Asymptotics of Shell Eigenvalue Problems

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Abstract

The asymptotic behaviour of the smallest eigenvalue in linear shell problems is studied, as the thickness parameter tends to zero. When pure bending is not inhibited, such a behaviour has been essentially studied in [9]. When pure bending is inhibited, the situation is more complex and some information can be obtained by using the Real Interpolation Theory. In order to cover the widest range of mid-surface geometry and boundary conditions, an abstract approach has been followed. A result concerning the ratio between the bending and the total elastic energy is also announced.

Résumé

Étude asymptotique des problèmes de valeur prope pour les coques. Dans cette note, on étudie le comportement asymptotique de la plus petite valeur propre pour un modèle de coque linéaire, lorsque l'épaisseur de la coque tend vers zéro. Pour les situations de flexion pure non-inhibée, ce comportement a déjà été largement étudié dans [8]. Dans un cas de flexion pure inhibée le problème est plus complexe et on peut employer la théorie de l'interpolation réelle pour l'aborder. Nous proposons une méthodologie abstraite qui vise à couvrir un éventail de configurations le plus large possible en matière de géométrie de la surface moyenne et de conditions aux limites. On annonce également un résultat portant sur le ratio entre énergie de flexion et énergie totale.

1. Introduction

In studying the eigenvalues for 2-D shell models, one is led to consider a problem of the following type (see [7], for instance):

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$$\begin{cases} \text{Find } (u_\varepsilon, \lambda_\varepsilon) \in V \times \mathbf{R} \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \lambda_\varepsilon (u_\varepsilon, v) \quad \forall v \in V \\ ||u_\varepsilon||_H = 1 \ . \end{cases} \tag{1}$$

Above, ε is the thickness parameter (for which we suppose that $0 < \varepsilon \leq 1$), $a^m(\cdot,\cdot)$ is the membrane (or the membrane-shear) bilinear form, and $a^b(\cdot,\cdot)$ is the bending bilinear form. Moreover, V is the admissible displacement space, which also takes into account the kinematical boundary conditions imposed to the structure, and H is an L^2-type space with inner product $(\cdot,\cdot)$.

The aim of this Note is to report our results, extensively proved in [3], about the asymptotics (as $\varepsilon \to 0^+$) of the *smallest* eigenvalue of problem (1). We remark that the main tool we use is the Real Interpolation Theory (see [4], for instance), which has been employed in [1] to classify the asymptotic behaviours of the shell *source* problem.

2. Setting of the problem

For problem (1) we will make the following assumptions (which are indeed shared by most of the shell models – see [5] or [6], for instance).

(i) V and H are both separable Hilbert spaces. Furthermore, we assume the compact and dense inclusion $V \subset H$.
(ii) The bilinear forms $a^m(\cdot,\cdot)$ and $a^b(\cdot,\cdot)$ are symmetric and continuous on V.
(iii) The sum $a^m(\cdot,\cdot) + a^b(\cdot,\cdot)$ is coercive on V.

Therefore, it follows that

(i) Using H as pivot space, we have the dense inclusion $H' \equiv H \subset V'$.
(ii) For each $\varepsilon > 0$, there exists a monotone non-decreasing sequence of real eigenvalues $\{\lambda_\varepsilon^n\}_{n=1}^{\infty}$, and corresponding eigenspaces $\Lambda_\varepsilon^n \subset V$, with $\dim \Lambda_\varepsilon^n < \infty$. Since we will focus only on the *smallest* eigenvalue, in the sequel λ_ε^1 will be simply denoted by λ_ε, and its eigenspace Λ_ε^1 by Λ_ε. We also recall that λ_ε is characterized by the Rayleigh quotient (see for instance [10])

$$\lambda_\varepsilon = \inf_{v \in V} \frac{\varepsilon a^m(v,v) + \varepsilon^3 a^b(v,v)}{||v||_H^2} \ . \tag{2}$$

(iii) The bilinear form $a^b(\cdot,\cdot)$ is coercive on the space of inextensional displacements (see [8]):

$$K = \{v \in V \ : \ a^m(v,w) = 0 \quad \forall w \in V\} \ . \tag{3}$$

We are interested in studying the real function $\varepsilon \to \lambda_\varepsilon$. By taking $v = u_\varepsilon$ in (1), we see that λ_ε is the elastic energy of *any* $u_\varepsilon \in \Lambda_\varepsilon$, with $||u_\varepsilon||_H = 1$:

$$\lambda_\varepsilon = \varepsilon a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^3 a^b(u_\varepsilon, u_\varepsilon) \ . \tag{4}$$

We now introduce the following definition.

