The importance of the exact satisfaction of the incompressibility constraint in nonlinear elasticity: mixed FEMs versus NURBS-based approximations.

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Abstract

In the present paper we investigate the capability of finite element methods to correctly reproduce the stability range of finite strain problems in the incompressible regime. To this end, we develop a numerical scheme, obtained combining a stream function formulation with an isogeometric NURBS approach, which is able to sharply estimate the stability limits of the continuum problem. Using such a method, we show a pair of benchmark problems on which various well-known finite element methods largely fail in approximating the correct stability range.

Key words: Incompressible nonlinear elasticity, Stability, NURBS-based isogeometric analysis, Stream function formulation, Mixed finite elements

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1 Introduction

Despite several finite element schemes perform very well both in terms of accuracy and stability for the case of linear elastic problems, also in the presence of highly constrained situations such as in incompressible cases, it is well known that extensions of such schemes to the case of nonlinear elasticity are not guaranteed to show the same properties in terms of accuracy and/or stability (see, e.g., [18,26]).

In previous works [3,5], we studied the capability of some mixed finite elements (well performing in the small strain regime) to correctly reproduce the stability range of the continuum finite strain problem on a simple model example. The model problem that we selected for our studies consisted of a bidimensional incompressible block, for which a trivial solution could be simply identified. Then, we were able to theoretically find a rough estimate of the stability limit and to make some comparisons with the results obtained on the discrete problem by means of the mixed finite element schemes under investigation. The conclusions of those works were that all the considered numerical schemes had problems in reproducing the correct stability limit, but a quantitative evaluation of such a failure was missing, since a sharp *theoretical* estimate of the limit for the continuum problem was not available.

In this paper, we study the original model problem proposed in [3] and a new variant obtained considering different boundary conditions. In particular, we show how to construct a reliable estimate for the stability range of the continuum problem, which can be used as a reference solution to evaluate in a precise way how a numerical scheme behaves in reproducing the stability properties of the continuum problem. Focusing on the *linearized* problem around a generic solution to the large strain (nonlinear) problem, such an estimate is obtained using an isogeometric “stream function” formulation (see [4]), which allows to exactly enforce the linearized incompressibility constraint. In fact, we show that the exact satisfaction of the linearized incompressibility constraint leads to a numerical approximation of the stability range that converges to the exact one (i.e. the one for the continuum problem). Within this “stream function” context, it is worth noticing that the high regularity of NURBS shape functions [14] is a key ingredient.

A noteworthy result we obtain is that, even with fine meshes, the investigated mixed finite element formulations show traction stability limits which are smaller than the 18% of the expected one in the first example, and smaller than the 35% in the second. We can therefore conclude that in both cases all the considered mixed finite element schemes completely fail in reproducing the stability range of the continuum problem, despite their small strain counterparts are absolutely reliable.

These conclusions show that, when dealing with highly constrained problems within the finite strain regime, it is not enough using the natural extensions of mixed formulations well performing in small strains, but it is of the highest importance to search for methods able to satisfy the internal constraints in an exact way.

A brief outline of the paper is as follows.

In Section 2 we present the (parameter-depending) family of large strain incompressible elasticity problems we are interested in. The Hellinger-Reissner variational principle is here used. We then derive the corresponding *linearized* problems around a generic solution. In addition, for $2D$ problems, we introduce a *stream function* formulation for the linearized problem (see section 2.1). In section 2.2 we present a definition of stability range, which is naturally based on the stability features of the corresponding *linearized* problems. We remark that the stability range can be equivalently defined using both the Hellinger-Reissner approach and the stream function formulation.

In Section 3 we state and prove a few results about the Galerkin approximation of the stability range. In particular, Proposition 1 concerns the schemes based on the Hellinger-Reissner formulation, as the *mixed finite element methods* presented in Section 5. This result is compatible with the observed phenomenon that mixed finite elements may detect a smaller stability range than the expected one (cf. Remark 2). However, we stress that an efficient approximation of the stability range might be possible, for certain particular cases. Proposition 3 concerns the schemes based on the stream function formulation, as the *NURBS approach* presented in Section 5. It is shown that those methods are able to correctly approximate the stability range of the problem at hand (cf. Remark 4).

In Section 4 we provide a description of our two model problems, while Section 5 presents detailed numerical computations, performed with several mixed finite elements and NURBS-based approximation schemes. As already mentioned, the numerical tests show a severe failure of the mixed finite elements in approximating the stability range of the corresponding continuous problems.

2 The finite strain incompressible elasticity problem and the stability range

In this paper we adopt the so-called *material description* to study the finite strain elasticity problem. Accordingly, we suppose that we are given a *reference configuration* $\Omega \subset \mathcal{R}^d$ for a d-dimensional bounded material body $\mathcal{B}$. Therefore, the deformation of $\mathcal{B}$ can be described by means of the map $\hat{\boldsymbol{\varphi}} : \Omega \to \mathcal{R}^d$

defined by

$$\widehat{\varphi}(\mathbf{X}) = \mathbf{X} + \widehat{\mathbf{u}}(\mathbf{X}) \ , \tag{1}$$

where $\mathbf{X} = (X_1, .., X_d)$ denotes the coordinates of a material point in the reference configuration and $\widehat{\mathbf{u}}(\mathbf{X})$ represents the corresponding displacement vector. Following standard notations (see [7,12,19,20], for instance) we introduce the deformation gradient $\widehat{\mathbf{F}} = \mathbf{F}(\widehat{\mathbf{u}})$ and the right Cauchy-Green deformation tensor $\widehat{\mathbf{C}} = \mathbf{C}(\widehat{\mathbf{u}})$ by setting

$$\widehat{\mathbf{F}} = \mathbf{I} + \nabla \widehat{\mathbf{u}} \qquad , \qquad \widehat{\mathbf{C}} = \widehat{\mathbf{F}}^T \widehat{\mathbf{F}} \ , \tag{2}$$

where $\mathbf{I}$ is the second-order identity tensor and ∇ is the gradient operator with respect to the coordinates $\mathbf{X}$.

For a homogeneous neo-Hookean material we define (see for example [7,9]) the potential energy function as

$$\Psi(\widehat{\mathbf{u}}) = \frac{1}{2}\mu \left[\mathbf{I} : \widehat{\mathbf{C}} - d \right] - \mu \ln \widehat{J} + \frac{\lambda}{2} (\ln \widehat{J})^2 \ , \tag{3}$$

where λ and μ are positive constants, " : " represents the usual inner product for second-order tensors and $\widehat{J} = \det \widehat{\mathbf{F}}$ is the deformation gradient jacobian.

Introducing the pressure-like variable (or simply pressure) $\widehat{p} = \lambda \ln \widehat{J}$, the potential energy (3) can be equivalently written as the following function of $\widehat{\mathbf{u}}$ and $\widehat{p}$ (still denoted as Ψ with a little abuse of notation)

$$\Psi(\widehat{\mathbf{u}}, \widehat{p}) = \frac{1}{2}\mu \left[\mathbf{I} : \widehat{\mathbf{C}} - d \right] - \mu \ln \widehat{J} + \widehat{p} \ln \widehat{J} - \frac{1}{2\lambda} \widehat{p}^2 \ . \tag{4}$$

We wish to study problems whose Hellinger-Reissner functionals are of the form

$$\Pi(\widehat{\mathbf{u}}, \widehat{p}; \gamma) = \int_\Omega \Psi(\widehat{\mathbf{u}}, \widehat{p}) - \mathcal{F}(\widehat{\mathbf{u}}; \gamma) \ , \tag{5}$$

where $\mathcal{F}(\widehat{\mathbf{u}}; \gamma)$ represents the work of a family of external loads, smoothly depending on a real parameter γ. In the sequel, we will always suppose that $\mathcal{F}(\cdot; \gamma)$ is a *linear* functional for every choice of γ.

According with the Hellinger-Reissner variational principle, equilibrium is derived by searching for critical points of (5) in suitable admissible displacement and pressure spaces $\widehat{U}$ and $\widehat{P}$. The corresponding Euler-Lagrange equations emanating from (5) lead to solve

$$\begin{cases} \text{Find } (\widehat{\mathbf{u}}, \widehat{p}) \in \widehat{U} \times \widehat{P} \text{ such that} \\ \\ \mu \displaystyle\int_\Omega \widehat{\mathbf{F}} : \nabla \mathbf{v} + \int_\Omega (\widehat{p} - \mu)\, \widehat{\mathbf{F}}^{-T} : \nabla \mathbf{v} = \mathcal{F}(\mathbf{v}; \gamma) \qquad \forall \mathbf{v} \in U \\ \\ \displaystyle\int_\Omega \left(\ln \widehat{J} - \frac{\widehat{p}}{\lambda} \right) q = 0 \qquad \forall q \in P \ , \end{cases} \tag{6}$$

where U and P are the admissible variation spaces for the displacements and the pressures, respectively. We note that a solution $(\widehat{\mathbf{u}}, \widehat{p}) \in \widehat{U} \times \widehat{P}$ may obviously depend on the parameter γ, though not explicitly written, for notational simplicity. We also note that in (6) we used that the linearization of the deformation gradient jacobian is

$$DJ(\widehat{\mathbf{u}})[\mathbf{v}] = J(\widehat{\mathbf{u}}) \mathbf{F}(\widehat{\mathbf{u}})^{-T} : \nabla \mathbf{v} = \widehat{J}\, \widehat{\mathbf{F}}^{-T} : \nabla \mathbf{v} \qquad \forall \mathbf{v} \in U \ . \tag{7}$$

