Mathematical Models and Methods in Applied Sciences

# AN INTERPOLATION THEORY APPROACH TO SHELL EIGENVALUE PROBLEMS

L. BEIRÃO DA VEIGA
*Dipartimento di Matematica, Università di Milano,*
*Via Saldini 50, I-20133 Milano, Italy*
*E-mail:* `beirao@mat.unimi.it`

C. LOVADINA*
*Dipartimento di Matematica, Università di Pavia,*
*and I.M.A.T.I.–C.N.R.,*
*Via Ferrata 1, I-27100 Pavia, Italy*
*E-mail:* `carlo.lovadina@unipv.it`

Received (Day Month Year)
Revised (Day Month Year)
Communicated by (xxxxxxxxxx)

The asymptotic behaviour of the smallest eigenvalue in linear shell problems is studied, as the thickness parameter tends to zero. In order to cover the widest range of mid-surface geometry and boundary conditions, an abstract approach has been followed, and the Real Interpolation Theory has been used as main tool. A result concerning the ratio between the bending and the total elastic energy is proved. Furthermore, an example of application to cylindrical shells is detailed.

*Keywords*: Koiter shell, Interpolation Theory.

AMS Subject Classification: 22E46, 53C35, 57S20

## 1. Introduction

In studying the eigenvalues for 2-D shell models, one is led to consider a problem of the following type (see Ref. 23 or Ref. 24, for instance):

$$\begin{cases} \text{Find } (u_\varepsilon, \lambda_\varepsilon) \in V \times \mathbb{R} \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \lambda_\varepsilon (u_\varepsilon, v) \quad \forall v \in V \\ ||u_\varepsilon||_H = 1 \ . \end{cases} \tag{1.1}$$

Above, $\varepsilon$ is the thickness parameter, $a^m(\cdot,\cdot)$ is the membrane bilinear form, and $a^b(\cdot,\cdot)$ is the bending bilinear form. Moreover, $V$ is the admissible displacement

---

[*] Corresponding author.

space, which also takes into account the kinematic boundary conditions imposed to the structure, and $H$ is an $L^2$-type space with inner product $(\cdot,\cdot)$. The aim of this paper is to study the asymptotics, as $\varepsilon \to 0^+$, of the *smallest* eigenvalue of problem (1.1), by using the Real Interpolation Theory (see Ref. 12, for example) as our main technical tool. We remark that a detailed knowledge of the asymptotic behavior may help in designing and analysing efficient and reliable numerical schemes based on the finite element technique. Indeed, such an information is often able to identify the roots of possible numerical pathologies, such as *locking* phenomena or *spurious mode occurrence*. This is the case of the shell *source* problem, for which there is a wide literature. We cite here, in a non-exhaustive way, the results detailed in Refs. 1, 2, Refs. 4–10, Refs. 16–19 and Refs. 25–31. However, we notice that a completely satisfactory mathematical analysis of shell elements is still missing.

A brief outline of the paper is as follows. In Section 2 we introduce our abstract framework, as well as the concept of *problem order* for the eigenvalue problem under consideration (see Definition 2.1). Roughly, we say that the eigenvalue problem is of order $\alpha \in \mathbb{R}$ if its smallest eigenvalue $\lambda_\varepsilon$ behaves like $\varepsilon^\alpha$. Section 3 deals with the asymptotic behaviour of $\lambda_\varepsilon$. We remark that in the cases corresponding to the so-called *non-inhibited pure bending shells*, the behaviour of $\lambda_\varepsilon$ is well-known (see Ref. 34). Instead, the cases corresponding to *inhibited pure bending shells* lead to a wide variety of different situations, which may be classified by means of the *problem order*. We also study the behaviour of the energy percentage stored in the bending term (cf. Section 3.2.1). Finally, Section 4 provides an example of application of our results. More precisely, we study a clamped cylindrical shell, and we determine its order. We notice that the same results, together with other several considerations and numerical tests, have been developed in Ref. 11 by using a Fourier expansion technique.