Definition 2.1 *We say that the eigenvalue problem* (1) *is of* order α *if*

$$\alpha = \inf \left\{ \beta \ : \ \varepsilon^\beta \lambda_\varepsilon^{-1} \in L^\infty(0,1) \right\} \ . \tag{5}$$

Remark 1 Definition 2.1 means that if the eigenvalue problem is of order α, then α is the "best" exponent in order to have $\lambda_\varepsilon \sim \varepsilon^\alpha$. Furthermore, it is easily seen that if the eigenvalue problem is of order α, then $1 \leq \alpha \leq 3$.

We will also consider the percentage of the elastic energy stored in the bending part. Accordingly, for $0 < \varepsilon \le 1$ and $u_\varepsilon \in \Lambda_\varepsilon$ with $||u_\varepsilon||_H = 1$, we define the function $R(\varepsilon, u_\varepsilon)$ as

$$R(\varepsilon, u_\varepsilon) := \frac{\varepsilon^3 a^b(u_\varepsilon, u_\varepsilon)}{\lambda_\varepsilon} \ . \tag{6}$$

3. Asymptotic behaviour of λ_ε and of $R(\varepsilon)$

In order to study the asymptotic behaviour of the shell eigenvalue problem, we distinguish two cases, depending whether the space K defined in (3) is reduced to $\{0\}$ or not.

3.1. *The case $K \neq \{0\}$: non-inhibited pure bending*

The following result is an easy consequence of the theory developed in [9].
Theorem 3.1 *Suppose that $K \neq \{0\}$. Then there exist constants C_1 and C_2, independent of ε such that*

$$C_1 \varepsilon^3 \le \lambda_\varepsilon \le C_2 \varepsilon^3. \tag{7}$$

Therefore, the eigenvalue problem is of order $\alpha = 3$ (cf. Definition 2.1). Furthermore, it holds

$$\lim_{\varepsilon \to 0^+} R(\varepsilon, u_\varepsilon) = 1 \ . \qquad \Box \tag{8}$$

Remark 2 We notice that (8) *is consistent with Proposition 3.4.*

3.2. *The case $K = \{0\}$: inhibited pure bending*

We first notice that in this case $a^m(\cdot, \cdot)$ defines a norm on V. We set W as the completion of V with the norm $a^m(v, v)^{1/2} := ||v||_W$. Therefore, we have the dense inclusion $V \subseteq W$, which implies $W' \subseteq V'$ densely. We have the following result, whose proof can be found in [3].
Theorem 3.2 *Suppose that $K = \{0\}$. Then, for $0 < \theta < 1$, we have*

$$H \subseteq (W', V')_{\theta,\infty} \ \text{ if and only if } \ \varepsilon^{2\theta+1} \lambda_\varepsilon^{-1} \in L^\infty(0, 1) \ . \qquad \Box \tag{9}$$

The following corollary is an immediate consequence of Theorem 3.2 and Definition 2.1.
Corollary 3.3 *The order α of the eigenvalue problem* (1) *is given by*

$$\alpha = \inf \left\{ 2\theta + 1 \ : \ H \subseteq (W', V')_{\theta,\infty} \right\} \ . \tag{10}$$

Concerning the ratio $R(\varepsilon, u_\varepsilon)$ defined by (6), in [3] we proved the following result.
Proposition 3.4 *Let the eigenvalue problem* (1) *be of order α. Suppose also that there exist*

$$\lim_{\varepsilon \to 0^+} \left(\varepsilon^{-\alpha} \lambda_\varepsilon \right) = l_0 > 0 \ \text{ and } \ \lim_{\varepsilon \to 0^+} R(\varepsilon, u_\varepsilon) \ge 0 \ , \tag{11}$$

where $u_\varepsilon \in \Lambda_\varepsilon$. Then it holds

$$\lim_{\varepsilon \to 0^+} R(\varepsilon, u_\varepsilon) = \frac{\alpha - 1}{2} \ . \qquad \Box \tag{12}$$

Remark 3 In [2] an asymptotic analysis of λ_ε and $R(\varepsilon, u_\varepsilon)$ for a clamped cylindrical shell has been developed, by using a Fourier expansion technique. Among the results of that paper, it has been proved and numerically tested that

$$\lambda_\varepsilon \sim \varepsilon^2 \ , \quad \lim_{\varepsilon \to 0^+} R(\varepsilon, u_\varepsilon) = \frac{1}{2} \ . \tag{13}$$

In the terminology of the present Note, Equations (13) *correspond to the choice* $\alpha = 2$. *The same result can be obtained by applying Corollary 3.3 and Proposition 3.4 to the clamped cylindrical shell, as detailed in [3].*

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