol{\eta}_h, v_h) \in \boldsymbol{\Theta}_h \times W_h, \\ (\boldsymbol{\nabla}_h w_h - \mathbf{R}_h\boldsymbol{\theta}_h, \boldsymbol{\tau}_h) - \lambda^{-1}t^2(\boldsymbol{\gamma}_h, \boldsymbol{\tau}_h) = 0 \qquad \boldsymbol{\tau}_h \in \boldsymbol{\Gamma}_h. \end{cases} \tag{11}$$

Above, the bilinear form $a_h(\cdot,\cdot)$ is defined by

$$a_h(\boldsymbol{\theta}, \boldsymbol{\eta}) := \sum_{T\in\mathcal{T}_h} \int_T \mathbf{C}\,\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta})\,\mathrm{d}x + p_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}), \tag{12}$$

where p_Θ is a penalty term given by

$$p_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}) := \sum_{e\in\mathcal{E}_h} \frac{\kappa_e}{|e|} \int_e [\boldsymbol{\theta}] : [\boldsymbol{\eta}]\,\mathrm{d}s \qquad (|e| := \text{length of the edge } e), \tag{13}$$

and κ_e is a positive constant having the same physical dimension as $\mathbf{C}$ (for instance, for smooth $\mathbf{C}$ one could take κ_e as $|\mathbf{C}|$ evaluated at the midpoint of e). Furthermore, $\mathbf{R}_h : \mathbf{H}^1(\mathcal{T}_h) \to \boldsymbol{\Gamma}_h$ is a suitable 'reduction' operator, to be defined case by case.

Remark 2.1 *We point out that eliminating $\boldsymbol{\gamma}_h$ from system* (11)*, our scheme is equivalent to the following problem involving only the rotations and the vertical displacements:*

$$\begin{cases} \textit{Find } (\boldsymbol{\theta}_h, w_h) \in \boldsymbol{\Theta}_h \times W_h : \\ a_h(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + \lambda t^{-2}(\boldsymbol{\nabla}_h w_h - \mathbf{R}_h\boldsymbol{\theta}_h, \boldsymbol{\nabla}_h v_h - \mathbf{R}_h\boldsymbol{\eta}_h) \\ \qquad\qquad = (g, v_h) \qquad \forall(\boldsymbol{\eta}_h, v_h) \in \boldsymbol{\Theta}_h \times W_h. \end{cases} \tag{14}$$

2.1 The "nonconforming bubble" element

This is the element presented and analyzed in [8], defined as follows. First, on a generic triangle $T \in \mathcal{T}_h$ we define:

$$B_2^{NC}(T) := \text{Span}\{\chi_2\}, \tag{15}$$

where χ_2 denotes the nonconforming bubble of P_2, i.e., the polynomial of degree 2 vanishing at the two Gauss points of each edge. In barycentric coordinates this bubble has the expression (for instance),

$$\chi_2 = 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 2. \tag{16}$$

The scheme is then given by the following choices.

- The finite element spaces are

$$\boldsymbol{\Theta}_h = \Big\{\boldsymbol{\eta} : \boldsymbol{\eta}_{|T} \in \Big(P_1(T) \oplus B_2^{NC}(T)\Big)^2,\ \int_e [\boldsymbol{\eta}]\,\mathrm{d}s = 0\ \forall e \in \mathcal{E}_h\Big\}, \tag{17}$$

$$W_h = \Big\{v : v_{|T} \in P_1(T) \oplus B_2^{NC}(T),\ \int_e [v]\,\mathrm{d}s = 0\ \forall e \in \mathcal{E}_h\Big\}, \tag{18}$$

$$\boldsymbol{\Gamma}_h = \Big\{\boldsymbol{\tau} : \boldsymbol{\tau}_{|T} \in P_0(T)^2 \oplus \boldsymbol{\nabla} B_2^{NC}(T)\Big\}. \tag{19}$$

- The reduction operator $\mathbf{R}_h : \mathbf{H}^1(\mathcal{T}_h) \to \boldsymbol{\Gamma}_h$ is defined locally by:

$$\int_T (\boldsymbol{\eta} - \mathbf{R}_h\boldsymbol{\eta})\,\mathrm{d}x = 0 \quad \forall T \in \mathcal{T}_h,\quad \boldsymbol{\eta} \in \mathbf{H}^1(\mathcal{T}_h), \tag{20}$$

$$\int_T \mathrm{div}(\boldsymbol{\eta} - \mathbf{R}_h\boldsymbol{\eta})\,\mathrm{d}x = 0 \quad \forall T \in \mathcal{T}_h,\quad \boldsymbol{\eta} \in \mathbf{H}^1(\mathcal{T}_h). \tag{21}$$

2.2 The "conforming bubble" element

This element is also mentioned in [8], and it differs from the previous one only for the choice of the type of bubble. Define, on a generic triangle $T \in \mathcal{T}_h$:

$$B_3(T) := \text{Span}\{b_3\}, \tag{22}$$

where b_3 denotes the standard cubic bubble. In barycentric coordinates its expression is, for instance,

$$b_3 = 27\lambda_1\lambda_2\lambda_3. \tag{23}$$

This scheme is characterized by the following choices.

- The finite element spaces are

$$\boldsymbol{\Theta}_h = \Big\{\boldsymbol{\eta} : \boldsymbol{\eta}_{|T} \in \Big(P_1(T) \oplus B_3(T)\Big)^2,\ \int_e [\boldsymbol{\eta}]\, \mathrm{d}s = 0\ \forall e \in \mathcal{E}_h\Big\}, \tag{24}$$

$$W_h = \Big\{v : v_{|T} \in P_1(T) \oplus B_3(T),\ \int_e [v]\, \mathrm{d}s = 0\ \forall e \in \mathcal{E}_h\Big\}, \tag{25}$$

$$\boldsymbol{\Gamma}_h = \Big\{\boldsymbol{\tau} : \boldsymbol{\tau}_{|T} \in P_0(T)^2 \oplus \boldsymbol{\nabla} B_3(T)\Big\}. \tag{26}$$

- The reduction operator $\mathbf{R}_h : \mathbf{H}^1(\mathcal{T}_h) \to \boldsymbol{\Gamma}_h$ is the same defined in (20)-(21).

