Positive Definite Balancing Neumann-Neumann preconditioners for Nearly Incompressible Elasticity

L. BEIRÃO DA VEIGA
Department of Mathematics, Università di Milano,
Via Saldini 50 20133 Milano, ITALY,
and I.M.A.T.I.–C.N.R., Via Ferrata 1, Pavia, Italy
E-mail address: beirao@mat.unimi.it

C. LOVADINA
Dipartimento di Matematica, Università di Pavia,
Via Ferrata 1 27100 Pavia, ITALY,
and I.M.A.T.I.–C.N.R., Via Ferrata 1, Pavia, Italy
E-mail address: carlo.lovadina@unipv.it,
URL: http://www-dimat.unipv.it/lovadina

L.F. PAVARINO
Department of Mathematics, Università di Milano,
Via Saldini 50 20133 Milano, ITALY.
E-mail address: pavarino@mat.unimi.it,
URL: http://mat.unimi.it/∼pavarino/

Abstract

In this paper, a positive definite Balancing Neumann-Neumann (BNN) solver for the linear elasticity system is constructed and analyzed. The solver implicitly eliminates the interior degrees of freedom in each subdomain and solves iteratively the resulting Schur complement, involving only interface displacements, using a BNN preconditioner based on the solution of a coarse elasticity problem and local elasticity problems with natural and essential boundary conditions. While the Schur complement becomes increasingly ill-conditioned as the materials becomes almost incompressible, the BNN preconditioned operator remains well-conditioned. The main theoretical result of the paper shows that the proposed BNN method is scalable and quasi-optimal in the constant coefficient case. This bound holds for material parameters arbitrarily close to the incompressible limit. While this result is due to an underlying mixed formulation of the problem, both the interface problem and the preconditioner are positive definite. Numerical results in two and three dimensions confirm these good convergence properties and the robusteness of the methods with respect to the almost incompressibility of the material.

Keywords: Almost Incompressible Elasticity; Domain Decomposition; Balancing Neumann-Neumann; Finite Element Methods; Spectral Methods.

AMS Subject Classification: 65F10, 65N30, 65N55

1 Introduction

The numerical approximation of the linear elasticity system using only displacement variables increasingly suffers from locking phenomena when the material approaches the incompressible limit. In such situation, the introduction of an additional pressure variable and a mixed finite element method is a well-known remedy, but the resulting discrete system becomes indefinite and its iterative solution by domain decomposition methods requires special techniques; see Ref. [37, 18] and the related works on Stokes problems [35, 39, 29, 41, 25, 42, 20, 2, 38, 26, 27]. In this paper, we instead construct and analyze a positive definite iterative solver based on an equivalent positive definite reformulation of the mixed system and on a positive definite Balancing Neumann-Neumann (BNN) algorithm. This work extends the mixed BNN solver studied in Goldfeld, Pavarino and Widlund [18] and the positive definite variant proposed in Goldfeld's Ph.D. Thesis [17], without a direct positive definite theory. A related idea for FETI-DP methods has been presented without proofs in Klawonn, Rheinbach and Wohlmuth [21]. Note that the positive definite formulation, which is simply obtained with a static condensation of the pressure variable, is the most used in the engineering practice for two reasons. First of all, the number of degrees of freedom is significantly lower; secondly, the ellipticity of the original problem, with all the related advantages in the linear system solution, is not lost.

Like all iterative substructuring methods, our BNN algorithm starts with the implicit elimination of the degrees of freedom associated with the interior of each subdomain. The resulting Schur complement involves only interface displacement unknowns and it is solved iteratively using a BNN preconditioner based on a coarse elasticity problem and local elasticity problems with natural and essential boundary conditions. While the Schur complement becomes increasingly ill-conditioned as the materials becomes almost incompressible, our BNN preconditioned operator remains well-conditioned. Our main theoretical result shows that our BNN method is scalable and quasi-optimal in the constant coefficient case, i.e., a polylogarithmic bound in the local number of degrees of freedom holds for the condition number of the preconditioned operator. Moreover, this bound holds for material parameters arbitrarily close to the incompressible limit. We remark that this property is shared by methods employing mixed formulations and mixed preconditioners, but it is here obtained employing positive definite reformulations of both the problem and preconditioner. The bound in our main result depends on the inf-sup constants of the underlying mixed methods chosen, but in our algorithm all coarse and local problems are positive definite. Numerical results in two and three dimensions confirm these good convergence properties and the robusteness of our methods with respect to the almost incompressibility of the material. As it is the case for mixed methods, numerical results show that these results also hold for heterogeneous materials with discontinuous coefficients across subdomains.

Neumann-Neumann methods were first introduced and analyzed for second order elliptic problems (see Cowsar et al. [9], Dryja and Widlund [11], Mandel [30], Mandel and Brezina [31], and Pavarino [36]) and later extended to plate and shell problems (Le Tallec et al. [24]), convection-diffusion problems (Achdou et al. [1] and Alart et al. [3]), vector field problems (Toselli [46]). We refer to Toselli and Widlund [45] for an overview of this family of domain decomposition methods and to Klawonn and Widlund [22] for their connection with FETI methods. More recent evolutions of Neumann-Neumann methods are the BDDC methods (see Dohrmann [10], Mandel and Dohrmann [32]), which can be interpreted as dual of the FETI-DP methods (see Farhat et al. [12], Mandel et al. [33], Li and Widlund [28, 27], Klawonn and Widlund [23] and the references therein).

Many mixed domain decomposition methods have been proposed for the incompressible Stokes equations and the incompressible limit of mixed elasticity. Iterative substructuring methods have been studied e.g. by Ainsworth and Sherwin [2], Bramble and Pasciak [6], Casarin [8], Fischer and Rønquist [15], Goldfeld et al. [18], Le Tallec and Patra [25], Li [26], Li and Widlund [27], Marini and Quarteroni [34], Pasciak [35], Pavarino and Widlund [37, 38], Quarteroni [39], and Rønquist [41]. Overlapping Schwarz methods have been considered by Fischer [13], Fischer et al. [14], Gervasio [16], Klawonn and Pavarino [20], and Rønquist [42].

For a general introduction to domain decomposition methods, we refer to Quarteroni and Valli [40], Smith, Bjørstad, and Gropp [43], Toselli and Widlund [45].

The remainder of this paper is organized as follows. In Section 2 we introduce the linear elasticity problem, and in Section 3 we present its discretization by means of finite/spectral elements, focusing our attention on the nearly incompressible case. In Section 4 and 5 we introduce the Schur complement system and our BNN preconditioner, respectively. In Section 6 we develop the analysis on the condition number of the preconditioned system. Finally, in Section 7 we show a set of numerical tests confirming the theoretical bounds.

Throughout the paper, we will denote with C a generic positive constant, possibly different at each occurrence, which depends only on the mesh regularity.

2 The linear elasticity problem

Let Ω be a polygonal domain in $\mathbb{R}^d$, $d = 2, 3$, representing the elastic material body. We assume that the boundary $\partial\Omega$ is split into two non-overlapping parts Γ_D and Γ_N. We suppose that the body is clamped on Γ_D and it is subjected to a given traction $\boldsymbol{g} : \Gamma_N \rightarrow \mathbb{R}^d$ on Γ_N, as well as to a body force density $\boldsymbol{f} : \Omega \rightarrow \mathbb{R}^d$. The linear elastic deformation problem then reads

$$\begin{cases} \text{Find the displacement field } \boldsymbol{u}_e : \Omega \rightarrow \mathbb{R}^d \text{ such that:} \\ \operatorname{div} \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_e) + \boldsymbol{f} = 0 & \text{in } \Omega \\ \boldsymbol{u}_e = 0 & \text{on } \Gamma_D \\ \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_e) \cdot \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma_N \end{cases} \tag{1}$$

Above, $\boldsymbol{n}$ is the unit outward normal at each point of the boundary and the fourth order tensor $\mathbb{C}$ is defined by

$$\mathbb{C}\boldsymbol{w} = 2\mu\boldsymbol{\tau} \ + \ \lambda \text{tr}(\boldsymbol{\tau})\mathbf{I} \tag{2}$$

$$\lambda = \frac{2\mu\nu}{1 - 2\nu} \tag{3}$$

for all second order tensors $\boldsymbol{\tau}$. Moreover, "tr" represents the trace operator and the functions $\mu = \mu(x) > 0$, $0 \leq \nu = \nu(x) < 1/2$ are the shear modulus and the Poisson's ratio, respectively.

Assuming for simplicity regular loadings $\boldsymbol{f} \in [L^2(\Omega)]^d$ and $\boldsymbol{g} \in [L^2(\Gamma_N)]^d$, we introduce

$$< \boldsymbol{\psi}, \boldsymbol{v} >= (\boldsymbol{f}, \boldsymbol{v})_\Omega + (\boldsymbol{g}, \boldsymbol{v})_{\Gamma_N} \qquad \forall \boldsymbol{v} \in [H^1(\Omega)]^d \tag{4}$$

where $(\,,\,)_\Omega$, $(\,,\,)_{\Gamma_N}$ indicate as usual the L^2 scalar product respectively on Ω and Γ_N. The variational formulation of problem (1) then reads:

$$\begin{cases} \text{Find } \boldsymbol{u}_e \in [H^1_{\Gamma_D}(\Omega)]^d \text{ such that:} \\ a^\mu(\boldsymbol{u}_e, \boldsymbol{v}) + (\lambda \operatorname{div} \boldsymbol{u}_e, \operatorname{div} \boldsymbol{v}) = <\boldsymbol{\psi}, \boldsymbol{v}> \qquad \forall \boldsymbol{v} \in [H^1_{\Gamma_D}(\Omega)]^d \end{cases} \tag{5}$$

where the parentheses indicate the $L^2(\Omega)$ scalar product and

$$[H^1_{\Gamma_D}(\Omega)]^d = \{\boldsymbol{v} \in [H^1(\Omega)]^d \mid \boldsymbol{v}|_{\Gamma_D} = 0\} \tag{6}$$

$$a^\mu(\boldsymbol{w}, \boldsymbol{v}) = \int_\Omega 2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \qquad \forall \boldsymbol{w}, \boldsymbol{v} \in [H^1_{\Gamma_D}(\Omega)]^d \tag{7}$$

3 Discretization of the problem

Let $V \subset [H^1_{\Gamma_D}(\Omega)]^d$ represent a finite dimensional space, typically built by means of a finite or spectral element approach. Then, the classical Galerkin discrete formulation of problem (5) reads

$$\begin{cases} \text{Find } \boldsymbol{u} \in V \text{ such that:} \\ a^\mu(\boldsymbol{u}, \boldsymbol{v}) + (\lambda \operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{v}) = <\boldsymbol{\psi}, \boldsymbol{v}> \qquad \forall \boldsymbol{v} \in V \end{cases} \tag{8}$$

Due to the elliptic form of the problem, whenever $\mu \sim \lambda$, the above method is well known to give results which are optimal with respect to the interpolation properties of the space V.