We focus on the case of an *incompressible* material, which corresponds to take the limit $\lambda \to +\infty$ in (6). Hence, our problem becomes:

$$\begin{cases} \textit{Problem } (\mathcal{P}_\gamma)\textit{:} \ \text{Find } (\widehat{\mathbf{u}}, \widehat{p}) \in \widehat{U} \times \widehat{P} \text{ such that} \\ \\ \mu \displaystyle\int_\Omega \widehat{\mathbf{F}} : \nabla \mathbf{v} + \int_\Omega (\widehat{p} - \mu) \widehat{\mathbf{F}}^{-T} : \nabla \mathbf{v} = \mathcal{F}(\mathbf{v}; \gamma) \qquad \forall \mathbf{v} \in U \\ \\ \displaystyle\int_\Omega q \, \ln \widehat{J} = 0 \qquad \forall q \in P \ . \end{cases} \tag{8}$$

We now derive the linearization of problem (8) around a solution $(\widehat{\mathbf{u}}, \widehat{p})$, corresponding to an infinitesimal increment $d\gamma$ of γ. Observing that

$$D\widehat{\mathbf{F}}^{-T}(\widehat{\mathbf{u}})[\mathbf{u}] = -\widehat{\mathbf{F}}^{-T} (\nabla \mathbf{u})^T \widehat{\mathbf{F}}^{-T} \qquad \forall \mathbf{u} \in U \ , \tag{9}$$

we easily get the problem for the *infinitesimal* increment $(\mathbf{u}, p)$

$$
\begin{cases}
\textit{Problem } (\mathcal{LP}_\gamma)\textit{:} \text{ Find } (\mathbf{u}, p) \in U \times P \text{ such that} \\[2ex]
\mu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} + \int_\Omega (\mu - \widehat{p})(\widehat{\mathbf{F}}^{-1} \nabla \mathbf{u})^T : \widehat{\mathbf{F}}^{-1} \nabla \mathbf{v} \\[2ex]
\qquad + \int_\Omega p \widehat{\mathbf{F}}^{-T} : \nabla \mathbf{v} = \dfrac{\partial}{\partial \gamma} \mathcal{F}(\mathbf{v}; \gamma)\, d\gamma \qquad \forall \mathbf{v} \in U \\[2ex]
\int_\Omega q \widehat{\mathbf{F}}^{-T} : \nabla \mathbf{u} = 0 \qquad \forall q \in P \ ,
\end{cases}
\tag{10}
$$

where the functional $\frac{\partial}{\partial \gamma} \mathcal{F}(\cdot\, ; \gamma) : U \to \mathcal{R}$ is defined by

$$
\frac{\partial}{\partial \gamma} \mathcal{F}(\mathbf{v}; \gamma) := \lim_{\Delta\gamma \to 0} \frac{\mathcal{F}(\mathbf{v}; \gamma + \Delta\gamma) - \mathcal{F}(\mathbf{v}; \gamma)}{\Delta\gamma} \qquad \forall \mathbf{v} \in U \ . \tag{11}
$$

Setting

$$
\begin{cases}
a_\gamma(\mathbf{u}, \mathbf{v}) := \mu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} + \int_\Omega (\mu - \widehat{p})(\widehat{\mathbf{F}}^{-1} \nabla \mathbf{u})^T : \widehat{\mathbf{F}}^{-1} \nabla \mathbf{v} \qquad \forall \mathbf{u}\,, \mathbf{v} \in U \\[2ex]
b_\gamma(\mathbf{v}, q) := \int_\Omega q \widehat{\mathbf{F}}^{-T} : \nabla \mathbf{v} \qquad \forall \mathbf{v} \in U \ , \ \forall q \in P \ ,
\end{cases}
\tag{12}
$$

Problem (10) can be written as

$$
\begin{cases}
\textit{Problem } (\mathcal{LP}_\gamma)\textit{:} \text{ Find } (\mathbf{u}, p) \in U \times P \text{ such that} \\[2ex]
a_\gamma(\mathbf{u}, \mathbf{v}) + b_\gamma(\mathbf{v}, p) = \dfrac{\partial}{\partial \gamma} \mathcal{F}(\mathbf{v}; \gamma)\, d\gamma \qquad \forall \mathbf{v} \in U \\[2ex]
b_\gamma(\mathbf{u}, q) = 0 \qquad \forall q \in P \ .
\end{cases}
\tag{13}
$$

We remark that the forms $a_\gamma(\cdot, \cdot)$ and $b_\gamma(\cdot, \cdot)$ are dependent on γ, since $\widehat{\mathbf{F}}$ and $\widehat{p}$ are so. We also note that problem (13) is a typical (parameter-dependent) saddle-point problem in variational form, where the second equation represents a kinematical constraint for the infinitesimal displacement increment $\mathbf{u}$.

2.1 *A "stream function" formulation for* Problem $(\mathcal{LP}_\gamma)$

We now present a reformulation of *Problem* $(\mathcal{LP}_\gamma)$ for which the infinitesimal displacement increment $\mathbf{u}$ automatically satisfies the constraint given by the second equation of (13). We here suppose that Ω is a simply connected domain of $\mathcal{R}^2$. For some hints on possible extensions to three-dimensional and multiply connected cases, see [4]. We first note that the Piola identity $\mathbf{div}(\widehat{J}\widehat{\mathbf{F}}^{-T}) = \mathbf{0}$ and $\widehat{J} = 1$ give $\mathbf{div}(\widehat{\mathbf{F}}^{-T}) = \mathbf{0}$. Hence, from the definition of $b_\gamma(\cdot,\cdot)$ in (12) it follows

$$\mathrm{div}(\widehat{\mathbf{F}}^{-1}\mathbf{v}) = \mathbf{div}(\widehat{\mathbf{F}}^{-T}) \cdot \mathbf{v} + \widehat{\mathbf{F}}^{-T} : \nabla \mathbf{v} = 0 \qquad \forall \mathbf{v} \in K_\gamma \ , \tag{14}$$

where

$$K_\gamma := \{\mathbf{v} \in U \ : \ b_\gamma(\mathbf{v}, q) = 0 \qquad \forall q \in P\} \ . \tag{15}$$

Therefore, given $\mathbf{v} \in K_\gamma$, there exists a "stream function" ψ, defined up to an additive constant, such that

$$\widehat{\mathbf{F}}^{-1}\mathbf{v} = \mathbf{curl}\,\psi \qquad \text{i.e.} \qquad \mathbf{v} = \widehat{\mathbf{F}}\,\mathbf{curl}\,\psi \ . \tag{16}$$

We set the *stream function space* as

$$\Phi_\gamma = \left\{\psi \ : \ \widehat{\mathbf{F}}\,\mathbf{curl}\,\psi \in U\right\} / \mathcal{R} \ , \tag{17}$$

where the index highlights the dependence of the space definition on the parameter γ (through $\widehat{\mathbf{F}}$).

Introducing the notation

$$\mathbf{curl}_{\widehat{\mathbf{F}}}\,\psi := \widehat{\mathbf{F}}\,\mathbf{curl}\,\psi \ , \tag{18}$$

Problem $(\mathcal{LP}_\gamma)$ (see (10)) can thus be written as

$$\begin{cases} \textit{Problem } (\mathcal{LP}^*_\gamma)\textit{:} \text{ Find } \varphi \in \Phi_\gamma \text{ such that} \\ \\ a^*_\gamma(\varphi, \psi) = \dfrac{\partial}{\partial \gamma}\mathcal{F}(\mathbf{curl}_{\widehat{\mathbf{F}}}\,\psi; \gamma)\, d\gamma \qquad \forall \psi \in \Phi_\gamma \ , \end{cases} \tag{19}$$

where

$$a^*_\gamma(\varphi,\psi) := \mu \int_\Omega \nabla(\mathbf{curl}_{\widehat{\mathbf{F}}}\,\varphi) : \nabla(\mathbf{curl}_{\widehat{\mathbf{F}}}\,\psi) + \int_\Omega (\mu - \widehat{p})(\widehat{\mathbf{F}}^{-1}\nabla(\mathbf{curl}_{\widehat{\mathbf{F}}}\,\varphi))^T : \widehat{\mathbf{F}}^{-1}\nabla(\mathbf{curl}_{\widehat{\mathbf{F}}}\,\psi)\ . \tag{20}$$