2.3 The $P_1^{NC} - P_1^{NC} - P_0$ element

This element has been introduced and analyzed in [13], and is characterized by the following choices.

- The finite element spaces are

$$\boldsymbol{\Theta}_h = \Big\{\boldsymbol{\eta} : \boldsymbol{\eta}_{|T} \in (P_1(T))^2,\ \int_e [\boldsymbol{\eta}]\, \mathrm{d}s = 0 \quad \forall e \in \mathcal{E}_h\Big\}, \tag{27}$$

$$W_h = \Big\{v : v_{|T} \in P_1(T),\ \int_e [v]\, \mathrm{d}s = 0 \quad \forall e \in \mathcal{E}_h\Big\}, \tag{28}$$

$$\boldsymbol{\Gamma}_h = \Big\{\boldsymbol{\tau} : \boldsymbol{\tau}_{|T} \in (P_0(T))^2\Big\}, \tag{29}$$

- $\mathbf{R}_h$ is simply the L^2-projection operator onto the piecewise constant functions (see (20)).

3 L^2-estimates for the $P_1^{NC} - P_1^{NC} - P_0$ element

In this section we shall derive optimal error estimates in L^2 for rotations and deflections, when the $P_1^{NC} - P_1^{NC} - P_0$ element is considered. Since in this case the reduction operator is just the L^2-projection onto $\boldsymbol{\Gamma}_h$, problem (11) simplifies in:

$$\begin{cases} \text{Find } (\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in \boldsymbol{\Theta}_h \times W_h \times \boldsymbol{\Gamma}_h \\ a_h(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + (\boldsymbol{\gamma}_h, \boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h) = (g, v_h) \qquad (\boldsymbol{\eta}_h, v_h) \in \boldsymbol{\Theta}_h \times W_h, \\ (\boldsymbol{\nabla}_h w_h - \boldsymbol{\theta}_h, \boldsymbol{\tau}_h) - \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \boldsymbol{\tau}_h) = 0 \qquad \boldsymbol{\tau}_h \in \boldsymbol{\Gamma}_h, \end{cases} \tag{30}$$

where $\boldsymbol{\Theta}_h$, W_h and $\boldsymbol{\Gamma}_h$ are given in Section 2.3. It has been proved in [13] that, when Ω is a convex polygon, it holds

$$||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_{1,h} + ||w - w_h||_{1,h} + ||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_{\Gamma} + t||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_0 \leq Ch\, ||g||_0. \tag{31}$$

Above, $||\cdot||_{1,h}$ denotes the H^1-broken norm and $||\cdot||_\Gamma$ is the norm given by

$$||\boldsymbol{\tau}||_\Gamma := (||\boldsymbol{\tau}||_{-1}^2 + ||\operatorname{div}\boldsymbol{\tau}||_{-1}^2)^{1/2}. \tag{32}$$

To derive our L^2 estimates we shall use duality arguments. Let then $(\boldsymbol{\varphi}, z, \boldsymbol{\rho})$ be the solution of the *dual problem:*

$$-\operatorname{div}\mathbf{C}\,\varepsilon(\boldsymbol{\varphi}) - \boldsymbol{\rho} = \boldsymbol{\theta} - \boldsymbol{\theta}_h \qquad \text{in } \Omega, \tag{33}$$

$$-\operatorname{div}\boldsymbol{\rho} = w - w_h \qquad \text{in } \Omega, \tag{34}$$

$$\boldsymbol{\nabla} z - \boldsymbol{\varphi} - \lambda^{-1}t^2\boldsymbol{\rho} = 0 \qquad \text{in } \Omega, \tag{35}$$

$$\boldsymbol{\varphi} = 0,\ z = 0 \qquad \text{on } \partial\Omega, \tag{36}$$

for which we have the estimate

$$||\boldsymbol{\varphi}||_2 + ||z||_2 + ||\boldsymbol{\rho}||_{H(\mathrm{div})} + t||\boldsymbol{\rho}||_1 \le C\left(||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0 + ||w - w_h||_0\right). \tag{37}$$

Theorem 3.1 *It holds*

$$||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0 + ||w - w_h||_0 \le Ch^2. \tag{38}$$

Proof. Multiplying equations (1)–(3) by $\boldsymbol{\eta}_h \in \boldsymbol{\Theta}_h$, $v_h \in W_h$, $\boldsymbol{\tau}_h \in \boldsymbol{\Gamma}_h$, and integrating by parts we obtain

$$\begin{cases} a_h(\boldsymbol{\theta}, \boldsymbol{\eta}_h) + (\boldsymbol{\gamma}, \boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h) = (g, v_h) + c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h) + c_W(\boldsymbol{\gamma}, v_h), \\ (\boldsymbol{\nabla} w - \boldsymbol{\theta}, \boldsymbol{\tau}_h) - \lambda^{-1}t^2(\boldsymbol{\gamma}, \boldsymbol{\tau}_h) = 0, \end{cases} \tag{39}$$

where $c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h)$ and $c_W(\boldsymbol{\gamma}, v_h)$ are given by

$$c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h) := \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\mathbf{n}\cdot\boldsymbol{\eta}_h\,\mathrm{d}s = \sum_{e\in\mathcal{E}_h}\int_e\{\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\} : [\boldsymbol{\eta}_h]\,\mathrm{d}s, \tag{40}$$

$$c_W(\boldsymbol{\gamma}, v_h) := \sum_{T\in\mathcal{T}_h}\int_{\partial T}\boldsymbol{\gamma}\cdot\mathbf{n}v_h\,\mathrm{d}s = \sum_{e\in\mathcal{E}_h}\int_e\{\boldsymbol{\gamma}\}\cdot[v_h]\,\mathrm{d}s. \tag{41}$$