On the other hand, whenever the Poisson ratio $\nu \to 1/2$, the ratio λ/μ in (5) tends to $+\infty$. The *almost incompressible materials* are modeled by constitutive laws for which $\lambda \gg \mu$. In such cases, due to the numerical locking effect of the "divergence-free" limit constraint, the classical Galerkin finite element/spectral method (8) give highly non-satisfactory approximation results (see for instance Ref. [7]). In the engineering practice, the most popular approach in order to cure this problem is to adopt a relaxation/under-integration of the second term in (8). This leads to the modified discrete problem, for piecewise constant λ:

$$\begin{cases} \text{Find } \boldsymbol{u} \in V \text{ such that:} \\ \chi(\boldsymbol{u}, \boldsymbol{v}) = <\boldsymbol{\psi}, \boldsymbol{v}> \qquad \forall \boldsymbol{v} \in V \end{cases} \tag{9}$$

where

$$\chi(\boldsymbol{u}, \boldsymbol{v}) := a^\mu(\boldsymbol{u}, \boldsymbol{v}) + (\lambda\, \Pi_h \operatorname{div} \boldsymbol{u}, \Pi_h \operatorname{div} \boldsymbol{v}) \ , \tag{10}$$

and

$$\Pi_h \ : \ L^2(\Omega) \ \longrightarrow \ Q \tag{11}$$

is the L^2 projection operator on a suitable auxiliary finite dimensional space $Q \subset L^2(\Omega)$. Problem (9) is equivalent to the mixed formulation

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) \in V \times Q \text{ such that:} \\ a^{\mu}(\boldsymbol{u}, \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p) = < \boldsymbol{\psi}, \boldsymbol{v} > \qquad \forall \boldsymbol{v} \in V \\ (\operatorname{div} \boldsymbol{u}, q) - (\lambda^{-1} p, q) = 0 \qquad \forall q \in Q \end{cases} \tag{12}$$

Therefore, as it follows from the classical theory of mixed finite element methods (see Ref. [7]), the approach shown in (9) is robust with respect to the ratio λ/μ whenever the inf-sup condition

$$\sup_{\boldsymbol{v} \in V, |\boldsymbol{v}|_1 \neq 0} \frac{(\operatorname{div} v, q)}{|\boldsymbol{v}|_1} \geq \beta_G \|q\|_0 \qquad \forall q \in Q \tag{13}$$

holds with $\beta_G > 0$ independent of the discrete space dimensions. Note that in the particular case $\Gamma_D = \partial\Omega$, Q must be taken in the space of the L^2 functions with zero average.

The advantage of formulation (9) upon (12) is twofold. First of all, the number of degrees of freedom is significantly lower; secondly, the ellipticity of the original problem, with all the related advantages in the linear system solution, is not lost. Moreover, in a finite element framework the auxiliary space Q usually consists of functions which are discontinuous across the mesh elements. As a consequence, the effect of the operator Π_h can be very easily computed by an element-wise procedure.

4 Iterative substructuring

We decompose the domain Ω into N open, nonoverlapping subdomains Ω_i of characteristic size H forming a shape-regular finite element mesh τ_H. In the finite element case, this coarse triangulation τ_H is further refined into a finer triangulation τ_h of characteristic size h; both meshes will typically be composed of tetrahedra or hexahedra (respectively triangles or quadrilaterals in 2 dimensions). In the spectral element case, each subdomain Ω_i is assumed to be the affine image of the reference cube or square $\Omega_{\text{ref}} = (-1,1)^d,\ d = 2,3$ and the nodes of the fine hexahedral mesh τ_h in each Ω_i are the affine images of the Gauss-Lobatto-Legendre nodes of degree n on Ω_{ref}. For simplicity of notation, in the sequel the parameter h will indicate the fine mesh size in the finite element case, and the ratio H/n in the spectral element case. We assume that both μ and λ are piecewise constant functions with respect to the coarse mesh. Accordingly, we denote with μ_i, λ_i the value of the functions μ, λ on the domain Ω_i, $i = 1, 2, .., N$. In the sequel, we will assume the following hypotheses on the chosen discrete spaces V and Q.

H1) For each $i = 1, 2, ..N$, there exists a positive constant β_i such that

$$\beta_i \, ||q_i||_{0,\Omega_i} \leq \sup_{\boldsymbol{v}_I \in V_i} \frac{(q_i, \operatorname{div} \boldsymbol{v}_I)_{\Omega_i}}{|\boldsymbol{v}_I|_{1,\Omega_i}} \qquad \forall\, q_i \in Q_{h/H,i} \;, \tag{14}$$

where

$$Q_{h/H,i} = \left\{ q \in Q : \quad q = 0 \text{ in } \Omega \setminus \Omega_i \quad , \quad \int_{\Omega_i} q = 0 \right\} \;. \tag{15}$$

Furthermore, it will be useful to denote with β the following constant:

$$\beta := \min \{ \, \beta_i \; : \; i = 1, ..., N \, \} \;. \tag{16}$$

H2) The auxiliary pressure space Q consists of functions which are discontinuous across the elements of τ_h.

As a consequence of hypothesis **H2)**, the projection operator Π_h in (11) can be *locally* defined in each subdomain Ω_i:

$$\Pi_h = \sum_{i=1}^{N} \Pi_h^i \quad \text{where } \Pi_h^i \text{ is the } L^2 \text{ projection operator on } Q_{|\Omega_i}. \tag{17}$$

Remark 4.1 *We remark that hypotheses* **H1)** *and* **H2)** *are indeed met by a wide class of discretization methods. For example, we may cite the triangular finite element method where V consists of piecewise quadratic continuous functions, and the auxiliary space Q contains the piecewise constant functions. More generally, any* inf-sup stable *finite element pairing (V, Q) for the Stokes problem with discontinuous pressure approximation satisfies* (14) *with β_i independent of h and H. We refer to Refs. [4, 7, 19] and the references therein for several instances of stable finite element schemes. We also notice that in the framework of spectral elements the well-known Q_n/Q_{n-2} approximation satisfies* (14) *with non-uniform constant $\beta_i \sim n^{(\frac{1-d}{2})}$, where $d = 2, 3$ is the space dimension (see Refs. [29, 44]). A possible alternative with uniform* inf-sup *constant β_i is the Q_n/P_{n-1} spectral element (cf. Ref. [5]), which is however less convenient from the implementation point of view.*

As it is standard in iterative substructuring methods, we first reduce the problem to the interface

$$\Gamma = \left(\bigcup_{i=1}^{N} \partial\Omega_i \right) \setminus \Gamma_D,$$

by implicitly eliminating the interior degrees of freedom, a process also known as static condensation. In variational form, this process consists in a suitable decomposition of the discrete space V. More precisely, let us define $W = V_{|\Gamma}$, i.e. the space of the traces of functions in V, as well as the local spaces

$$V_i = V \cap H_0^1(\Omega_i) \ .$$

The space V can be decomposed as

$$V = \oplus_{i=1}^{N} V_i \oplus \mathcal{EH}(W) \ .$$

Here $\mathcal{EH} : W \longrightarrow V$ is the discrete "elastic-harmonic" extension operator defined by solving the problem

$$\begin{cases} \text{Find } \mathcal{EH}(\boldsymbol{u}_\Gamma) \in V \text{ such that:} \\ \chi(\mathcal{EH}(\boldsymbol{u}_\Gamma), \boldsymbol{v}_I) = 0 \qquad \forall \, \boldsymbol{v}_I \in V_i \quad i = 1, 2, ..., N \\ \mathcal{EH}(\boldsymbol{u}_\Gamma)|_\Gamma = \boldsymbol{u}_\Gamma \ . \end{cases} \tag{18}$$

Defining the Schur complement bilinear form

$$s(\boldsymbol{u}_\Gamma, \boldsymbol{v}_\Gamma) = \chi(\mathcal{EH}(\boldsymbol{u}_\Gamma), \mathcal{EH}(\boldsymbol{v}_\Gamma)),$$

it follows that the interface component of the discrete solution satisfies the reduced system

$$s(\boldsymbol{u}_\Gamma, \boldsymbol{v}_\Gamma) = < \tilde{\boldsymbol{\psi}}, \boldsymbol{v}_\Gamma > \qquad \forall \, \boldsymbol{v}_\Gamma \in W. \tag{19}$$

In the sequel, to simplify the notation we will drop the index Γ for functions in W if there is no risk of confusion. Furthermore, we will adopt the following notation: given any two functions v, w in the same generic discrete space X, we indicate with vw the unique function of X having the value $v(x)w(x)$ at all the nodes x which define the degrees of freedom of X.

5 A Balancing Neumann-Neumann (BNN) method

Our BNN preconditioner is defined by further decomposing the interface space W as

$$W = R_0^T W_0 + \sum_{i=1}^{N} R_i^T W_i,$$

where $R_0^T : W_0 \longrightarrow W$, $R_i^T : W_i \longrightarrow W$ are prolongation operators defined below.