We remark that *Problem* $(\mathcal{LP}^*_\gamma)$ above corresponds to a *fourth order PDE problem*. We also remark that, once $\varphi \in \Phi_\gamma$ has been found by solving (19), the infinitesimal displacement increment $\mathbf{u}$ can be obtained by computing (cf. (18))

$$\mathbf{u} = \mathbf{curl}_{\widehat{\mathbf{F}}}\,\varphi\ . \tag{21}$$

2.2 The stability range

We now make the assumption

$$\mathcal{F}(\mathbf{v};0) = 0 \qquad \forall \mathbf{v} \in U\ , \tag{22}$$

which means to consider a loading-free problem at $\gamma = 0$. Therefore, *Problem* $(\mathcal{P}_0)$ (see (8)) admits the trivial solution $(\widehat{\mathbf{u}},\widehat{p}) = (\mathbf{0},0)$. Furthermore, its linearization becomes (cf. (10)):

$$\begin{cases} \textit{Problem } (\mathcal{LP}_0)\textit{: } \text{Find } (\mathbf{u},p) \in U \times P \text{ such that} \\ 2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \int_\Omega p \ \mathrm{div}\,\mathbf{v} = \dfrac{\partial}{\partial\gamma}\mathcal{F}(\mathbf{v};0)\,d\gamma \qquad \forall \mathbf{v} \in U \\ \int_\Omega q \ \mathrm{div}\,\mathbf{u} = 0 \qquad \forall q \in P\ , \end{cases} \tag{23}$$

where $\boldsymbol{\varepsilon}(\cdot)$ denotes the symmetric gradient operator. It is well-known (see [8], for instance) that Problem (23) corresponds to a well-posed *coercive* constrained problem, the constraint being the divergence-free condition. As a consequence, the trivial solution $(\mathbf{0},0)$ is an *isolated stable* solution to *Problem* $(\mathcal{P}_0)$. More precisely, the point $(\widehat{\mathbf{u}},\widehat{p};\gamma) = (\mathbf{0},0;0)$ fits the following Definition.

Definition 1 *Let* $\gamma \in \mathcal{R}$*, and let* $(\widehat{\mathbf{u}},\widehat{p}) \in U \times P$ *be a solution to* Problem $(\mathcal{P}_\gamma)$ *(see* (8)*). We say that* $(\widehat{\mathbf{u}},\widehat{p};\gamma)$ *is a* linearization stable point *for* Problem $(\mathcal{P}_\gamma)$ *if the corresponding linearization* (13) *satisfies:*

- the inf-sup condition, *i.e. it holds*

$$\beta_\gamma := \inf_{q\in P}\sup_{\mathbf{v}\in U}\frac{b_\gamma(\mathbf{v},q)}{||\mathbf{v}||_U||q||_P} > 0 \; ; \tag{24}$$

- the coercivity on the kernel condition, *i.e. it holds*

$$\alpha_\gamma := \inf_{\mathbf{v}\in K_\gamma}\frac{a_\gamma(\mathbf{v},\mathbf{v})}{||\mathbf{v}||_U^2} > 0 \; . \qquad \square \tag{25}$$

We are now ready to introduce our definition of *stability range*.

Definition 2 *We define the* stability range *of* Problem $(\mathcal{P}_\gamma)$ *as the interval* $S(\mathcal{P}_\gamma) = (\gamma_m, \gamma_M) \subseteq \mathcal{R}$*, where*

$$\begin{cases} \gamma_m = \inf\Big\{\gamma \in \mathcal{R} : \; \exists \textit{ a continuous path of linearization stable points} \\ \qquad\qquad\qquad \textit{joining } (\mathbf{0},0;0) \textit{ and } (\widehat{\mathbf{u}},\widehat{p};\gamma)\Big\} \\ \\ \gamma_M = \sup\Big\{\gamma \in \mathcal{R} : \; \exists \textit{ a continuous path of linearization stable points} \\ \qquad\qquad\qquad \textit{joining } (\mathbf{0},0;0) \textit{ and } (\widehat{\mathbf{u}},\widehat{p};\gamma)\Big\} \; . \qquad \square \end{cases} \tag{26}$$

For $2D$ problems, we notice that using the *stream function* formulation (19)–(20), Definition 1 can be rephrased into the following single

- *coercivity condition*: it holds

$$\alpha_\gamma := \inf_{\psi\in\Phi_\gamma}\frac{a^*_\gamma(\psi,\psi)}{||\psi||^2_{\Phi_\gamma}} > 0 \; , \tag{27}$$

  where

$$||\psi||_{\Phi_\gamma} := ||\mathbf{v}||_U \qquad \text{if } \mathbf{v} = \widehat{\mathbf{F}}\,\mathbf{curl}\,\psi \text{ (cf. (16)).} \tag{28}$$

Accordingly, the *stability range* of *Problem* $(\mathcal{P}_\gamma)$ introduced by Definition 2 can be equivalently given in terms of the *stream function* formulation (19)–(20).

A typical stability range is schematically depicted in Figure 1. We remark that outside the stability range (γ_m, γ_M) we expect multiple solutions of the nonlinear problem (8). Therefore, $\gamma \to \alpha_\gamma$ is a multi-valued function outside (γ_m, γ_M), in general. However, in Figure 1 a single branch has been displayed, for simplicity.

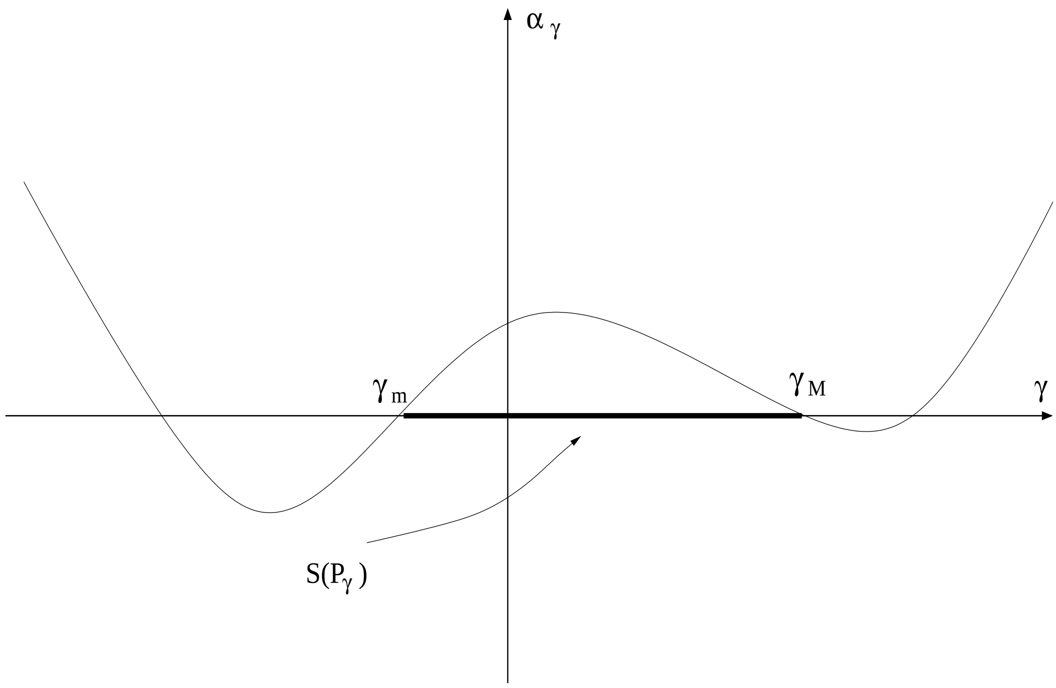

Fig. 1. Stability range.

3 Galerkin approximation of the stability range

In this Section we prove a few results about the numerical approximation of the stability range, as defined by Definitions 1 and 2 (cf. also (27)–(28)) when we discretize the continuous problem through a Galerkin approximation. In what follows we will exclusively focus on the stability properties of the discretization of the *linearized Problem* $(\mathcal{LP}_\gamma)$ (see (13)). Therefore, we will always suppose that the analytical solution $(\widehat{\mathbf{u}}, \widehat{p})$ of the non-linear Problem $(\mathcal{P}_\gamma)$, corresponding to γ (see (8)), is *exactly available*. Of course, this assumption is unrealistic in most practical situations, because only an approximation $(\widehat{\mathbf{u}}_h, \widehat{p}_h)$ of $(\widehat{\mathbf{u}}, \widehat{p})$ is generally at hand. However, it allows a simpler, though meaningful, study on possible discretization troubles.

3.1 Approximation by means of the Hellinger-Reissner formulation

We now consider a Galerkin approximation of the linearized problem (13). We thus select a family of finite dimensional subspaces $U_h \subset U$ and $P_h \subset P$ $(h > 0)$, with the basic approximation property:

$$\bigcup_{h>0} U_h \ \text{ is dense in } \ U \qquad , \qquad \bigcup_{h>0} P_h \ \text{ is dense in } \ P \,. \tag{29}$$

We then consider the following discrete problem.

$$\begin{cases} \textit{Problem } (\mathcal{LP}_{\gamma,h})\textit{:} \text{ Find } (\mathbf{u}_h, p_h) \in U_h \times P_h \text{ such that} \\ \\ a_\gamma(\mathbf{u}_h, \mathbf{v}_h) + b_\gamma(\mathbf{v}_h, p_h) = \dfrac{\partial}{\partial \gamma} \mathcal{F}(\mathbf{v}_h; \gamma)\, d\gamma \qquad \forall \mathbf{v} \in U_h \\ \\ b_\gamma(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in P_h \ . \end{cases} \tag{30}$$

Within this framework, we can now introduce the dicrete counterparts of Definitions 1 and 2.