Subtracting (30) from (39) we obtain the error equations

$$\begin{cases} a_h(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\nabla}_h v_h - \boldsymbol{\eta}_h) = c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}_h) + c_W(\boldsymbol{\gamma}, v_h) \\ \Big(\boldsymbol{\nabla}_h(w - w_h) - (\boldsymbol{\theta} - \boldsymbol{\theta}_h), \boldsymbol{\tau}_h\Big) - \lambda^{-1}t^2(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\tau}_h) = 0. \end{cases} \tag{42}$$

Choose now $\boldsymbol{\eta}_h = \boldsymbol{\varphi}_I$ as the P_1 *conforming* interpolant of $\boldsymbol{\varphi}$, $v_h = z_I$ as the P_1 *conforming* interpolant of z, and $\boldsymbol{\tau}_h = \boldsymbol{\rho}_I$ as a *suitable* interpolant of $\boldsymbol{\rho}$ (to be defined later). From (42) we get

$$\begin{cases} a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\varphi}_I)+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\nabla} z_I-\boldsymbol{\varphi}_I)=0 \\ \Big(\boldsymbol{\nabla}_h(w-w_h)-(\boldsymbol{\theta}-\boldsymbol{\theta}_h),\boldsymbol{\rho}_I\Big)-\lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\rho}_I)=0. \end{cases} \tag{43}$$

Testing the *dual problem* (33)–(36) with $\boldsymbol{\theta}-\boldsymbol{\theta}_h$, $w-w_h$ and $\boldsymbol{\gamma}-\boldsymbol{\gamma}_h$ we have

$$\begin{cases} a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\varphi})+\Big(\boldsymbol{\nabla}_h(w-w_h)-(\boldsymbol{\theta}-\boldsymbol{\theta}_h),\boldsymbol{\rho}\Big)=||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0^2+||w-w_h||_0^2 \\ \qquad\qquad +c_\Theta(\boldsymbol{\varphi},\boldsymbol{\theta}-\boldsymbol{\theta}_h)+c_W(\boldsymbol{\rho},w-w_h) \\ (\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\nabla} z-\boldsymbol{\varphi})-\lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\rho})=0. \end{cases} \tag{44}$$

Adding the two equations in (44) and using (43), we infer

$$\begin{aligned} a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,&\boldsymbol{\varphi}-\boldsymbol{\varphi}_I)+\Big(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\nabla}(z-z_I)-(\boldsymbol{\varphi}-\boldsymbol{\varphi}_I)\Big) \\ &+\Big(\boldsymbol{\nabla}_h(w-w_h)-(\boldsymbol{\theta}-\boldsymbol{\theta}_h),\boldsymbol{\rho}-\boldsymbol{\rho}_I\Big)-\lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\rho}-\boldsymbol{\rho}_I) \\ &=||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0^2+||w-w_h||_0^2+c_\Theta(\boldsymbol{\varphi},\boldsymbol{\theta}-\boldsymbol{\theta}_h)+c_W(\boldsymbol{\rho},w-w_h). \end{aligned} \tag{45}$$

Therefore, it holds

$$\begin{aligned} &||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0^2+||w-w_h||_0^2 \\ &=a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\varphi}-\boldsymbol{\varphi}_I)-c_\Theta(\boldsymbol{\varphi},\boldsymbol{\theta}-\boldsymbol{\theta}_h)+\Big(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\nabla}(z-z_I)-(\boldsymbol{\varphi}-\boldsymbol{\varphi}_I)\Big) \\ &\qquad+\Big\{\Big(\boldsymbol{\nabla}_h(w-w_h)-(\boldsymbol{\theta}-\boldsymbol{\theta}_h),\boldsymbol{\rho}-\boldsymbol{\rho}_I\Big)-c_W(\boldsymbol{\rho},w-w_h)\Big\} \\ &\qquad\qquad-\lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\rho}-\boldsymbol{\rho}_I) \\ &=: T_1+T_2+T_3+T_4+T_5. \end{aligned} \tag{46}$$

Before proceeding, let us recall a useful trace inequality (see [1], [2]). Let T be a triangle, and let e be an edge of T. Then, if $v\in H^1(T)$,

$$||v||_{0,e}^2\le C(|e|^{-1}||v||_{0,T}^2+|e|\,|v|_{1,T}^2), \tag{47}$$

with C a constant only depending on the minimum angle of T. Inequality (47) implies, in particular, that, for $\boldsymbol{\eta}\in\mathbf{H}_0^1(\Omega)$, $\boldsymbol{\eta}_h\in\boldsymbol{\Theta}_h$:

$$\sum_{e\in\mathcal{E}_h}|e|^{-1}||[\boldsymbol{\eta}-\boldsymbol{\eta}_h]||_{0,e}^2=\sum_{e\in\mathcal{E}_h}|e|^{-1}||[(\boldsymbol{\eta}-\boldsymbol{\eta}_h)-P_e^0(\boldsymbol{\eta}-\boldsymbol{\eta}_h)]||_{0,e}^2\le C||\boldsymbol{\eta}-\boldsymbol{\eta}_h||_{1,h}^2, \tag{48}$$

where P_e^0 denotes the $L^2(e)-$projection onto constant functions. Similarly, via Cauchy-Schwarz inequality and (47), for piecewise smooth functions $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$

we deduce:

$$
\begin{aligned}
c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}) &= \sum_{e\in\mathcal{E}_h}\int_e \{\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\} : [\boldsymbol{\eta}]\,\mathrm{d}s \\
&\le \Big(\sum_{e\in\mathcal{E}_h}|e|\,||\{\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\}||^2_{0,e}\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}||[\boldsymbol{\eta}]||^2_{0,e}\Big)^{1/2} \qquad (49)\\
&\le C\sum_{T\in\mathcal{T}_h}(|\boldsymbol{\theta}|^2_{1,T}+|e|^2|\boldsymbol{\theta}|^2_{2,T})^{1/2}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}||[\boldsymbol{\eta}]||^2_{0,e}\Big)^{1/2}.
\end{aligned}
$$