The local spaces W_i are spaces of discrete functions living on $\partial\Omega_i \cap \Gamma$, defined as $W_i = W_{|\partial\Omega_i}$, $i = 1, 2, .., N$. The prolongation operators R_i^T, $i = 1, 2, .., N$ are maps which extend any function of W_i to the function of W which is zero at all the nodes not in $\partial\Omega_i \cap \Gamma$. It will be useful to denote with $a_i^{\mu}(\cdot,\cdot)$, $\chi_i(\cdot,\cdot)$ and $s_i(\cdot,\cdot)$ the corresponding bilinear forms when restricted to the subdomain Ω_i. Accordingly, the local discrete "elastic-harmonic" extension operator $\mathcal{EH}_i : W_i \longrightarrow V_{|\Omega_i}$ is defined by solving the problem

$$\begin{cases} \text{Find } \mathcal{EH}_i(\boldsymbol{u}_i) \in V_{|\Omega_i} \text{ such that:} \\ \chi_i(\mathcal{EH}_i(\boldsymbol{u}_i), \boldsymbol{v}_I) = 0 \qquad \forall\, \boldsymbol{v}_I \in V_i \\ \mathcal{EH}(\boldsymbol{u}_i)|_{\partial\Omega_i \cap \Gamma} = \boldsymbol{u}_i \ . \end{cases} \tag{20}$$

In order to introduce W_0, we need the following definition (see for example Section 6.2.1 of Ref. [45]). Given any node $x \in \Gamma$, let $\mathcal{N}_x = \{ j \in \mathbb{N} \mid x \in \partial\Omega_j \}$. Then, the weighted counting functions δ_i (and their pseudo-inverses $\delta_i^{\dagger}$) are functions in W_i defined by

$$\delta_i(x) = \frac{\sum_{j \in \mathcal{N}_x} \mu_j}{\mu_i} \quad \forall x \text{ node of } \partial\Omega_i \cap \Gamma \tag{21}$$

$$\delta_i^{\dagger}(x) = (\delta_i(x))^{-1} \quad \forall x \text{ node of } \partial\Omega_i \cap \Gamma \tag{22}$$

Note that the extended pseudo-inverses $\left\{ R_i^T \delta_i^{\dagger} \ , \ \ i = 1, .., N \right\}$ provide a partition of unity.

We assume that the space W_0 satisfies the following two hypotheses:

H3) The coarse space $R_0^T W_0$ contains the space

$$W_0^0 = \left\{ \boldsymbol{v} \in W : \boldsymbol{v} \in \left(\operatorname{span} \left\{ R_i^T \delta_i^{\dagger} \boldsymbol{z} \right\}, \boldsymbol{z} \in \ker(\chi_i) \right) \right\} \ . \tag{23}$$

Above, the prolongation operator R_0^T represents the injection of W_0 into W; in terms of linear algebra, it is simply the operator which writes functions of the coarse space W_0 in terms of the basis of W. Note also that $\ker(\chi_i) = \ker(a_i^{\mu})$, which consists of the local rigid body motions.

H4) The coarse inf-sup condition holds

$$\sup_{\boldsymbol{v} \in W_0, |\boldsymbol{v}|_1 \neq 0} \frac{(\operatorname{div} v, q_0)}{|\boldsymbol{v}|_1} \geq \beta_0 \|q_0\|_0 \qquad \forall q \in Q_0 \tag{24}$$

with β_0 positive constant independent of h, H and where Q_0 is the space of piecewise constant functions with respect to the coarse triangulation τ_H. In the case $\Gamma_D = \partial\Omega$ the L^2 norm in the right hand side must be substituted with $\|\cdot\|_{0/\mathbb{R}}$.
An example of coarse space which satisfies the hypotheses **H3)** and **H4)**, both in the case of hexahedral and tetrahedral coarse meshes τ_H, is:

$$\begin{aligned} W_0^2 &= W_0^0 + \{\text{piecewise linear coarse edge functions}\} \\ &+ \{\text{quadratic coarse edge bubbles in the normal direction}\}. \end{aligned} \tag{25}$$

Define now the coarse operator $P_0 = R_0^T \tilde{P}_0 : W \longrightarrow R_0^T W_0$ by

$$s(\tilde{P}_0 \boldsymbol{u}, \boldsymbol{v}) = s(\boldsymbol{u}, R_0^T \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in W_0, \tag{26}$$

and the local operators $P_i = R_i^T \tilde{P}_i : W \longrightarrow R_i^T W_i$ by

$$\widetilde{s}_i(\tilde{P}_i \boldsymbol{u}, \boldsymbol{v}_i) = s(\boldsymbol{u}, R_i^T \boldsymbol{v}_i) \qquad \forall\, \boldsymbol{v}_i \in W_i, \tag{27}$$

where the inexact bilinear forms $\widetilde{s}_i(\cdot,\cdot)$ are given by

$$\widetilde{s}_i(\boldsymbol{u}_i, \boldsymbol{v}_i) = \chi_i(\mathcal{EH}(\delta_i \boldsymbol{u}_i), \mathcal{EH}(\delta_i \boldsymbol{v}_i)) \ , \tag{28}$$

with $\mathcal{EH}$ the discrete elastic-harmonic extension operator defined in (18).

Then our Balancing Neumann-Neumann method is defined by the preconditioned operator

$$P = P_0 + (I - P_0) \sum_{i=1}^{N} P_i (I - P_0). \tag{29}$$

Remark 5.1 *The bilinear forms appearing in the problems* (27) *have a non trivial kernel, given by the space* $K_i = \left\{\delta_i^\dagger \boldsymbol{z}\ ,\ \boldsymbol{z} \in \ker(a^\mu)\right\}$. *On the other hand, the operators* P_i *are always applied to functions of type* $(I - P_0)\boldsymbol{z}$, $\boldsymbol{z} \in W$. *Therefore, what one really needs to solve are the problems*

$$\widetilde{s}_i(\tilde{P}_i \boldsymbol{u}, \boldsymbol{v}_i) = s((I - P_0)\boldsymbol{z}, R_i^T \boldsymbol{v}_i) \qquad \forall\, \boldsymbol{v}_i \in W_i. \tag{30}$$

A simple calculation shows that, due to property **H3)** *of the coarse space* W_0*, the right hand side is compatible (i.e.* $s((I - P_0)\boldsymbol{z}, R_0^T \boldsymbol{v}_i) = 0$ *if* $\boldsymbol{v}_i \in K_i$*). Therefore there is a (non unique) solution of problem* (30)*; for the choice of the particular solution there are various classical possibilities, see for example Section 6.2.2 of Ref. [45].*

5.1 Matrix form of the BNN method

The Schur complement system (19) has the matrix representation

$$S \boldsymbol{u}_\Gamma = \tilde{b}_\Gamma, \tag{31}$$

where $\tilde{b}_\Gamma = b_\Gamma - K_{I\Gamma} K_{II}^{-1} b_I$ and $S = K_{\Gamma\Gamma} - K_{I\Gamma} K_{II}^{-1} K_{I\Gamma}^T$ is the Schur complement of the original stiffness matrix

$$\begin{bmatrix} K_{II} & K_{I\Gamma}^T \\ K_{I\Gamma} & K_{\Gamma\Gamma} \end{bmatrix}.$$

The coarse problem (26) has the matrix form

$$P_0 = Q_H S, \quad \text{with} \quad Q_H = R_0^T S_0^{-1} R_0 \quad \text{and} \quad S_0 = R_0 S R_0^T, \tag{32}$$

where R_0^T is defined according to the choice of coarse space; in other words its columns are the functions of the coarse space written in terms of the basis function of W.

The local problems (27) have the matrix form

$$P_i = Q_i S, \quad \text{with} \quad Q_i = R_i^T D_i S_i^\dagger D_i R_i, \tag{33}$$

where R_i^T are $0, 1$ prolongation matrices mapping vectors in W_i into vectors in W, D_i are diagonal matrices with elements $\delta_i^\dagger(x)$ at the node $x \in \partial\Omega_i$ and S_i is the local Schur complement for subdomain Ω_i. As it is standard in Neumann-Neumann methods, the pseudo-inverse $S_i^\dagger$ is computed by solving a local problem on Ω_i with Neumann boundary conditions on $\partial\Omega_i \setminus \Gamma_D$, using the original local stiffness matrices (associated with $\chi_i(\cdot, \cdot)$).

Therefore the preconditioned operator P defined by (29) can be represented in matrix form as a preconditioner Q applied to the Schur complement S

$$\begin{aligned} P &= Q_H S + (I - Q_H S) \sum_{i=1}^N Q_i S (I - Q_H S) \\ &= [Q_H + (I - Q_H S) \sum_{i=1}^N Q_i (I - S Q_H)] S \\ &= QS. \end{aligned}$$

6 Analysis of the method

In this section we focus on the three-dimensional elastic problem and we prove the following result.

Theorem 6.1 *For the condition number of our preconditioned operator it holds*

$$\text{Cond}(P) \leq 1 + C\,(1 + 2\beta^{-1}) \left(1 + 2\beta_0^{-1} \frac{\mu_{max}}{\mu_{min}}\right)^2 \left(1 + \sqrt{3}\,\beta^{-1}\right)^2 C_{ap}^2\, \rho(\mathbb{E}) \;, \tag{34}$$

where the constant $\rho(\mathbb{E})$ is defined in Lemma 6.2. Moreover, the constant C_{ap} is such that

$$C_{ap} \sim \begin{cases} 1 + \log(H/h) & \textit{for finite elements} \\ 1 + \log(n) & \textit{for hexahedral spectral elements.} \end{cases} \tag{35}$$

Proof. The result is an easy consequence of the upper and lower bounds shown below in Theorems 6.2 and 6.3. □

Remark 6.1 *Theorem 6.1 highlights that the proposed Neumann-Neumann preconditioner is robust with respect to the material parameter λ. Moreover, it is scalable and quasi-optimal, whenever one considers an* inf-sup stable *discretization method (cf. Remark 4.1).*

6.1 Local stability

First, for each $i = 1, ..., N$ we recall the definition of the space:

$$Q_{h/H,i} = \left\{ q \in Q : \quad q = 0 \text{ in } \Omega \setminus \Omega_i \ , \ \int_{\Omega_i} q = 0 \right\} \ . \tag{36}$$

From hypothesis **H1)** of Section 4 and recalling (16), we deduce that

$$\beta \, ||q_i||_{0,\Omega_i} \leq \sup_{\boldsymbol{v}_I \in V_i} \frac{(q_i, \operatorname{div} \boldsymbol{v}_I)_{\Omega_i}}{|\boldsymbol{v}_I|_{1,\Omega_i}} \qquad \forall \, q_i \in Q_{h/H,i} \ . \tag{37}$$

We now prove the following result (see also Ref. [38]).