Definition 3 *Let $\gamma \in \mathcal{R}$, and let $(\widehat{\mathbf{u}}, \widehat{p}) \in U \times P$ be a solution to* Problem $(\mathcal{P}_\gamma)$ *(see* (8)*). For a given $h > 0$, we say that $(\widehat{\mathbf{u}}, \widehat{p}; \gamma)$ is a* discrete linearization stable point *for* Problem $(\mathcal{P}_\gamma)$ *if the corresponding discrete linearization* (30) *satisfies:*

- the discrete inf-sup condition*, i.e. it holds*

$$\inf_{q_h \in P_h} \sup_{\mathbf{v}_h \in U_h} \frac{b_\gamma(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_U ||q_h||_P} > 0 \ ; \tag{31}$$

- the discrete coercivity on the kernel condition*, i.e. it holds*

$$\inf_{\mathbf{v}_h \in K_{\gamma,h}} \frac{a_\gamma(\mathbf{v}_h, \mathbf{v}_h)}{||\mathbf{v}_h||_U^2} > 0 \ , \tag{32}$$

where the discrete kernel $K_{\gamma,h}$ is defined by (cf. (15)*)*

$$K_{\gamma,h} := \{\mathbf{v}_h \in U_h \ : \ b_\gamma(\mathbf{v}_h, q_h) = 0 \qquad \forall q_h \in P_h\} \ . \tag{33}$$

Definition 4 *For a given $h > 0$, we define the* discrete stability range *of* Problem $(\mathcal{P}_\gamma)$ *as the interval $S_h(\mathcal{P}_\gamma) = (\gamma_{m,h} \, , \, \gamma_{M,h}) \subseteq \mathcal{R}$, where*

$$\begin{cases} \gamma_{m,h} = \inf \Big\{ \gamma \in \mathcal{R} : \ \exists \textit{ a continuous path of discrete linearization stable} \\ \qquad\qquad \textit{points joining } (\mathbf{0}, 0; 0) \textit{ and } (\widehat{\mathbf{u}}, \widehat{p}; \gamma) \Big\} \\ \\ \gamma_{M,h} = \sup \Big\{ \gamma \in \mathcal{R} : \ \exists \textit{ a continuous path of discrete linearization stable} \\ \qquad\qquad \textit{points joining } (\mathbf{0}, 0; 0) \textit{ and } (\widehat{\mathbf{u}}, \widehat{p}; \gamma) \Big\} \ . \qquad \square \end{cases} \tag{34}$$

We will always suppose that the spaces U_h and P_h satisfy a condition even stronger than the *discrete inf-sup condition* (31). More precisely, we assume that the following *h-uniform discrete inf-sup condition* holds, for every γ:

$$\inf_{h>0} \left\{ \inf_{q_h \in P_h} \sup_{\mathbf{v}_h \in U_h} \frac{b_\gamma(\mathbf{v}_h, q_h)}{||\mathbf{v}_h||_U ||q_h||_P} \right\} > 0 \ . \tag{35}$$

This assumption is very reasonable, since for every γ it essentially corresponds to a classical *inf-sup* condition in the deformed configuration (see [16] for details). Therefore, a crucial role to determine the *discrete stability range* $S_h(\mathcal{P}_\gamma)$ will be played by the *h-uniform discrete coercivity on the kernel condition.* With this respect, we can now prove the following proposition.

Proposition 1 *Let* $(\widehat{\mathbf{u}}, \widehat{p}) \in \widehat{U} \times \widehat{P}$ *be a solution of Problem* (8)*, for a given* γ*. Suppose that the corresponding bilinear form* $a_\gamma(\cdot,\cdot)$ *is* indefinite *on* K_γ*, i.e. (cf.* (25)*)*

$$\alpha_\gamma := \inf_{\mathbf{v} \in K_\gamma} \frac{a_\gamma(\mathbf{v}, \mathbf{v})}{||\mathbf{v}||_U^2} < 0 \ . \tag{36}$$

Then for the discrete counterpart (30) *it holds*

$$\inf_{h>0} \left\{ \inf_{\mathbf{v}_h \in K_{\gamma,h}} \frac{a_\gamma(\mathbf{v}_h, \mathbf{v}_h)}{||\mathbf{v}_h||_U^2} \right\} < 0 \ , \tag{37}$$

where the discrete kernel $K_{\gamma,h}$ *is defined by* (33)*.*

Proof. By (36) there exists $\tilde{\mathbf{v}} \in K_\gamma$ such that

$$\frac{a_\gamma(\tilde{\mathbf{v}}, \tilde{\mathbf{v}})}{||\tilde{\mathbf{v}}||_U^2} < 0 \ . \tag{38}$$

The h-uniform discrete *inf-sup* condition (35) implies (see [8]):

$$\inf_{\mathbf{v}_h \in K_{\gamma,h}} ||\mathbf{v} - \mathbf{v}_h||_U \leq C_\gamma \inf_{\mathbf{v}_h \in U_h} ||\mathbf{v} - \mathbf{v}_h||_U \qquad \forall \mathbf{v} \in K_\gamma \ . \tag{39}$$

Therefore, from (39) and (29) there exists $\tilde{\mathbf{v}}_h \in K_{\gamma,h}$ such that

$$\tilde{\mathbf{v}}_h \rightarrow \tilde{\mathbf{v}} \qquad \text{in } U, \text{ as } h \rightarrow 0 \ . \tag{40}$$

Hence it holds, as $h \rightarrow 0$:

$$a_\gamma(\tilde{\mathbf{v}}_h, \tilde{\mathbf{v}}_h) \rightarrow a_\gamma(\tilde{\mathbf{v}}, \tilde{\mathbf{v}}) \qquad \text{and} \qquad ||\tilde{\mathbf{v}}_h||_U^2 \rightarrow ||\tilde{\mathbf{v}}||_U^2 \ . \tag{41}$$

Estimate (37) now follows from (38) and (41). □

Remark 2 *Proposition 1 reveals that the following pathology may occur when a conforming approximation based on the Hellinger-Reissner principle is used. It may happen that a discretization procedure detects a stability range which is strictly contained in the stability range for the continuous problem (cf. Definitions 1 and 2), even asymptotically as* $h \to 0$*. This means that the discretized problem might get unstable "too early" (see Figure 2). Instances of such cases are presented in Section 5. However, we remark that in very particular situations the numerical scheme could detect a larger stability range. This might happen only if the function* $\gamma \to \alpha_\gamma$ *exhibits a zero plateau region at the boundary of the stability range* $S(\mathcal{P}_\gamma)$*, as depicted in Figure 3. We note that a zero plateau region may mechanically correspond to a region where the body is placed in* indifferent equilibrium *positions. Finally, we also remark that Proposition 1 does not imply that the stability range approximation* always *fail: situations showing an accurate discrete stability range might occur.*

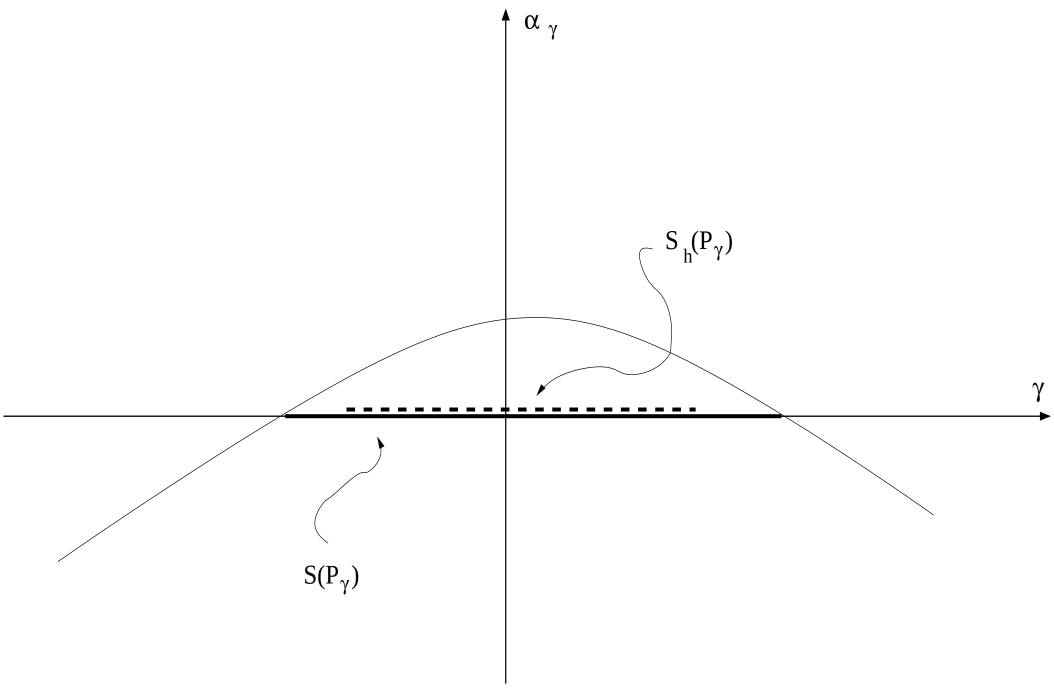

Fig. 2. A first possible pathology of the stability range for the Hellinger-Reissner discretization with h "small" (dashed line). The stability range for the continuous problem is also displayed (solid line).