<u>Estimate for T_1</u>. Classical interpolation results, (31) and (37) give

$$
a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\varphi}-\boldsymbol{\varphi}_I) \le C||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{1,h}\,|\boldsymbol{\varphi}-\boldsymbol{\varphi}_I|_1 \le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0+||w-w_h||_0\right). \qquad (50)
$$

<u>Estimate for T_2</u>. Since $[\boldsymbol{\theta}]_{|e}=0\ \forall e$, and $\int_e[\boldsymbol{\theta}_h]\,\mathrm{d}s=0\ \forall e$ by definition (27), recalling that $\mathbf{C}$ is constant, and using (49), (48), (31) and (37) we obtain:

$$
\begin{aligned}
T_2 &= -c_\Theta(\boldsymbol{\varphi},\boldsymbol{\theta}-\boldsymbol{\theta}_h) = -\sum_{e\in\mathcal{E}_h}\int_e\{\mathbf{C}\,\varepsilon(\boldsymbol{\varphi})\} : [\boldsymbol{\theta}-\boldsymbol{\theta}_h]\,\mathrm{d}s \\
&= -\sum_{e\in\mathcal{E}_h}\int_e\{\mathbf{C}\,\varepsilon(\boldsymbol{\varphi}-\boldsymbol{\varphi}_I)\} : [\boldsymbol{\theta}-\boldsymbol{\theta}_h]\,\mathrm{d}s \qquad (51)\\
&\le C\sum_{T\in\mathcal{T}_h}(|\boldsymbol{\varphi}-\boldsymbol{\varphi}_I|^2_{1,T}+|e|^2|\boldsymbol{\varphi}|^2_{2,T})^{1/2}\Big(\sum_{e\in\mathcal{E}_h}|e|^{-1}||[\boldsymbol{\theta}-\boldsymbol{\theta}_h]||^2_{0,e}\Big)^{1/2}\\
&\le Ch||\boldsymbol{\varphi}||_2||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{1,h} \le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0+||w-w_h||_0\right).
\end{aligned}
$$

<u>Estimate for T_3</u>. We have, recalling that $\boldsymbol{\Gamma}=H^{-1}(\mathrm{div})$ (see (32)):

$$
\begin{aligned}
\Big(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\nabla}(z-z_I)-(\boldsymbol{\varphi}-\boldsymbol{\varphi}_I)\Big) &\le C||\boldsymbol{\gamma}-\boldsymbol{\gamma}_h||_\Gamma\left(||z-z_I||_1+||\boldsymbol{\varphi}-\boldsymbol{\varphi}_I||_1\right)\\
&\le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0+||w-w_h||_0\right). \qquad (52)
\end{aligned}
$$

<u>Estimate for T_4</u>. This term is more difficult. We first consider the Helmholtz decomposition for $\boldsymbol{\rho}$ (see [6]):

$$
\boldsymbol{\rho} = \boldsymbol{\nabla}p+\mathbf{curl}\,q \qquad p\in H^2(\Omega)\cap H^1_0(\Omega),\quad q\in H^1(\Omega)/\mathbf{R}, \qquad (53)
$$

for which it holds

$$
\Big(||p||^2_2+||q||^2_1\Big)^{1/2} \le C||\boldsymbol{\rho}||_{H(\mathrm{div})}. \qquad (54)
$$

We now take p_I as the piecewise linear and continuous Lagrange interpolant of p, and q_I as the Clément interpolant of q. We finally set $\boldsymbol{\rho}_I \in \boldsymbol{\Gamma}_h$ as

$$\boldsymbol{\rho}_I = \boldsymbol{\nabla} p_I + \mathbf{curl}\, q_I. \tag{55}$$

The following interpolation estimates hold (cf. [9]):

$$||\boldsymbol{\rho} - \boldsymbol{\rho}_I||_0 \leq Ch\,||\boldsymbol{\rho}||_1, \tag{56}$$

$$||\boldsymbol{\rho} - \boldsymbol{\rho}_I||_\Gamma \leq Ch\,||\boldsymbol{\rho}||_{H(\mathrm{div})}. \tag{57}$$

Using (53) and (55) we get

$$\begin{aligned} T_4 =& \Big(\boldsymbol{\nabla}_h(w-w_h) - (\boldsymbol{\theta}-\boldsymbol{\theta}_h), \boldsymbol{\nabla}(p-p_I) + \mathbf{curl}\,(q-q_I)\Big) - c_W(\boldsymbol{\rho}, w-w_h) \\ =& \Big(\boldsymbol{\nabla}_h(w-w_h), \boldsymbol{\nabla}(p-p_I)\Big) + \Big\{\Big(\boldsymbol{\nabla}_h(w-w_h), \mathbf{curl}\,(q-q_I)\Big) \\ & - c_W(\boldsymbol{\rho}, w-w_h)\Big\} - \Big(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\nabla}(p-p_I)\Big) - \Big(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \mathbf{curl}\,(q-q_I)\Big) \\ =& T_4^1 + T_4^2 + T_4^3 + T_4^4. \end{aligned} \tag{58}$$