Lemma 6.1 *For each subdomain Ω_i it holds*

$$a_i^\mu(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i)^{1/2} \leq \sqrt{2} \mu_i^{1/2} \left(1 + \sqrt{3} \, \beta^{-1}\right) ||\nabla \mathcal{H}_i \boldsymbol{u}_i||_{0,\Omega_i} \qquad \forall \boldsymbol{u}_i \in W_i \ , \tag{38}$$

where $\mathcal{H}_i : W_i \longrightarrow V_{|\Omega_i}$ is the local discrete harmonic operator defined by

$$\begin{cases} (\nabla \mathcal{H}_i \boldsymbol{u}_i, \nabla \boldsymbol{v}_I)_{\Omega_i} = 0 & \forall \, \boldsymbol{v}_I \in V_i \\ \mathcal{H}_i \boldsymbol{u}_i|_{\partial\Omega_i \cap \Gamma} = \boldsymbol{u}_i \ . \end{cases} \tag{39}$$

Proof. We first recall that $\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i \in V_{|\Omega_i}$ is defined by

$$\begin{cases} \chi_i(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \boldsymbol{v}_I) = 0 & \forall \, \boldsymbol{v}_I \in V_i \\ \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i|_{\partial\Omega_i \cap \Gamma} = \boldsymbol{u}_i \ , \end{cases} \tag{40}$$

i.e. explicitly

$$\begin{cases} a_i^\mu(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \boldsymbol{v}_I) + \lambda_i \, (\Pi_h \operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \Pi_h \operatorname{div} \boldsymbol{v}_I)_{\Omega_i} = 0 & \forall \, \boldsymbol{v}_I \in V_i \\ \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i|_{\partial\Omega_i \cap \Gamma} = \boldsymbol{u}_i \ . \end{cases} \tag{41}$$

By using that $\int_{\Omega_i} \operatorname{div} \boldsymbol{v}_I = 0$, we obtain

$$\begin{aligned} \lambda_i \, (\Pi_h \operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \Pi_h \operatorname{div} \boldsymbol{v}_I)_{\Omega_i} &= \lambda_i \, (\Pi_h \operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \operatorname{div} \boldsymbol{v}_I)_{\Omega_i} \\ &= \lambda_i \left(\Pi^i_{h/H} \operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \operatorname{div} \boldsymbol{v}_I\right)_{\Omega_i} \ , \end{aligned} \tag{42}$$

where $\Pi^i_{h/H}$ denotes the L^2-projection operator $\Pi^i_{h/H} : L^2(\Omega_i) \longrightarrow Q_{h/H,i}$. From (41) and (42), it follows that $\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i$ is equivalently defined by

$$\begin{cases} a_i^\mu(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \boldsymbol{v}_I) + \lambda_i \left(\Pi^i_{h/H} \operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \operatorname{div} \boldsymbol{v}_I\right)_{\Omega_i} = 0 & \forall \, \boldsymbol{v}_I \in V_i \\ \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i|_{\partial\Omega_i \cap \Gamma} = \boldsymbol{u}_i \ . \end{cases} \tag{43}$$

By setting $p_i = \lambda \, \Pi^i_{h/H} \operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i \in Q_{h/H,i}$, we may write Problem (43) in mixed form:

$$\begin{cases} a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \boldsymbol{v}_I) + (p_i, \operatorname{div} \boldsymbol{v}_I)_{\Omega_i} = 0 & \forall\, \boldsymbol{v}_I \in V_i \\ (\operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, q_i)_{\Omega_i} - \lambda_i^{-1} (p_i, q_i)_{\Omega_i} = 0 & \forall\, q_i \in Q_{h/H,i} \\ \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i|_{\partial\Omega_i \cap \Gamma} = \boldsymbol{u}_i \ . \end{cases} \tag{44}$$

From (44) and (37) we get

$$\begin{aligned} ||p_i||_{0,\Omega_i} &\leq \beta^{-1} \sup_{\boldsymbol{v}_I \in V_i} \frac{(p_i, \operatorname{div} \boldsymbol{v}_I)_{\Omega_i}}{|\boldsymbol{v}_I|_{1,\Omega_i}} \leq \beta^{-1} \sup_{\boldsymbol{v}_I \in V_i} \frac{a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \boldsymbol{v}_I)}{|\boldsymbol{v}_I|_{1,\Omega_i}} \\ &\leq \beta^{-1} \sup_{\boldsymbol{v}_I \in V_i} \frac{a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i)^{1/2} a_i^{\mu}(\boldsymbol{v}_I, \boldsymbol{v}_I)^{1/2}}{|\boldsymbol{v}_I|_{1,\Omega_i}} \\ &\leq \sqrt{2} \mu_i^{1/2}\, \beta^{-1} a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i)^{1/2} \ . \end{aligned} \tag{45}$$

We now choose the admissible test function $\boldsymbol{v}_I = \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i - \mathcal{H}_i \boldsymbol{u}_i$ in (44) to obtain

$$a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i) + (p_i, \operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i)_{\Omega_i} = a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{H}_i \boldsymbol{u}_i) + (p_i, \operatorname{div} \mathcal{H}_i \boldsymbol{u}_i)_{\Omega_i} \ . \tag{46}$$

We notice that from the second equation of (44) it follows that $(p_i, \operatorname{div} \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i)_{\Omega_i} \geq 0$. Therefore, from (46) we infer

$$\begin{aligned} a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i) &\leq a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{H}_i \boldsymbol{u}_i) + (p_i, \operatorname{div} \mathcal{H}_i \boldsymbol{u}_i)_{\Omega_i} \\ &\leq a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i)^{1/2} a_i^{\mu}(\mathcal{H}_i \boldsymbol{u}_i, \mathcal{H}_i \boldsymbol{u}_i)^{1/2} + \sqrt{3} ||p_i||_{0,\Omega_i} ||\nabla \mathcal{H}_i \boldsymbol{u}_i||_{0,\Omega_i} \\ &\leq \left(\sqrt{2} \mu_i^{1/2}\, a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i)^{1/2} + \sqrt{3} ||p_i||_{0,\Omega_i} \right) ||\nabla \mathcal{H}_i \boldsymbol{u}_i||_{0,\Omega_i} \ . \end{aligned} \tag{47}$$

Estimates (45) and (47) lead to

$$a_i^{\mu}(\mathcal{E}\mathcal{H}_i \boldsymbol{u}_i, \mathcal{E}\mathcal{H}_i \boldsymbol{u}_i)^{1/2} \leq \sqrt{2} \mu_i^{1/2} \left(1 + \sqrt{3}\, \beta^{-1}\right) ||\nabla \mathcal{H}_i \boldsymbol{u}_i||_{0,\Omega_i} \ . \tag{48}$$

The proof is complete. $\square$

Proposition 6.1 *For $1 \leq i \leq N$ we have*

$$a^{\mu}\big(\mathcal{E}\mathcal{H}(R_i^T \boldsymbol{u}_i), \mathcal{E}\mathcal{H}(R_i^T \boldsymbol{u}_i)\big) \leq \omega \tilde{s}_i(\boldsymbol{u}_i, \boldsymbol{u}_i) \qquad \forall \boldsymbol{u}_i \in range(\widetilde{P}_i(I - P_0)) \subset W_i \ , \tag{49}$$

where ω is given by

$$\omega = C \left(1 + \sqrt{3}\, \beta^{-1}\right)^2 C_{ap}^2 \ , \tag{50}$$

where C_{ap} has been defined in (35)*.*

Proof. Setting $\boldsymbol{v}_i = \delta_i \boldsymbol{u}_i$ we first notice that (49) is equivalent to

$$a^{\mu}\big(\mathcal{E}\mathcal{H}(R_i^T(\delta_i^{\dagger} \boldsymbol{v}_i)), \mathcal{E}\mathcal{H}(R_i^T(\delta_i^{\dagger} \boldsymbol{v}_i))\big) \leq \omega s_i(\boldsymbol{v}_i, \boldsymbol{v}_i) \qquad \forall \boldsymbol{v}_i \in range(\delta_i \widetilde{P}_i(I - P_0)) \ . \tag{51}$$

Given any face in $\mathcal{F} \subset \partial\Omega_i$, let $\theta_{\mathcal{F}} \subset W_i$ represent the discrete function which is equal to one at all the nodes *inside* $\mathcal{F}$ and zero at all other nodes of $\partial\Omega_i$. Define analogously $\theta_{\mathcal{E}}$ for all edges $\mathcal{E}$. Moreover, given any vertex $\mathcal{V} \subset \partial\Omega_i$, let $\theta_{\mathcal{V}} \in W_i$ be the discrete function which is equal to one at $\mathcal{V}$ and zero on all other nodes . The sum of all these functions define a partition of unity on $\partial\Omega_i$. We can therefore write

$$\boldsymbol{v}_i = \sum_{\mathcal{F}\subset\partial\Omega_i} \theta_{\mathcal{F}}\boldsymbol{v}_i + \sum_{\mathcal{E}\subset\partial\Omega_i} \theta_{\mathcal{E}}\boldsymbol{v}_i + \sum_{\mathcal{V}\subset\partial\Omega_{i,h}} \theta_{\mathcal{V}}\boldsymbol{v}_i(\mathcal{V}) \ . \tag{52}$$