3.2 Approximation by means of the stream function formulation

We next consider a Galerkin approximation of Problem (19). We thus select a family of finite dimensional subspaces $\Phi_{\gamma,h} \subset \Phi_\gamma$ ($h > 0$), with the basic approximation property:

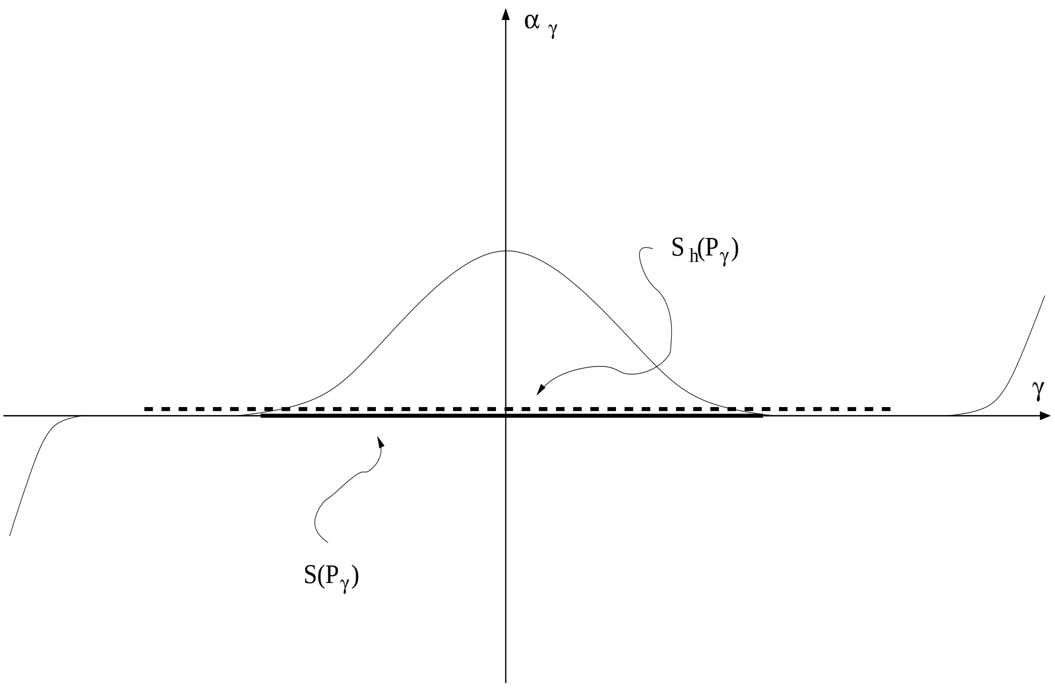

Fig. 3. A second possible pathology of the stability range for the Hellinger-Reissner discretization with h "small" (dashed line). The stability range for the continuous problem is also displayed (solid line).

$$\bigcup_{h>0} \Phi_{\gamma,h} \quad \text{is dense in} \quad \Phi_\gamma \ . \tag{42}$$

We then consider the following discrete problem.

$$\begin{cases} \textit{Problem } (\mathcal{LP}^*_{\gamma,h})\text{: Find } \varphi_h \in \Phi_{\gamma,h} \text{ such that} \\ \\ a^*_\gamma(\varphi_h, \psi_h) = \dfrac{\partial}{\partial \gamma} \mathcal{F}(\mathbf{curl}_{\widehat{\mathbf{F}}}\, \psi_h; \gamma)\, d\gamma \qquad \forall \psi_h \in \Phi_{\gamma,h} \ . \end{cases} \tag{43}$$

Once $\varphi_h \in \Phi_{\gamma,h}$ has been found by solving (43), the displacement infinitesimal increment $\mathbf{u}_h$ can be obtained by computing

$$\mathbf{u}_h = \mathbf{curl}_{\widehat{\mathbf{F}}}\, \varphi_h \ . \tag{44}$$

Within this framework, the discrete counterparts of Definitions 1 and 2 becomes (cf. (27))

Definition 5 *Let $\gamma \in \mathcal{R}$, and let $(\widehat{\mathbf{u}}, \widehat{p}) \in U \times P$ be a solution to* Problem $(\mathcal{P}_\gamma)$ *(see* (8)*). For a given $h > 0$, we say that $(\widehat{\mathbf{u}}, \widehat{p}; \gamma)$ is a* discrete linearization stable point *for* Problem $(\mathcal{P}_\gamma)$ *if the corresponding discrete linearization* (43) *satisfies:*

- the discrete coercivity on the kernel condition*, i.e. it holds*

$$\inf_{\psi_h \in \Phi_{\gamma,h}} \frac{a_\gamma^*(\psi_h, \psi_h)}{||\psi_h||^2_{\Phi_\gamma}} > 0 \ . \tag{45}$$

Definition 6 *For a given* $h > 0$*, we define the* discrete stability range *of* Problem $(\mathcal{P}_\gamma)$ *as the interval* $S_h^*(\mathcal{P}_\gamma) = (\gamma^*_{m,h}\, ,\, \gamma^*_{M,h}) \subseteq \mathcal{R}$*, where*

$$\begin{cases} \gamma^*_{m,h} = \inf \Big\{ \gamma \in \mathcal{R} : \ \exists \textit{ a continuous path of discrete linearization stable} \\ \qquad\qquad \textit{points joining } (\mathbf{0}, 0; 0) \textit{ and } (\widehat{\mathbf{u}}, \widehat{p}; \gamma) \Big\} \\ \\ \gamma^*_{M,h} = \sup \Big\{ \gamma \in \mathcal{R} : \ \exists \textit{ a continuous path of discrete linearization stable} \\ \qquad\qquad \textit{points joining } (\mathbf{0}, 0; 0) \textit{ and } (\widehat{\mathbf{u}}, \widehat{p}; \gamma) \Big\} \ . \qquad \Box \end{cases} \tag{46}$$

Proposition 3 *Let* $(\widehat{\mathbf{u}}, \widehat{p}) \in \widehat{U} \times \widehat{P}$ *be a solution of Problem* (8)*, for a given* γ*. Consider the corresponding* stream function *formulation* (19) *and its discrete counterpart* (43)*. Then it holds (cf. also* (27)*)*

$$\alpha_\gamma := \inf_{\psi \in \Phi_\gamma} \frac{a_\gamma^*(\psi, \psi)}{||\psi||^2_{\Phi_\gamma}} = \inf_{h>0} \left\{ \inf_{\psi_h \in \Phi_{\gamma,h}} \frac{a_\gamma^*(\psi_h, \psi_h)}{||\psi_h||^2_{\Phi_\gamma}} \right\} \ . \tag{47}$$

Proof. We first set

$$\alpha_\gamma(h) := \inf_{\psi_h \in \Phi_{\gamma,h}} \frac{a_\gamma^*(\psi_h, \psi_h)}{||\psi_h||^2_{\Phi_\gamma}} \ . \tag{48}$$

Since $\Phi_{\gamma,h} \subset \Phi_\gamma$ for every $h > 0$, we have

$$\alpha_\gamma := \inf_{\psi \in \Phi_\gamma} \frac{a_\gamma^*(\psi, \psi)}{||\psi||^2_{\Phi_\gamma}} \leq \inf_{h>0} \{\alpha_\gamma(h)\} \ . \tag{49}$$

Suppose now that it holds

$$\alpha_\gamma < \inf_{h>0} \{\alpha_\gamma(h)\} \ . \tag{50}$$

Hence, there exists $\varepsilon > 0$ and a $\tilde{\psi} \in \Phi_\gamma$ (depending on ε) such that

$$\alpha_\gamma \le \frac{a^*_\gamma(\tilde\psi,\tilde\psi)}{||\tilde\psi||^2_{\Phi_\gamma}} < \alpha_\gamma + \varepsilon < \inf_{h>0}\{\alpha_\gamma(h)\} \ . \tag{51}$$

We now choose $\tilde\psi_h \in \Phi_{\gamma,h}$ as the best approximation of $\tilde\psi$ with respect to the Φ_γ-norm. On the one hand, from (51) we have

$$\alpha_\gamma \le \frac{a^*_\gamma(\tilde\psi,\tilde\psi)}{||\tilde\psi||^2_{\Phi_\gamma}} < \alpha_\gamma + \varepsilon < \inf_{h>0}\{\alpha_\gamma(h)\} \le \frac{a^*_\gamma(\tilde\psi_h,\tilde\psi_h)}{||\tilde\psi_h||^2_{\Phi_\gamma}} \ . \tag{52}$$

On the other hand, since $\tilde\psi_h \to \tilde\psi$ in Φ_γ as $h \to 0$ (cf. (42)), it holds

$$\lim_{h\to 0} \frac{a^*_\gamma(\tilde\psi_h,\tilde\psi_h)}{||\tilde\psi_h||^2_{\Phi_\gamma}} = \frac{a^*_\gamma(\tilde\psi,\tilde\psi)}{||\tilde\psi||^2_{\Phi_\gamma}} \ . \tag{53}$$