The term T_4^1 is standard, by simply observing that

$$\begin{aligned} T_4^1 &= \Big(\boldsymbol{\nabla}_h(w-w_h), \boldsymbol{\nabla}(p-p_I)\Big) \leq Ch\,||\boldsymbol{\nabla} p||_1 ||w-w_h||_{1,h} \\ &\leq Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0 + ||w-w_h||_0\right), \end{aligned} \tag{59}$$

and so is T_4^3:

$$\begin{aligned} T_4^3 &= -\Big(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\nabla}(p-p_I)\Big) \leq Ch||\boldsymbol{\nabla} p||_1||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{1,h} \\ &\leq Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0 + ||w-w_h||_0\right). \end{aligned} \tag{60}$$

We now treat the term T_4^2, and notice that

$$\Big(\boldsymbol{\nabla}_h w, \mathbf{curl}\,(q-q_I)\Big) = \Big(\boldsymbol{\nabla} w, \mathbf{curl}\,(q-q_I)\Big) = 0, \tag{61}$$

$$c_W(\boldsymbol{\rho}, w) = 0, \tag{62}$$

and

$$(\boldsymbol{\nabla}_h w_h, \mathbf{curl}\, q_I) = 0. \tag{63}$$

Therefore, we obtain

$$\begin{aligned} T_4^2 =& -(\boldsymbol{\nabla}_h w_h, \mathbf{curl}\, q) + c_W(\boldsymbol{\rho}, w_h) = -\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\nabla}_h w_h \cdot \mathbf{curl}\, q\,\mathrm{d}x \\ & + \sum_{e\in\mathcal{E}_h}\int_e \{\mathbf{curl}\, q\}\cdot[w_h]\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h}\int_e \{\boldsymbol{\nabla} p\}\cdot[w_h]\,\mathrm{d}s. \end{aligned} \tag{64}$$

Since

$$-\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\nabla}_h w_h\cdot\mathbf{curl}\,q\,\mathrm{d}x+\sum_{e\in\mathcal{E}_h}\int_e\{\mathbf{curl}\,q\}\cdot[w_h]\,\mathrm{d}s=0,$$

it follows

$$\begin{aligned}T_4^2&=\sum_{e\in\mathcal{E}_h}\int_e\{\boldsymbol{\nabla}p\}\cdot[w_h]\,\mathrm{d}s=-\sum_{e\in\mathcal{E}_h}\int_e\{\boldsymbol{\nabla}p\}\cdot[w-w_h]\,\mathrm{d}s\\&=-\sum_{e\in\mathcal{E}_h}\int_e\{\boldsymbol{\nabla}p-\boldsymbol{\nabla}p_I\}\cdot[w-w_h]\,\mathrm{d}s,\end{aligned}\tag{65}$$

where, in the last step, we used $[w]=0$ and the definition (28) of W_h. Consequently, proceeding as we did for (51), and using (31) and (54), we obtain:

$$T_4^2\le Ch\,||p||_2||w-w_h||_{1,h}\le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0+||w-w_h||_0\right).\tag{66}$$

It remains to treat the term T_4^4. Integrating by parts we get

$$\begin{aligned}T_4^4&=-\sum_{T\in\mathcal{T}_h}\left\{\int_T(q-q_I)\operatorname{rot}(\boldsymbol{\theta}-\boldsymbol{\theta}_h)\,\mathrm{d}x-\int_{\partial T}(q-q_I)(\boldsymbol{\theta}-\boldsymbol{\theta}_h)\cdot\mathbf{t}_T\,\mathrm{d}s\right\}\\&=-\sum_{T\in\mathcal{T}_h}\int_T(q-q_I)\operatorname{rot}(\boldsymbol{\theta}-\boldsymbol{\theta}_h)\,\mathrm{d}x+\sum_{e\in\mathcal{E}_h}\int_e\mathbf{t}_e\otimes\mathbf{n}_e:[\boldsymbol{\theta}-\boldsymbol{\theta}_h]\{q-q_I\}\,\mathrm{d}s.\end{aligned}\tag{67}$$

On one hand, we have

$$\begin{aligned}-\sum_{T\in\mathcal{T}_h}\int_T(q-q_I)\operatorname{rot}(\boldsymbol{\theta}-\boldsymbol{\theta}_h)\,\mathrm{d}x&\le Ch||q||_1||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{1,h}\\&\le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0+||w-w_h||_0\right).\end{aligned}\tag{68}$$

On the other hand, it holds

$$\begin{aligned}&\sum_{e\in\mathcal{E}_h}\int_e\mathbf{t}_e\otimes\mathbf{n}_e:[\boldsymbol{\theta}-\boldsymbol{\theta}_h]\{q-q_I\}\\&\le\left(\sum_{e\in\mathcal{E}_h}|e|\,||\{q-q_I\}||_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}|e|^{-1}\,||[\boldsymbol{\theta}-\boldsymbol{\theta}_h]||_{0,e}^2\right)^{1/2}\\&\le C\left(\sum_{T\in\mathcal{T}_h}\left(||q-q_I||_{0,T}^2+h_T^2|q-q_I|_{1,T}^2\right)\right)^{1/2}||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{1,h}\\&\le Ch||q||_1||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{1,h}\le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0+||w-w_h||_0\right).\end{aligned}\tag{69}$$

Therefore, from (67)–(69) and (54) we obtain

$$T_4^4\le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0+||w-w_h||_0\right).\tag{70}$$

From (59), (60), (66) and (70) we infer

$$T_4 \leq Ch^2 \left(||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0 + ||w - w_h||_0 \right). \tag{71}$$