It is easy to realize that estimate (51) will follow, using Schwarz inequality, if one is able to prove

$$\begin{aligned}
&a^\mu\big(\mathcal{E}\mathcal{H}(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i))\big) \le \omega s_i(\boldsymbol{v}_i,\boldsymbol{v}_i)\\
&a^\mu\big(\mathcal{E}\mathcal{H}(R_i^T(\delta_i^\dagger\theta_{\mathcal{E}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}(R_i^T(\delta_i^\dagger\theta_{\mathcal{E}}\boldsymbol{v}_i))\big) \le \omega s_i(\boldsymbol{v}_i,\boldsymbol{v}_i)\\
&a^\mu\big(\mathcal{E}\mathcal{H}(R_i^T(\delta_i^\dagger\theta_{\mathcal{V}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}(R_i^T(\delta_i^\dagger\theta_{\mathcal{V}}\boldsymbol{v}_i))\big) \le \omega s_i(\boldsymbol{v}_i,\boldsymbol{v}_i)\ ,
\end{aligned} \tag{53}$$

for each face $\mathcal{F}$, edge $\mathcal{E}$, and vertex $\mathcal{V}$ entering in the splitting (52). Once we have Lemma 6.1 at our disposal, the bounds (53) are rather standard (see Section 6.2.3 of Ref. [45]). For completeness, we detail the proof of the estimate involving the *face* term. Let then $\mathcal{F}$ be a face shared by the subdomains Ω_i and Ω_j. Since the support of $R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i)$ is contained in $\overline{\Omega_i \cup \Omega_j}$, it follows that

$$\begin{aligned}
a^\mu\big(\mathcal{E}\mathcal{H}(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i))\big) &= a_i^\mu\big(\mathcal{E}\mathcal{H}_i(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i),\mathcal{E}\mathcal{H}_i(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i)\big)\\
&+ a_j^\mu\big(\mathcal{E}\mathcal{H}_j(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}_j(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i))\big)\ .
\end{aligned} \tag{54}$$

We now treat the second term in the right hand side of (54) Since $\delta_i^\dagger$ takes the *constant* value $\mu_i/(\mu_i+\mu_j)$ at *all* the nodes of $\mathcal{F}$, we infer that

$$\begin{aligned}
&a_j^\mu\big(\mathcal{E}\mathcal{H}_j(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}_j(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i))\big)\\
&\qquad = \frac{\mu_i^2}{(\mu_i+\mu_j)^2} a_j^\mu\big(\mathcal{E}\mathcal{H}_j(R_i^T(\theta_{\mathcal{F}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}_j(R_i^T(\theta_{\mathcal{F}}\boldsymbol{v}_i))\big)\ .
\end{aligned} \tag{55}$$

Using Lemma 6.1 we obtain

$$a_j^\mu\big(\mathcal{E}\mathcal{H}_j(R_i^T(\theta_{\mathcal{F}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}_j(R_i^T(\theta_{\mathcal{F}}\boldsymbol{v}_i))\big) \le \rho_j ||\nabla\mathcal{H}_j(R_i^T(\theta_{\mathcal{F}}\boldsymbol{v}_i))||_{0,\Omega_j}^2\ , \tag{56}$$

where

$$\rho_j = 2\frac{\mu_i^2\mu_j}{(\mu_i+\mu_j)^2}\left(1+\sqrt{3}\,\beta^{-1}\right)^2\ . \tag{57}$$

Therefore, we have

$$\begin{aligned}
a_j^\mu\big(\mathcal{E}\mathcal{H}_j(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}_j(R_i^T(\delta_i^\dagger\theta_{\mathcal{F}}\boldsymbol{v}_i))\big) &\le \rho_j ||\nabla\mathcal{H}_j(R_i^T(\theta_{\mathcal{F}}\boldsymbol{v}_i))||_{0,\Omega_j}^2\\
&\le C\rho_j |R_i^T(\theta_{\mathcal{F}}\boldsymbol{v}_i)|_{1/2,\partial\Omega_j}^2 \le C\rho_j ||\theta_{\mathcal{F}}\boldsymbol{v}_i||_{H_{00}^{1/2}(\mathcal{F})}^2\ .
\end{aligned} \tag{58}$$

Since $\mathcal{F} \subset \partial\Omega_i$, it holds (see Lemma 4.26 of Ref. [45])

$$||\theta_{\mathcal{F}} \boldsymbol{v}_i||^2_{H^{1/2}_{00}(\mathcal{F})} \leq C^2_{ap} ||\boldsymbol{v}_i||^2_{1/2,\partial\Omega_i} \ . \tag{59}$$

From (58)–(59) we infer

$$a^{\mu}_j\big(\mathcal{E}\mathcal{H}_i(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{F}}\boldsymbol{v}_i)), \mathcal{E}\mathcal{H}_i(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{F}}\boldsymbol{v}_i))\big) \leq \rho_j C^2_{ap} ||\boldsymbol{v}_i||^2_{1/2,\partial\Omega_i} \ . \tag{60}$$

Moreover, one has

$$||\boldsymbol{v}_i||^2_{1/2,\partial\Omega_i} \leq C|\boldsymbol{v}_i|^2_{1/2,\partial\Omega_i} \leq C|\mathcal{H}(\boldsymbol{v}_i)|^2_{1,\Omega_i} \tag{61}$$

and

$$|\mathcal{H}(\boldsymbol{v}_i)|^2_{1,\Omega_i} \leq C\mu_i^{-1}\, a^{\mu}_i(\mathcal{E}\mathcal{H}_i(\boldsymbol{v}_i), \mathcal{E}\mathcal{H}_i(\boldsymbol{v}_i)) \leq C\mu_i^{-1}\, s_i(\boldsymbol{v}_i, \boldsymbol{v}_i) \ , \tag{62}$$

where we have used $\boldsymbol{v}_i \in range(\delta_i \widetilde{P}_i(I - P_0))$ in the first estimates of (61) and (62); see Remark 5.1. Therefore, from (61) and (62) we get

$$||\boldsymbol{v}_i||^2_{1/2,\partial\Omega_i} \leq C\mu_i^{-1}\, s_i(\boldsymbol{v}_i, \boldsymbol{v}_i) \ . \tag{63}$$

A combination of (60), (63) and (57) gives

$$\begin{aligned} a^{\mu}_j\big(\mathcal{E}\mathcal{H}_j(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{F}}\boldsymbol{v}_i)),\mathcal{E}\mathcal{H}_j(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{F}}\boldsymbol{v}_i))\big) &\leq \rho_j \mu_i^{-1} C^2_{ap} s_i(\boldsymbol{v}_i, \boldsymbol{v}_i) \\ &\leq \left(1 + \sqrt{3}\,\beta^{-1}\right)^2 C^2_{ap} s_i(\boldsymbol{v}_i, \boldsymbol{v}_i) \ . \end{aligned} \tag{64}$$

A similar computation can be applied to the first term in the right hand side of (54), to eventually obtain

$$\begin{aligned} &a^{\mu}\big(\mathcal{E}\mathcal{H}(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{F}}\boldsymbol{v}_i)), \mathcal{E}\mathcal{H}(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{F}}\boldsymbol{v}_i))\big) \\ &\qquad \leq \left(1 + \sqrt{3}\,\beta^{-1}\right)^2 C^2_{ap} s_i(\boldsymbol{v}_i, \boldsymbol{v}_i) \ . \end{aligned} \tag{65}$$

□

Remark 6.2 *We remark that, still using Lemma 6.1 and the techniques detailed in Ref. [45], one could obtain the improved estimates for the edge and vertex contributions:*

$$\begin{aligned} a^{\mu}\big(\mathcal{E}\mathcal{H}(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{E}}\boldsymbol{v}_i)), \mathcal{E}\mathcal{H}(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{E}}\boldsymbol{v}_i))\big) &\leq \omega^* s_i(\boldsymbol{v}_i, \boldsymbol{v}_i) \\ a^{\mu}\big(\mathcal{E}\mathcal{H}(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{V}}\boldsymbol{v}_i)), \mathcal{E}\mathcal{H}(R_i^T(\delta_i^{\dagger}\theta_{\mathcal{V}}\boldsymbol{v}_i))\big) &\leq \omega^* s_i(\boldsymbol{v}_i, \boldsymbol{v}_i) \ , \end{aligned} \tag{66}$$

with

$$\omega^* = \left(1 + \sqrt{3}\,\beta^{-1}\right)^2 C_{ap} \ .$$

However, the square in the logarithmic term of the face contributions cannot be avoided.

6.2 Upper bounds

In this section we prove an upper bound for our Neumann-Neumann preconditioned operator. It will turn useful to define the bilinear form $\bar{s}(\cdot,\cdot)$ as

$$\bar{s}(\boldsymbol{u},\boldsymbol{v}) = a^\mu(\mathcal{EH}\boldsymbol{u},\mathcal{EH}\boldsymbol{v}) + (\lambda\Pi_{h/H}\,\mathrm{div}\,\mathcal{EH}\boldsymbol{u}\,,\,\Pi_{h/H}\,\mathrm{div}\,\mathcal{EH}\boldsymbol{v}) \qquad \forall\,\boldsymbol{u},\boldsymbol{v}\in W\ , \tag{67}$$

where $\Pi_{h/H}$ denotes the L^2-projection operator:

$$\begin{cases} \Pi_{h/H}\ :\ L^2(\Omega) \longrightarrow Q_{h/H} \\ Q_{h/H} := \left\{ q\in Q\ :\quad \displaystyle\int_{\Omega_i} q = 0 \quad \text{for } i=1,...,N \right\}\ . \end{cases} \tag{68}$$

Before proceeding, we prove the following well-known lemma, for the sake of completeness.