Throughout the paper we will use standard notations for Sobolev norms. Moreover, we will denote with $C$ a generic constant, *independent* of $\varepsilon$, which may take different values in different occurrences.

## 2. Problem setting

In order to incorporate the widest range of shell mid-surface geometry and boundary conditions, it is convenient to face the problem by means of an abstract approach. Accordingly, we suppose that we are given two separable Hilbert spaces $V$ and $H$, with $V \subset H$, together with two bilinear forms:

$$a^m(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R} , \qquad a^b(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R} . \tag{2.1}$$

We make the following assumptions (which are indeed shared by most of the shell models – see Ref. 15 or Ref. 21, for instance).

(1) We assume that the inclusion $V \subset H$ is compact and dense.
(2) The bilinear forms $a^m(\cdot,\cdot)$ and $a^b(\cdot,\cdot)$ are symmetric and continuous on $V$.

(3) The sum $a^m(\cdot,\cdot) + a^b(\cdot,\cdot)$ is coercive on $V$.

For each $\varepsilon \in (0,1]$, we then consider the abstract eigenvalue problem, written in variational form:

$$\begin{cases} \text{Find } (u_\varepsilon, \lambda_\varepsilon) \in V \times \mathbb{R} \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \lambda_\varepsilon (u_\varepsilon, v) \quad \forall v \in V \\ ||u_\varepsilon||_H = 1 \ , \end{cases} \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the inner product of $H$.

From assumptions 1, 2 and 3, it follows that

(1) For each $\varepsilon > 0$, there exists a monotone non-decreasing sequence of real eigenvalues $\left\{\lambda_\varepsilon^{(n)}\right\}_{n=1}^{\infty}$, and corresponding eigenspaces $\Lambda_\varepsilon^{(n)} \subset V$, with $\dim \Lambda_\varepsilon^{(n)} < \infty$. Since we will focus only on the *smallest* eigenvalue, in the sequel $\lambda_\varepsilon^{(1)}$ will be simply denoted by $\lambda_\varepsilon$, and its eigenspace $\Lambda_\varepsilon^{(1)}$ by $\Lambda_\varepsilon$. We also recall that $\lambda_\varepsilon$ is characterized by the Rayleigh quotient (see for instance Ref. 36)

$$\lambda_\varepsilon = \inf_{v \in V} \frac{\varepsilon a^m(v,v) + \varepsilon^3 a^b(v,v)}{||v||_H^2} \ . \tag{2.3}$$

(2) The bilinear form $a^b(\cdot,\cdot)$ is coercive on the space of inextensional displacements (see Ref. 33):

$$K = \{ v \in V \ : \ a^m(v,w) = 0 \quad \forall w \in V \} \ . \tag{2.4}$$

We are interested in studying the real function $\varepsilon \to \lambda_\varepsilon$. By taking $v = u_\varepsilon$ in (2.2), we see that $\lambda_\varepsilon$ is the elastic energy of *any* $u_\varepsilon \in \Lambda_\varepsilon$, with $||u_\varepsilon||_H = 1$:

$$\lambda_\varepsilon = \varepsilon a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^3 a^b(u_\varepsilon, u_\varepsilon) \ . \tag{2.5}$$

We now introduce the following definition.

**Definition 2.1.** We say that the eigenvalue problem (2.2) is of *order* $\alpha$ if

$$\alpha = \inf \left\{ \beta \ : \ \varepsilon^\beta \lambda_\varepsilon^{-1} \in L^\infty(0,1) \right\} \ . \tag{2.6}$$

**Remark 2.1.** Definition 2.1 means that if the eigenvalue problem is of order $\alpha$, then $\alpha$ is the “best” exponent in order to have $\lambda_\varepsilon \sim \varepsilon^\alpha$.