Estimate for T_5. From (31), (37) and (56) we have:

$$-\lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\rho} - \boldsymbol{\rho}_I) \leq Ch^2 t ||\boldsymbol{\rho}||_1 \leq Ch^2 \left(||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0 + ||w - w_h||_0 \right). \tag{72}$$

Collecting (50), (51), (52), (71) and (72), from (46) we get

$$||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0^2 + ||w - w_h||_0^2 \leq Ch^2 \left(||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0 + ||w - w_h||_0 \right), \tag{73}$$

by which

$$||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0 + ||w - w_h||_0 \leq Ch^2. \tag{74}$$

The proof is complete. □

Remark 3.1 *In the more general case when* $\mathbf{C}$ *is a smooth function (not necessarily constant), the term* T_2 *in* (46) *may be estimated as follows. We notice that (see* (40)*):*

$$\begin{aligned} T_2 &= -c_\Theta(\boldsymbol{\varphi}, \boldsymbol{\theta} - \boldsymbol{\theta}_h) = -\sum_{e \in \mathcal{E}_h} \int_e \{\mathbf{C}\, \varepsilon(\boldsymbol{\varphi})\} : [\boldsymbol{\theta} - \boldsymbol{\theta}_h] \, \mathrm{d}s \\ &= -\sum_{e \in \mathcal{E}_h} \int_e \{\mathbf{C}\, \varepsilon(\boldsymbol{\varphi} - \boldsymbol{\varphi}_I)\} : [\boldsymbol{\theta} - \boldsymbol{\theta}_h] \, \mathrm{d}s - \sum_{e \in \mathcal{E}_h} \int_e \{(\mathbf{C} - \overline{\mathbf{C}})\, \varepsilon(\boldsymbol{\varphi}_I)\} : [\boldsymbol{\theta} - \boldsymbol{\theta}_h] \, \mathrm{d}s \\ &= T_2^1 + T_2^2, \end{aligned} \tag{75}$$

where $\overline{\mathbf{C}} \in (L^2(\mathcal{E}_h))_s^4$ *denotes the* L^2*-projection of* $\mathbf{C}_{|\mathcal{E}_h}$ *onto the piecewise constant functions. The term* T_2^1 *is exactly as in* (51)*:*

$$T_2^1 = -\sum_{e \in \mathcal{E}_h} \int_e \{\mathbf{C}\, \varepsilon(\boldsymbol{\varphi} - \boldsymbol{\varphi}_I)\} : [\boldsymbol{\theta} - \boldsymbol{\theta}_h] \, \mathrm{d}s \leq Ch^2 \left(||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0 + ||w - w_h||_0 \right). \tag{76}$$

Similarly to (51), *we have*

$$
\begin{aligned}
T_2^2 &= -\sum_{e\in\mathcal{E}_h}\int_e \{(\mathbf{C}-\overline{\mathbf{C}})\,\varepsilon(\boldsymbol{\varphi}_I)\} : [\boldsymbol{\theta}-\boldsymbol{\theta}_h]\,\mathrm{d}s \\
&\le \Big(\sum_{e\in\mathcal{E}_h} |e|\,||\{(\mathbf{C}-\overline{\mathbf{C}})\,\varepsilon(\boldsymbol{\varphi}_I)\}||_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h} |e|^{-1}||[\boldsymbol{\theta}-\boldsymbol{\theta}_h]||_{0,e}^2\Big)^{1/2} \\
&\le C\Big(\sum_{e\in\mathcal{E}_h} |e|^3||\{\varepsilon(\boldsymbol{\varphi}_I)\}||_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h} |e|^{-1}||[\boldsymbol{\theta}-\boldsymbol{\theta}_h]||_{0,e}^2\Big)^{1/2} \\
&\le Ch|\boldsymbol{\varphi}_I|_1\Big(\sum_{e\in\mathcal{E}_h} |e|^{-1}||[\boldsymbol{\theta}-\boldsymbol{\theta}_h]||_{0,e}^2\Big)^{1/2} \\
&\le Ch||\boldsymbol{\varphi}||_2||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_{1,h} \le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0 + ||w-w_h||_0\right).
\end{aligned}
\tag{77}
$$

From (75), (76) *and* (77) *we obtain*

$$
T_2 \le Ch^2\left(||\boldsymbol{\theta}-\boldsymbol{\theta}_h||_0 + ||w-w_h||_0\right). \tag{78}
$$

4 Numerical results

In this section we present some numerical results showing the behavior of the nonconforming elements introduced in Section 2. Following [10], as a test problem we take an isotropic and homogeneous plate $\Omega = (0,1)\times(0,1)$, clamped on the whole boundary. The analytical solution is explicitly known, and given below. The bending moduli tensor $\mathbf{C}$ and the shear modulus λ are (choosing 5/6 as shear correction factor):

$$
\mathbf{C}\,\boldsymbol{\tau} := \frac{E}{12(1-\nu^2)}\Big((1-\nu)\boldsymbol{\tau} + \nu\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I}\Big) \qquad , \qquad \lambda := \frac{5E}{12(1+\nu)}, \tag{79}
$$

where $\boldsymbol{\tau}$ is a generic symmetric second-order tensor, $\mathbf{I}$ is the forth-order identity tensor, and, as usual, E is the Young's modulus and ν the Poisson's ratio. Choosing the transversal load g as

$$
\begin{aligned}
g(x,y) = \frac{E}{12(1-\nu^2)}\Big[&12y(y-1)(5x^2-5x+1)\Big(2y^2(y-1)^2 \\
&+ x(x-1)(5y^2-5y+1)\Big) \\
&+ 12x(x-1)(5y^2-5y+1)\Big(2x^2(x-1)^2 \\
&+ y(y-1)(5x^2-5x+1)\Big)\Big],
\end{aligned}
\tag{80}
$$

the solution $(\boldsymbol{\theta}, w)$ of (1)-(4) is given by

$$
\begin{aligned}
\theta_1(x,y) &= y^3(y-1)^3x^2(x-1)^2(2x-1) \\
\theta_2(x,y) &= x^3(x-1)^3y^2(y-1)^2(2y-1) \\
w(x,y) &= \frac{1}{3}1x^3(x-1)^3y^3(y-1)^3 \\
&- \frac{2t^2}{5(1-\nu)}\Big[y^3(y-1)^3x(x-1)(5x^2-5x+1) \\
&+ x^3(x-1)^3y(y-1)(5y^2-5y+1)\Big].
\end{aligned}
\tag{81}
$$