Lemma 6.2 *There exists a square matrix $\mathbb{E}\in\mathbb{R}^{N\times N}$ of components ϵ_{ij}, $i,j=1,2,...,N$, such that*

$$\begin{aligned} &a^\mu\big(\mathcal{EH}(R_i^T\boldsymbol{u}_i),\mathcal{EH}(R_j^T\boldsymbol{u}_j)\big) \\ &\leq \epsilon_{ij}a^\mu\big(\mathcal{EH}(R_i^T\boldsymbol{u}_i),\mathcal{EH}(R_i^T\boldsymbol{u}_i)\big)^{1/2}a^\mu\big(\mathcal{EH}(R_j^T\boldsymbol{u}_j),\mathcal{EH}(R_j^T\boldsymbol{u}_j)\big)^{1/2} \\ &\qquad\qquad\qquad\qquad\qquad \forall\boldsymbol{u}_i\in W_i,\boldsymbol{u}_j\in W_j \end{aligned} \tag{69}$$

with the spectral radius $\rho(\mathbb{E})$ bounded from above independently of h,H.

Proof. The proof is a classical coloring argument and will be presented briefly. Simply, the support of a function $\mathcal{EH}(R_i^T\boldsymbol{u}_i)$, $\boldsymbol{u}_i\in W_i$, is given by the interior of

$$\bigcup_{j\in\Xi_i}\bar{\Omega}_j\ , \qquad \Xi_i = \big\{ j\in\mathbb{N}\,|\,\bar{\Omega}_j\cap\bar{\Omega}_i\neq\emptyset \big\}\ .$$

Therefore, using a Cauchy-Schwarz inequality, it easily follows that there exists a matrix $\mathbb{E}$ which satisfies (69), each row of which has at most k elements equal to 1, and the remaining terms are zeros; the integer k is independent of h,H.

As a consequence it holds

$$\rho(\mathbb{E}) \leq \|\mathbb{E}\|_{L^\infty} \leq k$$

The result is proved. □

We now prove the following Lemma, which is a key point in our analysis.

Lemma 6.3 *It holds*

$$s((I-P_0)\boldsymbol{z},(I-P_0)\boldsymbol{z}) \leq C_1 a^\mu(\mathcal{EH}\boldsymbol{z},\mathcal{EH}\boldsymbol{z}) \qquad \forall\boldsymbol{z}\in V\ , \tag{70}$$

where

$$C_1 = (1+2\beta^{-1})(1+C_0)^2\ ,\quad C_0 = 2\beta_0^{-1}\frac{\mu_{max}}{\mu_{min}}.$$

Proof. Recalling that P_0 is a projection, we immediately have

$$s((I-P_0)\boldsymbol{z},(I-P_0)\boldsymbol{z}) \le s(\boldsymbol{z}-\bar{\boldsymbol{z}},\boldsymbol{z}-\bar{\boldsymbol{z}}) \ , \tag{71}$$

where $\bar{\boldsymbol{z}}$ is any element of $R_0^T W_0$. Let $\bar{\boldsymbol{z}}$ be the solution of the following mixed problem
Find $(\bar{\boldsymbol{z}}, p_0) \in W_0 \times Q_0$ such that

$$\begin{cases} a^\mu(\mathcal{EH}\bar{\boldsymbol{z}},\mathcal{EH}\boldsymbol{v}_0) + (p_0, \operatorname{div}\mathcal{EH}\boldsymbol{v}_0) = 0 & \forall \boldsymbol{v}_0 \in W_0 \\ (\operatorname{div}\mathcal{EH}\bar{\boldsymbol{z}}, q_0) = (\operatorname{div}\mathcal{EH}\boldsymbol{z}, q_0) & \forall q_0 \in Q_0 \ . \end{cases} \tag{72}$$

Then, due to the stability of the coarse problem (property **H4)** in Section 5), $\bar{\boldsymbol{z}}$ is well defined and it holds

$$\|\mu\boldsymbol{\varepsilon}(\mathcal{EH}\bar{\boldsymbol{z}})\|_{0,\Omega}^2 \le C_0 \|\mu\boldsymbol{\varepsilon}(\mathcal{EH}\boldsymbol{z})\|_{0,\Omega}^2 \ , \tag{73}$$

$$C_0 = 2\beta_0^{-1}\frac{\mu_{max}}{\mu_{min}}. \tag{74}$$

As a direct consequence of the second equation in (72), we have

$$\int_{\Omega_i} \operatorname{div}\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}) = 0 \qquad \forall\, i = 1, ..., N \ , \tag{75}$$

which immediately implies (see (68))

$$\Pi_h \operatorname{div}\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}) = \Pi_{h/H}\operatorname{div}\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}) \ . \tag{76}$$

Therefore, one has (see (67))

$$s(\boldsymbol{z}-\bar{\boldsymbol{z}},\boldsymbol{z}-\bar{\boldsymbol{z}}) = \bar{s}(\boldsymbol{z}-\bar{\boldsymbol{z}},\boldsymbol{z}-\bar{\boldsymbol{z}}). \tag{77}$$

By the definition of $\mathcal{EH}$ (see (18)), it holds

$$a^\mu(\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}),\boldsymbol{v}_I) + (\lambda\Pi_h \operatorname{div}\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}), \operatorname{div}\boldsymbol{v}_I) = 0 \quad \forall v_I \in \oplus_{i=1}^N V_i \ . \tag{78}$$

Integrating by parts on each subdomain, it is immediate to obtain

$$(\operatorname{div}\boldsymbol{v}_I, q_0) = 0 \quad \forall q_0 \in Q_0 \ . \tag{79}$$

As a consequence, equation (78) becomes

$$a^\mu(\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}),\boldsymbol{v}_I) + (\lambda\Pi_{h/H} \operatorname{div}\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}), \operatorname{div}\boldsymbol{v}_I) = 0 \quad \forall v_I \in \oplus_{i=1}^N V_i \ . \tag{80}$$

Now, due to the hypothesis **H1)** of Section 4 and equation (80), it follows

$$\begin{aligned} \|\lambda_i\Pi_{h/H}\operatorname{div}\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}})\|_{0,\Omega_i} &\le \beta^{-1}\sup_{v_I\in V_i}\frac{(\lambda_i\Pi_{h/H}\operatorname{div}\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}),\operatorname{div}\boldsymbol{v}_I)}{\|\boldsymbol{\varepsilon}(\boldsymbol{v}_I)\|_{0,\Omega}} \\ = \beta^{-1}\sup_{v_I\in V_i}\frac{a^\mu(\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}),\boldsymbol{v}_I)}{\|\boldsymbol{\varepsilon}(\boldsymbol{v}_I)\|_{0,\Omega}} &\le 2\beta^{-1}\|\mu_i\boldsymbol{\varepsilon}(\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}))\|_{0,\Omega_i} \ , \end{aligned} \tag{81}$$

for all $i = 1, 2, .., N$. Observing that $\Pi_{h/H}$ is a local L^2 projection on each subdomain and due to the definition of $\boldsymbol{\varepsilon}$, it holds

$$\|\Pi_{h/H}\operatorname{div}\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}})\|_{0,\Omega_i} \le 2\|\boldsymbol{\varepsilon}(\mathcal{EH}(\boldsymbol{z}-\bar{\boldsymbol{z}}))\|_{0,\Omega_i} \ , \tag{82}$$

which, combined with (81), easily gives

$$\|\lambda_i^{1/2}\Pi_{h/H}\,\mathrm{div}\,\mathcal{E}\mathcal{H}(\boldsymbol{z}-\bar{\boldsymbol{z}})\|^2_{0,\Omega_i} \leq 4\beta^{-1}\|\mu_i^{1/2}\boldsymbol{\varepsilon}(\mathcal{E}\mathcal{H}(\boldsymbol{z}-\bar{\boldsymbol{z}}))\|^2_{0,\Omega_i}\ . \tag{83}$$

Finally, using (71), (77) and (83) it immediately follows

$$\begin{aligned} &s((I-P_0)\boldsymbol{z},(I-P_0)\boldsymbol{z}) \leq \bar{s}(\boldsymbol{z}-\bar{\boldsymbol{z}},\boldsymbol{z}-\bar{\boldsymbol{z}}) = a^\mu(\boldsymbol{z}-\bar{\boldsymbol{z}},\boldsymbol{z}-\bar{\boldsymbol{z}}) + \\ &\|\lambda^{1/2}\Pi_{h/H}\,\mathrm{div}\,\mathcal{E}\mathcal{H}(\boldsymbol{z}-\bar{\boldsymbol{z}})\|^2_{0,\Omega} \leq 2(1+2\beta^{-1})\|\mu^{1/2}\boldsymbol{\varepsilon}(\mathcal{E}\mathcal{H}(\boldsymbol{z}-\bar{\boldsymbol{z}}))\|^2_{0,\Omega}\ . \end{aligned} \tag{84}$$

The result follows from (73), (84), a simple inequality and observing that by definition

$$2\|\mu^{1/2}\boldsymbol{\varepsilon}(\mathcal{E}\mathcal{H}\boldsymbol{z})\|^2_{0,\Omega} = a^\mu(\mathcal{E}\mathcal{H}\boldsymbol{z},\mathcal{E}\mathcal{H}\boldsymbol{z})$$

□

We can now state the main result of this section:

Theorem 6.2 *For the preconditioned operator P, defined in* (29)*, it holds*

$$s(P\boldsymbol{u},\boldsymbol{u}) \leq C_U\, s(\boldsymbol{u},\boldsymbol{u}) \qquad \forall \boldsymbol{u}\in V\ , \tag{85}$$

where

$$C_U = 1 + C_1\left(1+\sqrt{3}\,\beta^{-1}\right)^2 C_{ap}^2\,\rho(\mathbb{E})\ .$$

The constants C_1 and C_{ap} above are defined in Lemma 6.3 and in (35)*, respectively.*