A combination of (52) and (53) yields

$$\frac{a^*_\gamma(\tilde\psi,\tilde\psi)}{||\tilde\psi||^2_{\Phi_\gamma}} < \alpha_\gamma + \varepsilon < \frac{a^*_\gamma(\tilde\psi,\tilde\psi)}{||\tilde\psi||^2_{\Phi_\gamma}} \ , \tag{54}$$

which is clearly a contradiction. □

Remark 4 *Proposition 3 reveals that any reasonable conforming approximation of the linearized problem based on the "stream function" formulation is capable to correctly detect the stability range $S(\mathcal{P}_\gamma)$, at least asymptotically as $h \to 0$ (see Figure 4). Indeed, Proposition 3 implies*

$$S(\mathcal{P}_\gamma) = \bigcap_{h>0} S^*_h(\mathcal{P}_\gamma) \ . \tag{55}$$

4 Two simple examples

In this Section we present two *simple* problems which will be used in Section 5 to discuss on the performance of some numerical schemes in approximating the stability range $S(\mathcal{P}_\gamma)$. Using the usual Cartesian coordinates (X,Y), we consider a square material body whose reference configuration is $\Omega = (-1,1)\times(-1,1)$; we denote with $\Gamma = [-1,1]\times\{1\}$ the upper part of its boundary, while the remaining part of $\partial\Omega$ is denoted with Γ_D. The total energy is assumed to be as in (4), where the external loads are given by the vertical uniform body forces:

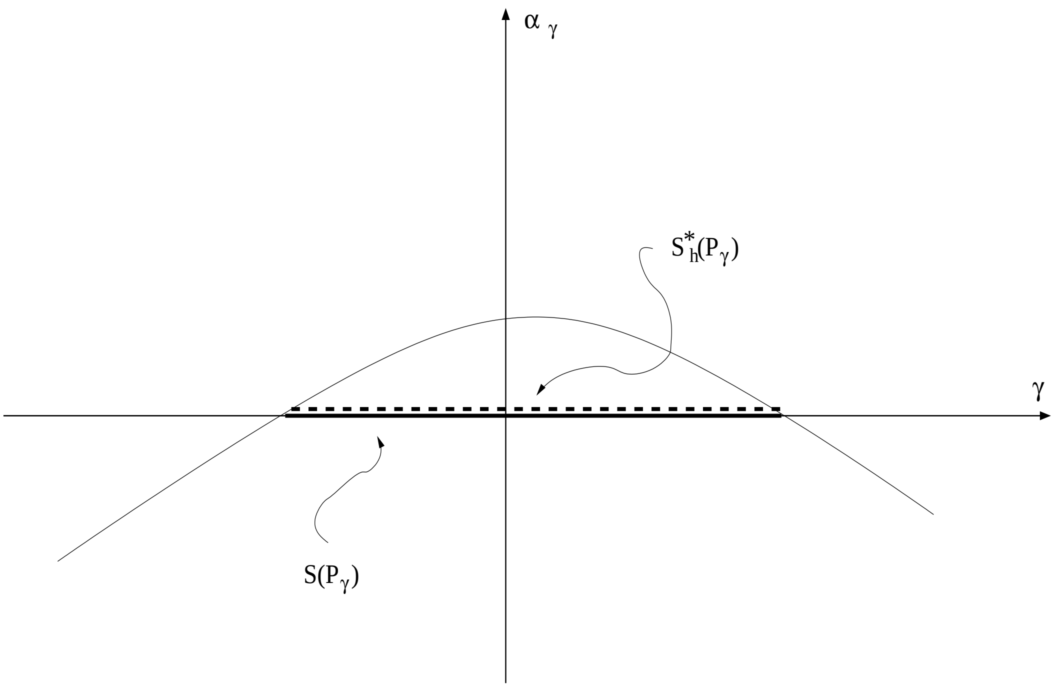

Fig. 4. Stability range for the stream-based discretization with h "small" (dashed line). The stability range for the continuous problem is also displayed (solid line).

$$\mathcal{F}(\mathbf{v};\gamma) = \gamma \int_\Omega \mathbf{f} \cdot \mathbf{v} \ , \tag{56}$$

with $\mathbf{f} = (0,1)^T$. The two problems differ for the imposed boundary conditions. More precisely:

- **Problem 1.** We set clamped boundary conditions on Γ_D, traction-free boundary conditions on Γ (cf. Fig. 5(left)).
- **Problem 2.** We set vanishing normal displacements on Γ_D, traction-free boundary conditions on Γ (cf. Fig. 5(right)).

It is easy to see that both the problems admit a trivial solution *for every* $\gamma \in \mathbf{R}$, i.e. $(\widehat{\mathbf{u}}, \widehat{p}) = (\mathbf{0}, \gamma r)$, where $r = r(X,Y) = 1 - Y$.

For the problems under investigation, the corresponding linearized problems (cf. (10)) may both be written as

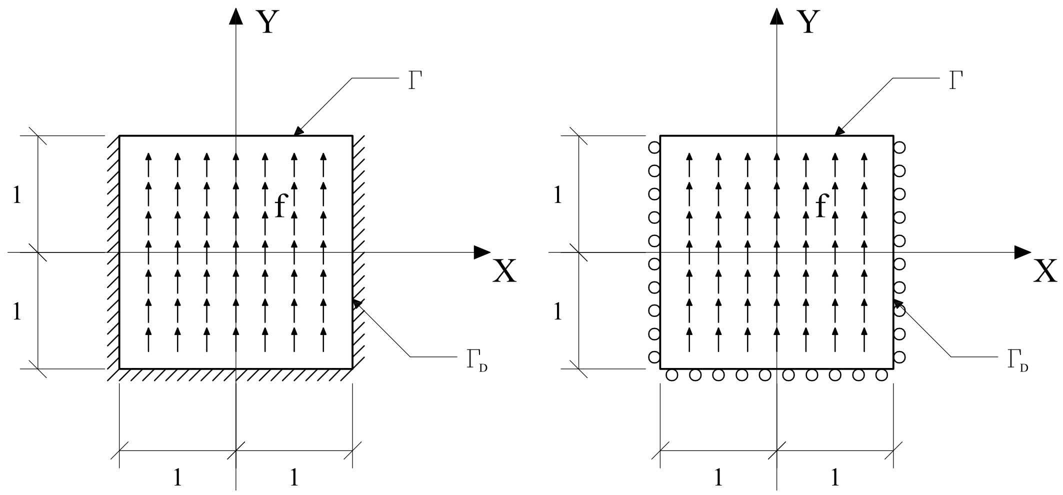

Fig. 5. Problem domain Ω.

$$\begin{cases} \textit{Problem } (\mathfrak{LP}_\gamma)\textit{:} \text{ Find } (\mathbf{u},p) \in U \times P \text{ such that} \\[2ex] 2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \gamma \int_\Omega r(\nabla \mathbf{u})^T : \nabla \mathbf{v} + \int_\Omega p \,\mathrm{div}\, \mathbf{v} = d\gamma \int_\Omega \mathbf{f} \cdot \mathbf{v} \\[2ex] \int_\Omega q \,\mathrm{div}\, \mathbf{u} = 0 \ , \end{cases} \tag{57}$$

for every $(\mathbf{v}, q) \in U \times P$. Accordingly with the two different problems, the spaces U and P are the following:

- **Problem 1.**

$$U = \left\{ \mathbf{v} \in H^1(\Omega)^2 \ : \ \mathbf{v}_{|\Gamma_D} = 0 \right\} \ ; \qquad P = L^2(\Omega) \ . \tag{58}$$

- **Problem 2.**

$$U = \left\{ \mathbf{v} \in H^1(\Omega)^2 \ : \ (\mathbf{v} \cdot \mathbf{n})_{|\Gamma_D} = 0 \right\} \ ; \qquad P = L^2(\Omega) \ , \tag{59}$$

  where $\mathbf{n}$ denotes the outward normal vector.

The "stream function" formulation reads as follows (see (19))

$$\begin{cases} \textit{Problem } (\mathfrak{LP}^*_\gamma)\textit{:} \text{ Find } \varphi \in \Phi \text{ such that} \\[2ex] a^*_\gamma(\varphi, \psi) = d\gamma \int_\Omega \mathbf{f} \cdot \mathbf{curl}\, \psi \qquad \forall \psi \in \Phi \ , \end{cases} \tag{60}$$

where

$$a^*_\gamma(\varphi,\psi) := 2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{curl}\,\varphi) : \boldsymbol{\varepsilon}(\mathbf{curl}\,\psi) - \gamma \int_\Omega r(\nabla(\mathbf{curl}\,\varphi))^T : \nabla(\mathbf{curl}\,\psi) \ . \tag{61}$$

Accordingly with the two different problems, the space Φ is the following:

- **Problem 1.**

$$\Phi = \left\{ \psi \in H^2(\Omega) \ : \ \psi = \frac{\partial \psi}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_D \right\} \ . \tag{62}$$

- **Problem 2.**

$$\Phi = \left\{ \psi \in H^2(\Omega) \ : \ \psi = 0 \text{ on } \Gamma_D \right\} \ . \tag{63}$$

Remark 5 *For Problem 1, it has been theoretically proved in [3] that the corresponding stability range $S(\mathcal{P}_\gamma)$ satisfies $(-\infty, 3\mu) \subseteq S(\mathcal{P}_\gamma)$.*

5 Numerical Results

In this Section we report our numerical computations about the approximation of the stability range for the two problems detailed in Section 4. We first present the stream formulation results, since they provide an accurate approximation of the continuum stability range (cf. Proposition 3) and, therefore, they can be used as reliable reference solutions for the schemes based on the Hellinger-Reissner formulation.