We introduce the relative errors in the discrete L^2 norm

$$
\begin{aligned}
(E_\theta)^2 &= \frac{\sum_{i=1}^N |\boldsymbol{\theta}(x_i,y_i) - \boldsymbol{\theta}_h(x_i,y_i)|^2}{\sum_{i=1}^N |\boldsymbol{\theta}(x_i,y_i)|^2}, \\
(E_w)^2 &= \frac{\sum_{i=1}^N (w(x_i,y_i) - w_h(x_i,y_i))^2}{\sum_{i=1}^N (w(x_i,y_i))^2},
\end{aligned}
\tag{82}
$$

N being the number of nodes in the decomposition, and the relative errors in the energy norm

$$
\begin{aligned}
(E_\theta^1)^2 &= \frac{a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h, \boldsymbol{\theta}-\boldsymbol{\theta}_h)}{a(\boldsymbol{\theta},\boldsymbol{\theta})}, \\
(E_w^1)^2 &= \frac{\|\boldsymbol{\nabla} w - \boldsymbol{\nabla}_h w_h\|_0^2}{\|\boldsymbol{\nabla} w\|_0^2}.
\end{aligned}
\tag{83}
$$

Above, $a_h(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are the bilinear forms defined in (12)-(13) and (7), respectively. Moreover, in the penalty term (13) we set $k_e = |\mathbf{C}| \simeq$ euclidean norm of $\mathbf{C}$.

We analyze the convergence properties of the elements by considering different uniform decompositions ($h = 1/4,\ h = 1/8,\ h = 1/16,\ h = 1/24$), and keeping the thickness sufficiently small ($t = .001$).

In Figures 1 and 2 we report the relative errors, in the L^2-norm, for rotations and deflection respectively.