Proof. Given $\boldsymbol{u}\in V$, let $\tilde{\boldsymbol{u}} = (I-P_0)\boldsymbol{u}$. Recalling the definitions of P_i and $\widetilde{s}_i(\cdot,\cdot)$ (cf. (27)–(28)), and using Proposition 6.1 it follows

$$\begin{aligned} &a^\mu\big(\mathcal{E}\mathcal{H}(P_i\tilde{\boldsymbol{u}}),\mathcal{E}\mathcal{H}(P_i\tilde{\boldsymbol{u}})\big) = a^\mu\big(\mathcal{E}\mathcal{H}(R_i^T\widetilde{P}_i\tilde{\boldsymbol{u}}),\mathcal{E}\mathcal{H}(R_i^T\widetilde{P}_i\tilde{\boldsymbol{u}})\big) \leq \omega\widetilde{s}_i(\widetilde{P}_i\tilde{\boldsymbol{u}},\widetilde{P}_i\tilde{\boldsymbol{u}}) \\ &= \omega s(\tilde{\boldsymbol{u}},R_i^T\widetilde{P}_i\tilde{\boldsymbol{u}}) = \omega s(\tilde{\boldsymbol{u}},P_i\tilde{\boldsymbol{u}})\ . \end{aligned} \tag{86}$$

Let

$$\widehat{P} =: \sum_{i=1}^N P_i\ . \tag{87}$$

Hence, $P = P_0 + (I-P_0)\widehat{P}(I-P_0)$; applying Lemma 6.3 and Lemma 6.2 we now get

$$\begin{aligned} s((I-P_0)\widehat{P}\tilde{\boldsymbol{u}},(I-P_0)\widehat{P}\tilde{\boldsymbol{u}}) &\leq C_1 a^\mu\big(\mathcal{E}\mathcal{H}(\widehat{P}\tilde{\boldsymbol{u}}),\mathcal{E}\mathcal{H}(\widehat{P}\tilde{\boldsymbol{u}})\big) \\ &= C_1\sum_{i,j=1}^N a^\mu\big(\mathcal{E}\mathcal{H}(P_i\tilde{\boldsymbol{u}}),\mathcal{E}\mathcal{H}(P_j\tilde{\boldsymbol{u}})\big) \\ &\leq C_1\sum_{i,j=1}^N \mathbb{E}_{ij}\, a^\mu\big(\mathcal{E}\mathcal{H}(P_i\tilde{\boldsymbol{u}}),\mathcal{E}\mathcal{H}(P_i\tilde{\boldsymbol{u}})\big)^{1/2} a^\mu\big(\mathcal{E}\mathcal{H}(P_j\tilde{\boldsymbol{u}}),\mathcal{E}\mathcal{H}(P_j\tilde{\boldsymbol{u}})\big)^{1/2}\ . \end{aligned} \tag{88}$$

From (88), applying (86) and basic linear algebra it follows

$$\begin{aligned} s((I-P_0)\widehat{P}\tilde{\boldsymbol{u}},(I-P_0)\widehat{P}\tilde{\boldsymbol{u}}) &\leq C_1\sum_{i,j=1}^N \mathbb{E}_{ij}\,\omega\, s(\tilde{\boldsymbol{u}},P_i\tilde{\boldsymbol{u}})^{1/2} s(\tilde{\boldsymbol{u}},P_j\tilde{\boldsymbol{u}})^{1/2} \\ &\leq C_1\omega\,\rho(\mathbb{E})\sum_{i=1}^N s(\tilde{\boldsymbol{u}},P_i\tilde{\boldsymbol{u}}) = C_1\omega\,\rho(\mathbb{E}) s(\tilde{\boldsymbol{u}},\widehat{P}\tilde{\boldsymbol{u}})\ . \end{aligned} \tag{89}$$

Recalling that $\tilde{\boldsymbol{u}} = (I - P_0)\boldsymbol{u}$ and that P_0 is an orthogonal projection with respect to $s(\cdot,\cdot)$, we obtain

$$\begin{aligned} &C_1\omega\, \rho(\mathbb{E}) s(\tilde{\boldsymbol{u}}, \widehat{P}\tilde{\boldsymbol{u}}) = C_1\omega\, \rho(\mathbb{E}) s(\tilde{\boldsymbol{u}}, (I-P_0)\widehat{P}\tilde{\boldsymbol{u}}) \\ &\leq C_1\omega\, \rho(\mathbb{E}) s(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}})^{1/2} s((I-P_0)\widehat{P}\tilde{\boldsymbol{u}}, (I-P_0)\widehat{P}\tilde{\boldsymbol{u}})^{1/2}\ . \end{aligned} \tag{90}$$

Bounds (89) and (90) immediately give

$$s((I-P_0)\widehat{P}\tilde{\boldsymbol{u}}, (I-P_0)\widehat{P}\tilde{\boldsymbol{u}}) \leq (C_1\omega\rho(\mathbb{E}))^2 s(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) \leq (C_1\omega\rho(\mathbb{E}))^2 s(\boldsymbol{u}, \boldsymbol{u})\ , \tag{91}$$

where we also used that P_0 is a projection. Using the Cauchy-Schwarz inequality, we then get

$$s((I-P_0)\widehat{P}(I-P_0)\boldsymbol{u}, \boldsymbol{u}) \leq C_1\omega\rho(\mathbb{E}) s(\boldsymbol{u}, \boldsymbol{u})\ . \tag{92}$$

The final result immediately follows recalling bound (50), the definition of P and that P_0 is a projection. □

6.3 Lower bounds

In this section we prove a lower bound for our Neumann-Neumann preconditioned operator. Differently from the mixed case described in Ref. [18], this result follows straightforwardly from the elliptic theory in Chapter 2 of Ref. [45]. We introduce the following lemma:

Lemma 6.4 *Every $\boldsymbol{u} \in range(I - P_0)$ admits a decomposition*

$$\boldsymbol{u} = \sum_{i=1}^{N} R_i^T \boldsymbol{u}_i\ , \qquad \{\boldsymbol{u}_i \in W_i,\ 1 \leq i \leq N\} \tag{93}$$

that satisfies

$$\sum_{i=1}^{N} \widetilde{s}_i(\boldsymbol{u}_i, \boldsymbol{u}_i) = s(\boldsymbol{u}, \boldsymbol{u})\ . \tag{94}$$

Proof. Let

$$\boldsymbol{u}_i = \delta_i^\dagger R_i \boldsymbol{u}\ . \tag{95}$$

Recalling that for all nodes x in Γ, it holds

$$\sum_{i=1}^{N} R_i^T \delta_i^\dagger(x) = 1\ , \tag{96}$$

it easily follows

$$\sum_{i=1}^{N} R_i^T \boldsymbol{u}_i = \sum_{i=1}^{N} R_i^T (\delta_i^\dagger R_i \boldsymbol{u}) = \boldsymbol{u}\ . \tag{97}$$

Moreover, by definition we have

$$\begin{aligned} &\sum_{i=1}^{N} \widetilde{s}_i(\boldsymbol{u}_i, \boldsymbol{u}_i) = \sum_{i=1}^{N} s_i(R_i^T \delta_i \boldsymbol{u}_i, R_i^T \delta_i \boldsymbol{u}_i) \\ &= s_i\big(R_i^T(\delta_i\delta_i^\dagger R_i\boldsymbol{u}), R_i^T(\delta_i\delta_i^\dagger R_i\boldsymbol{u})\big) = s_i(R_i^T R_i\boldsymbol{u}, R_i^T R_i\boldsymbol{u}) = s(\boldsymbol{u}, \boldsymbol{u})\ . \end{aligned} \tag{98}$$

The proof is complete. $\square$

As proved in the second part of Theorem 2.13 in Ref. [45], from Lemma 6.4 a lower bound for our preconditioned operator easily follows:

Theorem 6.3 *It holds*

$$s(P\boldsymbol{u},\boldsymbol{u}) \geq s(\boldsymbol{u},\boldsymbol{u}) \qquad \forall \boldsymbol{u} \in V \ . \tag{99}$$

7 Numerical results

We discretize our model problem with Q_n spectral elements, subdividing the domain Ω into $\sqrt{N}\times\sqrt{N}$ square subdomains or $N^{1/3}\times N^{1/3}\times N^{1/3}$ cubic subdomains. The auxiliary space Q is defined as the space of $L^2(\Omega)$ functions which are piecewise Q_{n-2}.

The boundary conditions are of homogeneous Dirichlet type. The interface problem (31) is solved iteratively by PCG with our BNN preconditioner, where each spectral element is considered as a subdomain (other choices with many elements per subdomain could be considered as well) and direct solvers are used for the local and coarse problems. The coarse space is

$$W_0^0 + \{\text{quadratic coarse functions}\} \ , \tag{100}$$

where W_0^0 is defined in hypothesis **H3)** of Section 5. We remark that different coarse spaces can be used, as in the mixed case studied in Ref. [18].

The algorithm has been implemented in Matlab, the initial guess is always zero, the right hand side is random and uniformly distributed, and the stopping criterion is $\|r_k\|_2/\|r_0\|_2 \leq 10^{-6}$, where r_k is the residual at the $k-$th iterate.

In Table 1, we consider homogeneous materials with constant Lamé parameters on a domain Ω in the plane. We consider four progressively more incompressible materials (one in each column), with Poisson ratio $\nu = 0.3,\ 0.4,\ 0.49,\ 0.5-10^{-6}$. The upper part of the table shows the results increasing the spectral degree n from 3 to 10 while keeping the number of subdomains $N = 3\times 3$ fixed; the lower part shows the results increasing N from 3×3 to 10×10 while keeping $n = 4$ fixed. In each run, we report the PCG iteration counts and in brackets the maximum eigenvalue of the preconditioned operator P (29) (the minimum eigenvalue, not reported, is always very close to 1). In agreement with our theory (Theorem 6.1), the results show that our BNN algorithm is quasi-optimal, i.e. it depends weakly (polylogarithmically) on the spectral degree n, it is scalable, i.e. it is independent on the number of subdomains N, and it is robust with respect to material properties.