5.1 NURBS for the stream function formulation

We assume that the physical domain Ω is a NURBS (see, e.g., [23,24]) region associated with a $n \times m$ net of control points $\mathbf{B}_{i,j}$, and we introduce the geometrical map $\mathbf{F} : [0,1] \times [0,1] \to \overline{\Omega}$ given by

$$\mathbf{F}(\xi,\eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R^{p,q}_{i,j}(\xi,\eta) \mathbf{B}_{i,j} \ , \tag{64}$$

where $R^{p,q}_{i,j}(\xi,\eta)$ are NURBS basis functions on the two-dimensional parametric space $[0,1] \times [0,1]$. The image of the elements in the parametric spaces are elements in the physical spaces. The physical mesh is therefore

$$\mathcal{T}_h = \{ \mathbf{F}((\xi_i, \xi_{i+1}) \times (\eta_j, \eta_{j+1})), \text{ with } i = 1, \ldots, n+p, \ j = 1, \ldots, m+q \} \ . \tag{65}$$

We denote by h the mesh-size, that is, the maximum diameter of the elements of $\mathcal{T}_h$.

Following the isoparametric approach, the space of NURBS functions on Ω is defined as the span of the *push-forward* of the basis functions $R^{p,q}_{i,j}(\xi, \eta)$

$$\mathcal{V}_h = \text{span}\{R^{p,q}_{i,j} \circ \mathbf{F}^{-1}\}_{i=1,\ldots,n;j=1,\ldots,m}. \tag{66}$$

This analysis framework is referred to as NURBS-based *isogeometric analysis*. The interested reader may find details and applications of isogeometric analysis for example in [4,6,10,11,14,15]; in particular, the approximation result (42) follows directly from the results of [6].

We introduce now the NURBS conforming discretization of the variational problem (60). Let $\mathcal{V}_h$ denote a NURBS space as introduced in (66), of regularity C^k, $k \geq 1$, and degree $p > k$. We have $\mathcal{V}_h \subset H^2(\Omega)$, and then we can introduce the following conforming discretizations of Φ

$$\Phi_h = \mathcal{V}_h \cap \Phi. \tag{67}$$

5.1.1 *NURBS approximation: numerical results*

We are now able to use the NURBS-based discretization introduced above in order to study the two model problems of Section 4, sketched in Figure 5, and find sharp estimates for the continuum stability range $S(\mathcal{P}_\gamma)$ (cf. Proposition 3).

We highlight that to find critical loads we study the eigenvalues of the (loading parameter-dependent) stiffness matrix for the NURBS-based stream formulation described by (60)–(63) for the problems under consideration. Indeed, the quantity (cf. (45))

$$\inf_{\psi_h \in \Phi_{\gamma,h}} \frac{a^*_\gamma(\psi_h, \psi_h)}{||\psi_h||^2_{\Phi_\gamma}} \tag{68}$$

is a kind of Rayligh quotient, whose sign is determined by the sign of the smallest eigenvalue of the matrix arising from the bilinear form $a^*_\gamma(\cdot, \cdot)$. The first loads for which we find a negative eigenvalue are the critical ones and, accordingly with Definition 6, they are indicated as $\gamma^*_{m,h}$ and $\gamma^*_{M,h}$, respectively.

Table 1 and Table 2 report the results we found for the two problems. For different approximation degrees (and, precisely, for $p = q = 2, 3, ..., 6$, being p and q the degrees in the two parametric directions) we used meshes of 4×4 to 32×32 elements. Due to the structure of the formulation (see [4]), this gives rise to $(n+p) \times (m+q)$ degrees-of-freedom (d.o.f.'s) for a mesh of $n \times m$ elements, so the total number of d.o.f.'s actually employed depends also on the approximation degrees (and is explicitly reported in the tables between brackets).

We moreover remark that, here and in the following, we express the numerical results in terms of the nondimensional quantity $\tilde{\gamma}^*_{m,h}$ (resp. $\tilde{\gamma}^*_{M,h}$) defined as $\tilde{\gamma}^*_{m,h} = \gamma^*_{m,h} L/\mu$ (resp. $\tilde{\gamma}^*_{M,h} = \gamma^*_{M,h} L/\mu$). Here, L is some problem characteristic length, set equal to 1 for simplicity, consistently also with the geometry of the model problems.

In Table 1 and Table 2, it is possible to observe excellent convergence paths both in h (i.e., mesh-refining) and in p (i.e., order-elevating). Therefore, recalling Remark 4, we can consider reliable the finest results obtained (i.e., for the case of degree $p = q = 6$ over a 32×32 mesh, corresponding to 38×38 d.o.f.'s), which are

- $\tilde{\gamma}^*_{M,h} = 6.60$ for **Problem 1**;
- $\tilde{\gamma}^*_{M,h} = 3.23$ for **Problem 2**.

In the following we indicate such results as "exact" (between quotes), referring to the fact that the stream function critical value converges to the one for the continuum problem, i.e. the exact one, as proved in Section 3.2.

We finally remark that we did not find any negative critical value (i.e., $\tilde{\gamma}^*_{m,h} = -\infty$), so, in the tables, we report only positive critical load values, $\tilde{\gamma}^*_{M,h}$. For Problem 1, this is in accordance with the theoretical result detailed in [3] (cf. also Remark 5).

	$p = q$				
elements	2	3	4	5	6
4×4	7.96 (6×6)	6.88 (7×7)	6.71 (8×8)	6.65 (9×9)	6.63 (10×10)
8×8	7.08 (10×10)	6.67 (11×11)	6.63 (12×12)	6.61 (13×13)	6.61 (14×14)
16×16	6.75 (18×18)	6.62 (19×19)	6.61 (20×20)	6.61 (21×21)	6.61 (22×22)
32×32	6.65 (34×34)	6.61 (35×35)	6.60 (36×36)	6.60 (37×37)	6.60 (38×38)

Table 1
Problem 1: critical load value $\tilde{\gamma}^*_{M,h}$ computed using the NURBS-based stream function formulation. The total number of employed d.o.f.'s is reported between brackets.

	$p = q$				
elements	2	3	4	5	6
4×4	3.41 (6×6)	3.24 (7×7)	3.23 (8×8)	3.23 (9×9)	3.23 (10×10)
8×8	3.28 (10×10)	3.23 (11×11)	3.23 (12×12)	3.23 (13×13)	3.23 (14×14)
16×16	3.24 (18×18)	3.23 (19×19)	3.23 (20×20)	3.23 (21×21)	3.23 (22×22)
32×32	3.24 (34×34)	3.23 (35×35)	3.23 (36×36)	3.23 (37×37)	3.23 (38×38)

Table 2
Problem 2: critical load value $\tilde{\gamma}^*_{M,h}$ computed using the NURBS-based stream function formulation. The total number of employed d.o.f.'s is reported between brackets.

5.2 *Mixed Finite Elements for the Hellinger-Reissner formulation*

We consider here the discretized counterpart of problem (57), using some mixed finite element formulations which are known to optimally perform in the linear elastic framework. In the following, we briefly present the numerical schemes under consideration and we then show the numerical results obtained using such elements.

5.2.1 *The MINI element*

The first considered scheme is the MINI element (cf. [1]). Let $\mathcal{T}_h$ be a triangular mesh of Ω, h being the meshsize. For the discretization of the displacement field, we take

$$U_h = \Big\{ \mathbf{v}_h \in U \ : \ \mathbf{v}_{h|T} \in \mathcal{P}_1(T)^2 + \mathcal{B}(T)^2 \qquad \forall T \in \mathcal{T}_h \Big\} \ , \tag{69}$$

where $\mathcal{P}_1(T)$ is the space of linear functions on T, and $\mathcal{B}(T)$ is the linear space generated by b_T, the standard cubic bubble function on T. For the pressure discretization, we take

$$P_h = \Big\{ q_h \in H^1(\Omega) \cap P \ : \ q_{h|T} \in \mathcal{P}_1(T) \qquad \forall T \in \mathcal{T}_h \Big\} \ . \tag{70}$$

5.2.2 *The $\mathcal{Q}_2/\mathcal{P}_1$ element*

We now consider the $\mathcal{Q}_2/\mathcal{P}_1$ element (see [8]). Let $\mathcal{T}_h$ be a quadrilateral mesh of Ω, h being the meshsize. For the discretization of the displacement field, we take

$$U_h = \Big\{ \mathbf{v}_h \in U \ : \ \mathbf{v}_{h|K} \in \mathcal{Q}_2(K)^2 \qquad \forall K \in \mathcal{T}_h \Big\} \ , \tag{71}$$

where $\mathcal{Q}_2(K)$ is the standard space of biquadratic functions. For the pressure discretization, we take

$$P_h = \Big\{ q_h \in P \ : \ q_{h|K} \in \mathcal{P}_1(K) \qquad \forall K \in \mathcal{T}_h \Big\} \ . \tag{72}$$