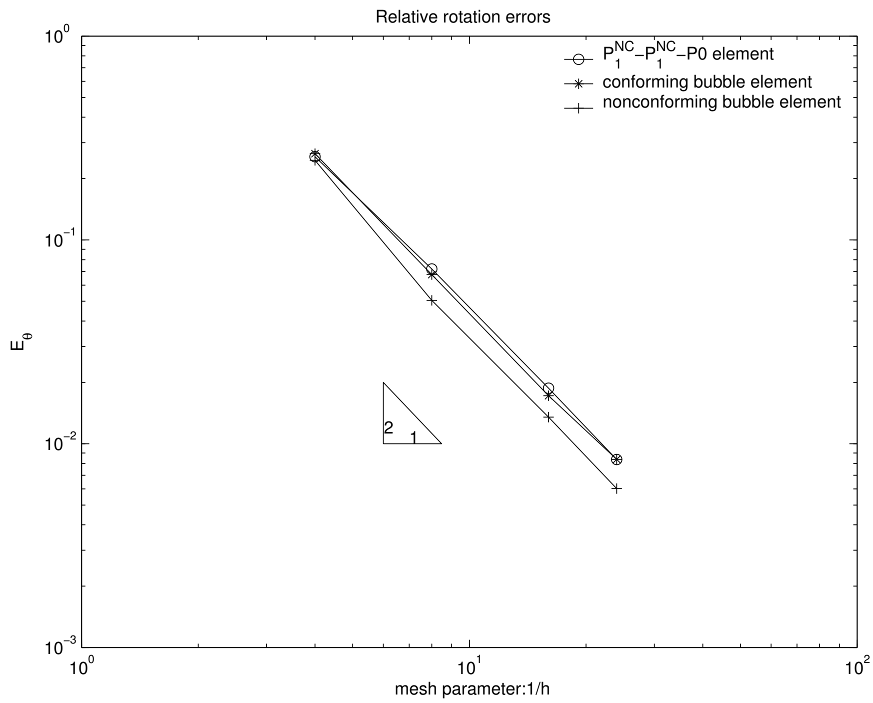

Fig. 1. Relative rotation errors in L^2-norm versus $1/h$

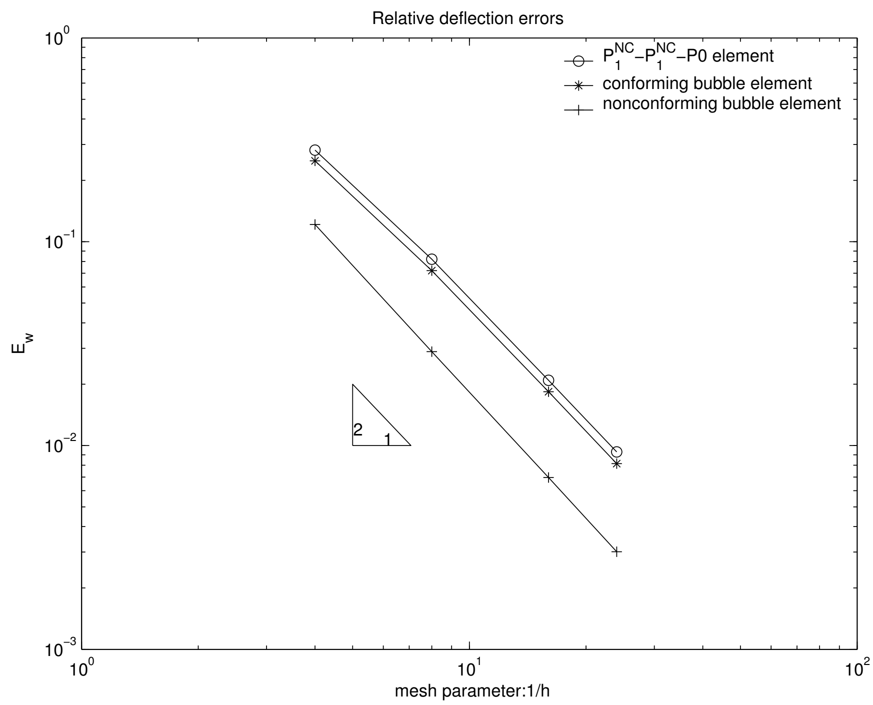

Fig. 2. Relative deflection errors in L^2-norm versus $1/h$

Figures 3 and 4 show the relative errors for rotations and deflection, respectively, in the energy norm.

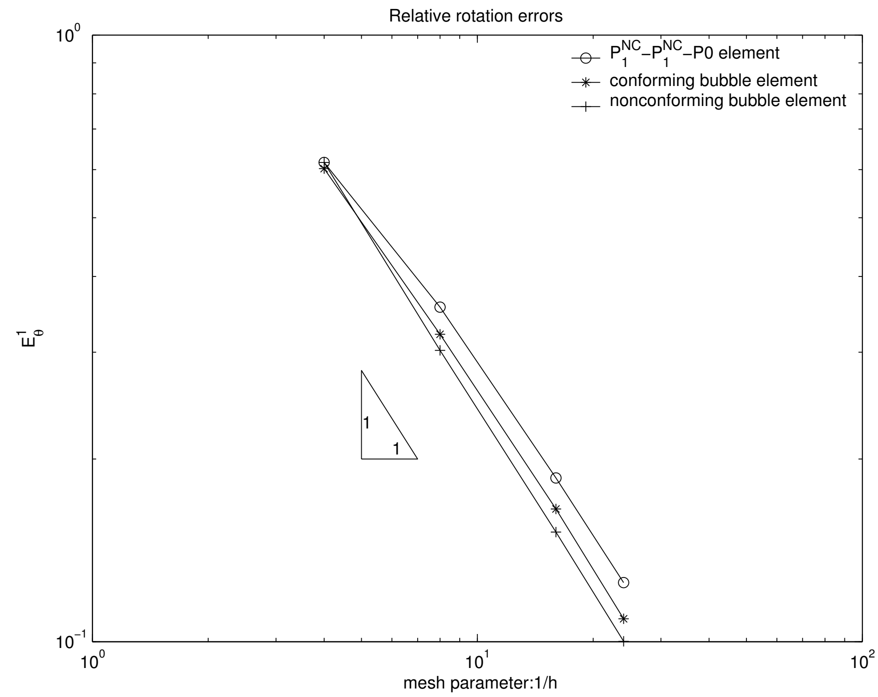

Fig. 3. Relative rotation errors in the energy norm versus $1/h$

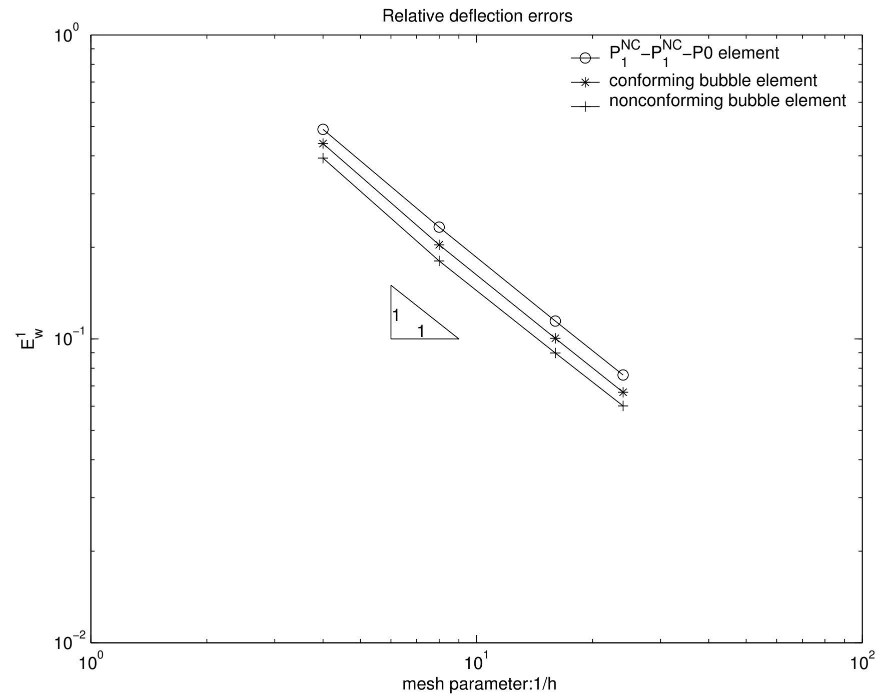

Fig. 4. Relative deflection errors in the energy norm versus $1/h$

Conclusions

We considered three *locking-free* plate elements, recently proposed and analyzed in [8] and [13]. For the scheme introduced in [13], optimal $O(h^2)$ L^2 error estimates for the kinematic variables were proved. Furthermore, some numerical tests were provided, confirming the theoretical predictions.

For all the schemes, both the rotations and the transversal displacement *share the same nodes*, which is a favorable feature in view of a possible generalization to shell problems.

The elements presented are all based on a *low-order* (i.e. P_1) approximation; it could be interesting to explore the possibility of extending the present approach to design higher-order elements, and to perform a suitable *hp*-analysis to derive error estimates.

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