In Table 2, we consider a test on a three-dimensional domain fixing $N = 3\times 3\times 3, n = 5$ and increasing the Poisson's ratio toward the incompressible limit. We report the PCG iteration counts, extreme eigenvalues and condition numbers for both the preconditioned operator P with BNN preconditioner and the unpreconditioned Schur complement S. The results clearly show that while the conditioning of the Schur complement S tends to infinity when the Poisson's ratio tends to 1/2, the conditioning of our BNN preconditioned operator remains bounded.

These results are analogous to the results with a mixed formulation presented in Pavarino and Widlund [38] for the Stokes case and in Goldfeld, Pavarino and Widlund [18] for the mixed elasticity case. In the latter, the interested reader can also find large-scale parallel results with mixed Q_2-Q_0 finite elements and results for heterogeneous materials

Table 1: 2D Elasticity system (homogeneous medium) and spectral elements: PCG iteration counts and maximum eigenvalue of P (in brackets) for the BNN preconditioner

Fixed number of subdomains $N = 3 \times 3$				
n	Poisson ratio ν			
	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.5 - 10^{-6}$
3	8 (2.10)	7 (2.05)	8 (2.03)	8 (2.03)
4	9 (2.57)	9 (2.42)	9 (2.37)	9 (2.38)
5	10 (3.18)	10 (2.98)	10 (2.90)	10 (2.91)
6	11 (3.72)	11 (3.48)	11 (3.45)	12 (3.47)
7	12 (4.25)	11 (3.98)	13 (3.94)	13 (3.96)
8	12 (4.76)	12 (4.45)	13 (4.46)	14 (4.50)
9	13 (5.22)	12 (4.88)	14 (4.89)	14 (4.93)
10	13 (5.68)	13 (5.32)	14 (5.37)	15 (5.42)
Fixed spectral degree $n = 4$				
N	Poisson ratio ν			
	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.5 - 10^{-6}$
9	9 (2.57)	9 (2.42)	9 (2.37)	9 (2.38)
16	10 (2.54)	10 (2.42)	9 (2.38)	9 (2.39)
25	10 (2.60)	10 (2.45)	9 (2.39)	9 (2.40)
36	10 (2.61)	9 (2.45)	9 (2.36)	9 (2.35)
49	10 (2.60)	10 (2.44)	9 (2.35)	9 (2.35)
64	10 (2.59)	10 (2.44)	9 (2.37)	9 (2.38)
81	10 (2.61)	10 (2.45)	9 (2.37)	9 (2.38)
100	10 (2.58)	10 (2.44)	9 (2.38)	9 (2.38)

Table 2: 3D Elasticity system (homogeneous medium) and spectral elements: PCG iteration counts (it.), extreme eigenvalues ($\lambda_{MAX}, \lambda_{min}$) and condition numbers (cond) of both the preconditioned operator P with BNN preconditioner and the unpreconditioned Schur complement S; $N = 3 \times 3 \times 3, n = 5$

Fixed $N = 3 \times 3 \times 3$ and $n = 5$				
	Poisson ratio ν			
	$\nu = 0.3$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.5 - 10^{-6}$
it. BNN	15	15	18	20
it. S	42	43	70	248
$\frac{\lambda_{MAX}}{\lambda_{min}}$(BNN)	$\frac{5.30}{1}$	$\frac{5.42}{1.01}$	$\frac{8.02}{1.01}$	$\frac{14.57}{1.01}$
$\frac{\lambda_{MAX}}{\lambda_{min}}(S)$	$\frac{2.11}{0.047}$	$\frac{2.65}{0.058}$	$\frac{15.87}{0.059}$	$\frac{1.53e+5}{0.059}$
cond(BNN)	5.27	5.40	7.98	14.48
cond(S)	44.07	45.69	267.74	2.5e+6

with discontinuous material parameters across the subdomain interfaces. These results show that the mixed BNN method retains its robusteness independently of the jumps of the material parameters and we conjecture that the same property holds for our positive definite BNN method. Additional large-scale parallel results using finite elements in 2D can be found in Goldfeld Ph.D. thesis [17], where it is also considered a positive definite implementation of the mixed formulation closely related to our BNN method.

8 Appendix – Connection with mixed formulation

Our BNN preconditioner can be interpreted as a positive definite reformulation of the saddle point BNN preconditioner proposed and studied in Ref. [18]. In fact, the first can be obtained from the second by eliminating the coarse pressure p_0 in all the saddle point problems involved. For notation simplicity, in this section we will restrict ourselves to the case μ, λ globally constant functions (as assumed in Ref. [18]).

a) First, we can easily check that the Schur complement matrix S in (31) is equal to the matrix obtained by eliminating the coarse pressure p_0 from the saddle point Schur complement (13) in Ref. [18]. We recall that in order to form these Schur complements, we reorder the vector of unknowns as

$$\begin{bmatrix} \mathbf{u}_I \\ p_I \\ p_0 \\ \mathbf{u}_\Gamma \end{bmatrix} \quad \begin{matrix} \text{interior displacements} \\ \text{interior pressures with zero average} \\ \text{constant pressures in each } \Omega_i. \\ \text{interface displacements} \end{matrix}$$

and partition the discrete system matrix as

$$\begin{bmatrix} \mu A_{II} & B_{II}^T & 0 & \mu A_{\Gamma I}^T \\ B_{II} & -1/\lambda\ C_{II} & 0 & B_{I\Gamma} \\ 0 & 0 & -1/\lambda\ C_0 & B_0 \\ \mu A_{\Gamma I} & B_{I\Gamma}^T & B_0^T & \mu A_{\Gamma\Gamma} \end{bmatrix},$$

where the zero blocks are due to the interior displacements having zero flux across the subdomain boundaries and the interior pressure having a zero average.

In Ref. [18], the interior unknowns $\boldsymbol{u}_I, p_I$ are eliminated by static condensation, obtaining the saddle point Schur complement

$$S_{\mu,\lambda} = \begin{bmatrix} -1/\lambda\ C_0 & B_0 \\ B_0^T & \mu S_{\Gamma,\mu,\lambda} \end{bmatrix},$$

where

$$\mu S_{\Gamma,\mu,\lambda} = \mu A_{\Gamma,\Gamma} - \begin{bmatrix} \mu A_{\Gamma I} & B_{I\Gamma}^T \end{bmatrix} \begin{bmatrix} \mu A_{II} & B_{II}^T \\ B_{II} & -1/\lambda\ C_{II} \end{bmatrix}^{-1} \begin{bmatrix} \mu A_{\Gamma I}^T \\ B_{I\Gamma} \end{bmatrix}$$

(see eqs. (13), (14) in Ref. [18] and take into account that the last two blocks of unknowns $p_0, \boldsymbol{u}_\Gamma$ are swapped). Eliminating the coarse pressure p_0 in this saddle point Schur complement we obtain the positive definite Schur complement

$$S^\star = \mu S_{\Gamma,\mu,\lambda} + \lambda B_0^T C_0^{-1} B_0.$$

Instead, in this paper we eliminate all the interior unknowns $\boldsymbol{u}_I, p_I, p_0$ from the beginning, obtaining the positive definite Schur complement

$$\begin{aligned} S &= \mu A_{\Gamma,\Gamma} - \left[\begin{array}{ccc} \mu A_{\Gamma I} & B_{I\Gamma}^T & B_0^T \end{array}\right] \left[\begin{array}{ccc} \mu A_{II} & B_{II}^T & 0 \\ B_{II} & -1/\lambda\ C_{II} & 0 \\ 0 & 0 & -1/\lambda\ C_0 \end{array}\right]^{-1} \left[\begin{array}{c} \mu A_{\Gamma I}^T \\ B_{I\Gamma} \\ B_0 \end{array}\right] \\ &= \mu A_{\Gamma,\Gamma} - \left[\begin{array}{ccc} \mu A_{\Gamma I} & B_{I\Gamma}^T & B_0^T \end{array}\right] \left[\begin{array}{cc} \left[\begin{array}{cc} \mu A_{II} & B_{II}^T \\ B_{II} & -1/\lambda\ C_{II} \end{array}\right]^{-1} & 0 \\ 0 & -\lambda\ C_0^{-1} \end{array}\right] \left[\begin{array}{c} \mu A_{\Gamma I}^T \\ B_{I\Gamma} \\ B_0 \end{array}\right] \\ &= \mu A_{\Gamma,\Gamma} - \left[\begin{array}{cc} \mu A_{\Gamma I} & B_{I\Gamma}^T \end{array}\right] \left[\begin{array}{cc} \mu A_{II} & B_{II}^T \\ B_{II} & -1/\lambda\ C_{II} \end{array}\right]^{-1} \left[\begin{array}{c} \mu A_{\Gamma I}^T \\ B_{I\Gamma} \end{array}\right] + \lambda B_0^T C_0^{-1} B_0 \\ &= \mu S_{\Gamma,\mu,\lambda} + \lambda B_0^T C_0^{-1} B_0 = S^\star, \end{aligned}$$

hence $S = S^\star$.

b) Analogously, eliminating the coarse pressure p_0 from the saddle point coarse matrix (22) in Ref. [18]

$$\left[\begin{array}{cc} \mu L_0^T S_{\Gamma,\mu,\lambda} L_0 & L_0^T B_0^T \\ B_0 L_0 & -1/\lambda\ C_0 \end{array}\right],$$

we obtain the matrix

$$\mu L_0^T S_{\Gamma,\mu,\lambda} L_0 + \lambda L_0^T B_0^T C_0^{-1} B_0 L_0 = L_0^T S^\star L_0.$$

Since $S = S^\star$ and in our notations $L_0 = R_0^T$, we then see that this matrix is indeed our coarse Schur complement $S_0 = R_0 S R_0^T = L_0^T S^\star L_0$ in (32).

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