**Remark 2.2.** We notice that, given $v \in V$ with $||v||_H = 1$, from (2.3) and (2.5) we have

$$\lambda_\varepsilon = \varepsilon a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^3 a^b(u_\varepsilon, u_\varepsilon) \le \varepsilon a^m(v,v) + \varepsilon^3 a^b(v,v) \le C\varepsilon ||v||_V^2 \ . \tag{2.7}$$

Furthermore, it holds

$$\begin{aligned}\lambda_\varepsilon = \varepsilon a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^3 a^b(u_\varepsilon, u_\varepsilon) &\geq \varepsilon^3 \left(a^m(u_\varepsilon, u_\varepsilon) + a^b(u_\varepsilon, u_\varepsilon)\right) \\ &\geq C\varepsilon^3 ||u_\varepsilon||_V^2 \geq C\varepsilon^3 \ .\end{aligned} \tag{2.8}$$

Therefore, from (2.7) and (2.8) we deduce that if the eigenvalue problem is of order $\alpha$, then $1 \leq \alpha \leq 3$.

We will also consider the percentage of the elastic energy stored in the bending part. Accordingly, for $0 < \varepsilon \leq 1$ and $u_\varepsilon \in \Lambda_\varepsilon$ with $||u_\varepsilon||_H = 1$, we define the function $R(\varepsilon, u_\varepsilon)$ as

$$R(\varepsilon, u_\varepsilon) := \frac{\varepsilon^3 a^b(u_\varepsilon, u_\varepsilon)}{\lambda_\varepsilon} \ . \tag{2.9}$$

## 3. Asymptotic behaviour of $\lambda_\varepsilon$ and of $R(\varepsilon)$

In order to study the asymptotic behaviour of the shell eigenvalue problem, we distinguish two cases, depending whether the space $K$ defined in (2.4) is reduced to $\{0\}$ or not.

### 3.1. *The case $K \neq \{0\}$: non-inhibited pure bending shells*

The following result is an easy consequence of the theory developed in Ref. 34. However, for the sake of completeness, we sketch its proof.

**Theorem 3.1.** *Suppose that $K \neq \{0\}$. Then there exist constants $C_1$ and $C_2$, independent of $\varepsilon$, such that*

$$C_1\varepsilon^3 \leq \lambda_\varepsilon \leq C_2\varepsilon^3. \tag{3.1}$$

*Therefore, the eigenvalue problem is of order $\alpha = 3$ (cf. Definition 2.1). Furthermore, it holds*

$$\lim_{\varepsilon \to 0^+} R(\varepsilon, u_\varepsilon) = 1 \ . \tag{3.2}$$

*Proof.* We have

$$\begin{aligned}\lambda_\varepsilon = \inf_{v \in V} \frac{\varepsilon a^m(v,v) + \varepsilon^3 a^b(v,v)}{||v||_H^2} &\leq \inf_{v \in K} \frac{\varepsilon a^m(v,v) + \varepsilon^3 a^b(v,v)}{||v||_H^2} \\ = \inf_{v \in K} \frac{\varepsilon^3 a^b(v,v)}{||v||_H^2} = C_2\varepsilon^3 \ .&\end{aligned} \tag{3.3}$$

To continue, we use the coercivity of $a^m(\cdot,\cdot) + a^b(\cdot,\cdot)$ to get

$$\begin{aligned}\lambda_\varepsilon &= \inf_{v\in V}\frac{\varepsilon a^m(v,v)+\varepsilon^3 a^b(v,v)}{||v||_H^2} \geq \inf_{v\in V}\frac{\varepsilon^3\left(a^m(v,v)+a^b(v,v)\right)}{||v||_H^2} \\ &\geq C\varepsilon^3 \inf_{v\in V}\frac{||v||_V^2}{||v||_H^2} = C_1\varepsilon^3 \ .\end{aligned} \tag{3.4}$$

Estimates (3.3) and (3.4) give (3.1). To prove (3.2) we notice that (cf. (2.9))

$$R(\varepsilon,u_\varepsilon)^{-1} = \frac{\lambda_\varepsilon}{\varepsilon^3 a^b(u_\varepsilon,u_\varepsilon)} = 1 + \frac{\varepsilon^{-2}a^m(u_\varepsilon,u_\varepsilon)}{a^b(u_\varepsilon,u_\varepsilon)} \ . \tag{3.5}$$