5.2.3 The QME element

We now focus on the QME quadrilateral method proposed by Pantuso and Bathe in [21,22], based on the *Enhanced Strain Technique* ([25]). This scheme optimally performs in small deformation regimes, as theoretically proved by [17]. Given $\mathcal{T}_h$, a quadrilateral mesh of Ω with meshsize h, the Pantuso-Bathe element is described by the following choice of spaces. For the discretization of the displacement field, we take

$$U_h = \Big\{ \mathbf{v}_h \in U \ : \ \mathbf{v}_{h|K} \in \mathcal{Q}_1(K)^2 \qquad \forall K \in \mathcal{T}_h \Big\} \ , \tag{73}$$

where $\mathcal{Q}_1(K)$ is the standard space of bilinear functions. For the pressure discretization, we take

$$P_h = \Big\{ q_h \in H^1(\Omega) \cap P \ : \ q_{h|K} \in \mathcal{Q}_1(K) \qquad \forall K \in \mathcal{T}_h \Big\} \ . \tag{74}$$

Furthermore, the *Enhanced Strain Technique* requires the introduction of an additional suitable *strain tensor* space, which in the present case is described by (see [22])

$$S_h = \Big\{ \mathbf{E}_h \in (L^2(\Omega))^4 \ : \ \mathbf{E}_{h|K} \in E_6(K) \qquad \forall K \in \mathcal{T}_h \Big\} \ . \tag{75}$$

Above, $E_6(K)$ is the space of tensor-valued functions defined on K, spanned by the following shape functions

$$\begin{bmatrix} \alpha_1\xi + \alpha_2\xi\eta \ ; & \alpha_3\xi \\ \alpha_4\eta & ; \alpha_5\eta + \alpha_6\xi\eta \end{bmatrix} \qquad \text{with } \alpha_i \in \mathbf{R} \ , \tag{76}$$

where (ξ, η) denotes the standard local coordinates on K.

5.2.4 The $\mathcal{P}1$-iso-$\mathcal{P}2/\mathcal{P}0$ element

The last considered scheme is the $\mathcal{P}1$-iso-$\mathcal{P}2/\mathcal{P}0$ element (see [2]). Let $\mathcal{T}_h$ be a triangular mesh of Ω, h being the meshsize. Let $\mathcal{T}_{h/2}$ be a refined mesh, obtained by splitting each triangle of $\mathcal{T}_h$ using the edge midpoints. For the discretization of the displacement field, we take

$$U_h = \left\{ \mathbf{v}_h \in U \ : \ \mathbf{v}_{h|T} \in \mathcal{P}_1(T)^2 \qquad \forall T \in \mathcal{T}_{h/2} \right\} \ . \tag{77}$$

For the pressure discretization, we take

$$P_h = \left\{ q_h \in P \ : \ q_{h|T} \in \mathcal{P}_0(T) \qquad \forall T \in \mathcal{T}_h \right\} \ , \tag{78}$$

where $\mathcal{P}_0(T)$ is the standard space of constant functions.

5.2.5 Mixed Finite Elements: numerical results

We now study the stability range of the discretized model problems by means of the mixed finite element formulations briefly described above.

To find critical loads we again study the eigenvalues of the (loading parameter-dependent) stiffness matrices derived by expressions (57)–(59) for the problems under consideration. Accordingly with Definition 4, we indicate as the critical loads $\gamma_{m,h}$ and $\gamma_{M,h}$ the first load values for which we find an eigenvalue which changes its sign. The corresponding nondimensional quantities are denotes with $\tilde{\gamma}^*_{m,h}$ and $\tilde{\gamma}^*_{M,h}$, respectively (cf. Section 5.1.1)

In the following tables, we report the results we found for the two problems (cf. respectively Table 3 and Table 4), compared with the "exact" results obtained using the stream function formulation.

For what concerns the negative stability limit, we may observe that, despite only the MINI element seems to be stable even for very large compressive loads, all the elements are able to represent rather well the stability range at least asymptotically. Instead, when focusing on the positive stability limit, it is immediately clear that all the considered elements fail in reproducing the correct limit. In fact, as highlighted in Table 5, they show critical load values which are between the 15% and the 18% of the "exact" one for **Problem 1** and between the 31% and the 34% for **Problem 2**.

Therefore, it is possible to conclude that, in both the studied model situations, all the investigated mixed finite element schemes fail in properly reproducing the stability range of the continuum problem.

	MINI		QME		$\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_0$		$\mathcal{Q}_2/\mathcal{P}_1$	
nodes	$\tilde{\gamma}_{m,h}$	$\tilde{\gamma}_{M,h}$	$\tilde{\gamma}_{m,h}$	$\tilde{\gamma}_{M,h}$	$\tilde{\gamma}_{m,h}$	$\tilde{\gamma}_{M,h}$	$\tilde{\gamma}_{m,h}$	$\tilde{\gamma}_{M,h}$
5×5	$-\infty$	1.29	-8.9	1.06	$-\infty$	1.54	$-\infty$	2.71
9×9	$-\infty$	1.23	-14.7	1.03	-225	1.21	-106	1.61
17×17	$-\infty$	1.17	-28.1	1.02	-240	1.10	-179	1.31
33×33	$-\infty$	1.11	-55.7	1.01	-428	1.06	-346	1.18
stream [38×38 d.o.f.'s, $p = q = 6$]:					$\tilde{\gamma}^*_{m,h} = -\infty$		$\tilde{\gamma}^*_{M,h} = 6.60$	

Table 3
Problem 1: critical load values computed using some mixed finite element formulations. On the last row, the "exact" critical value from the stream function formulation is reported.

	MINI		QME		$\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_0$		$\mathcal{Q}_2/\mathcal{P}_1$	
nodes	$\tilde{\gamma}_{m,h}$	$\tilde{\gamma}_{M,h}$	$\tilde{\gamma}_{m,h}$	$\tilde{\gamma}_{M,h}$	$\tilde{\gamma}_{m,h}$	$\tilde{\gamma}_{M,h}$	$\tilde{\gamma}_{m,h}$	$\tilde{\gamma}_{M,h}$
5×5	$-\infty$	1.29	-8.8	1.05	-48.3	1.28	-42.1	1.40
9×9	$-\infty$	1.22	-14.8	1.02	-104	1.15	-85.6	1.22
17×17	$-\infty$	1.15	-28.2	1.02	-196	1.08	-172	1.14
33×33	$-\infty$	1.10	-55.8	1.01	-381	1.05	-342	1.10
stream [38×38 d.o.f.'s, $p = q = 6$]:					$\tilde{\gamma}^*_{m,h} = -\infty$		$\tilde{\gamma}^*_{M,h} = 3.23$	

Table 4
Problem 2: critical load values computed using some mixed finite element formulations. On the last row, the "exact" critical value from the stream function formulation is reported.

	"exact"	MINI		QME		$\mathcal{P}_1$-iso-$\mathcal{P}_2/\mathcal{P}_0$		$\mathcal{Q}_2/\mathcal{P}_1$	
Problem	$\tilde{\gamma}^*_{M,h}$	$\tilde{\gamma}_{M,h}$	$\frac{\tilde{\gamma}_{M,h}}{\tilde{\gamma}^*_{M,h}}$	$\tilde{\gamma}_{M,h}$	$\frac{\tilde{\gamma}_{M,h}}{\tilde{\gamma}^*_{M,h}}$	$\tilde{\gamma}_{M,h}$	$\frac{\tilde{\gamma}_{M,h}}{\tilde{\gamma}^*_{M,h}}$	$\tilde{\gamma}_{M,h}$	$\frac{\tilde{\gamma}_{M,h}}{\tilde{\gamma}^*_{M,h}}$
1	6.60	1.11	16.8%	1.01	15.3%	1.06	16.1%	1.18	17.9%
2	3.23	1.10	34.1%	1.01	31.3%	1.05	32.5%	1.10	34.1%

Table 5
Comparison among the "exact" positive critical load values and those obtained with the considered mixed finite element formulations over the finest mesh used (33×33 nodes).

6 Conclusions

Within the framework of finite elasticity for incompressible materials, in this paper we focus on the problem of the stability range approximation for the continuum problem linearization.

In particular, we show how to construct a reliable estimate for the continuum stability limit, which is a fundamental ingredient in order to assess the performance of a numerical scheme in reproducing the stability properties of the continuum problem. Such an estimate is obtained using an isogeometric stream function formulation on the linearized problem around the exact solution. This allows to exactly enforce the linearized incompressibility constraint, using the high regularity of NURBS shape functions.

We then consider a couple of model problems in order to study the performance of a number of mixed finite element schemes known to optimally behave in the small strain regime. Our numerical tests show that, on the model problems, all the considered finite elements completely fail in reproducing the continuum stability range.

Therefore, we may conclude that, when dealing with highly constrained finite elasticity problems, the natural extensions of finite element schemes well performing in small strains are not guaranteed to correctly reproduce the stability range of the continuum problem. In fact, it seems to be a crucial issue the capability of the method to satisfy the internal constraints exactly, in particular when dealing with the incompressibility constraint.

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