By using the techniques of Ref. 20, it is easily seen that it holds

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon^{-2}a^m(u_\varepsilon,u_\varepsilon)}{a^b(u_\varepsilon,u_\varepsilon)} = 0 \ . \tag{3.6}$$

Therefore, (3.2) follows from (3.5) and (3.6). □

**Remark 3.1.** We notice that (3.2) is consistent with Proposition 3.1.

### 3.2. *The case $K = \{0\}$: inhibited pure bending shells*

We first notice that in this case $a^m(\cdot,\cdot)$ defines a norm on $V$. We set $W$ as the completion of $V$ with the norm $a^m(v,v)^{1/2} := ||v||_W$. Therefore, we have the dense inclusion $V \subseteq W$. The main technical tool we will use is the Real Interpolation Theory (see Ref. 12, for instance), whic was already employed in Ref. 3 to classify the asymptotic behaviours of the shell *source* problem. We have the following result.

**Theorem 3.2.** *Suppose that $K = \{0\}$. Then, for $0 < \theta < 1$, we have*

$$(V,W)_{1-\theta,1} \subseteq H \quad \textit{if and only if} \quad \varepsilon^{2\theta+1}\lambda_\varepsilon^{-1} \in L^\infty(0,1) \ . \tag{3.7}$$

*Proof.* First, suppose that $\varepsilon^{2\theta+1}\lambda_\varepsilon^{-1} \in L^\infty(0,1)$, and fix a generic $v \in V$. Using (2.3), we get

$$\begin{aligned}||v||_H^2 &\leq \frac{\varepsilon a^m(v,v)+\varepsilon^3 a^b(v,v)}{\lambda_\varepsilon} \leq C\left(\varepsilon^{-2\theta}||v||_W^2 + \varepsilon^{2(1-\theta)}||v||_V^2\right)\varepsilon^{2\theta+1}\lambda_\varepsilon^{-1} \\ &\leq C\left(\varepsilon^{-2\theta}||v||_W^2 + \varepsilon^{2(1-\theta)}||v||_V^2\right) \ .\end{aligned} \tag{3.8}$$

Hence, we have

$$||v||_H \leq C\left(\varepsilon^{-\theta}||v||_W + \varepsilon^{1-\theta}||v||_V\right) \ . \tag{3.9}$$

By taking $\varepsilon := c||v||_W||v||_V^{-1}$ ($c$ suitable to have $\varepsilon \leq 1$), from (3.9) we infer

$$||v||_H \leq C||v||_W^{1-\theta}||v||_V^{\theta} \qquad \forall v \in V \ . \tag{3.10}$$

Furthermore, we obviously have

$$||v||_W = ||v||_W^{1-\theta}||v||_W^{\theta} \leq C||v||_W^{1-\theta}||v||_V^{\theta} \qquad \forall v \in V \ . \tag{3.11}$$

Therefore, from (3.10) and (3.11) we deduce

$$\max(||v||_W, ||v||_H) = ||v||_{W\cap H} \leq C||v||_W^{1-\theta}||v||_V^{\theta} \qquad \forall v \in V \ . \tag{3.12}$$

Since $W \cap H$ is an intermediate space with respect to the couple $(V, W)$, estimate (3.12) implies (see Ref. 12)

$$(V, W)_{1-\theta,1} \subseteq W \cap H \ , \tag{3.13}$$

by which we obtain $(V, W)_{1-\theta,1} \subseteq H$.

Conversely, let $(V, W)_{1-\theta,1} \subseteq H$. Then, we have (see Ref. 12)

$$||v||_H^2 \leq C||v||_W^{2(1-\theta)}||v||_V^{2\theta} \qquad \forall v \in V \ . \tag{3.14}$$

We now notice that for every $\varepsilon \in (0, 1]$ it holds

$$||v||_W^{2(1-\theta)}||v||_V^{2\theta} \leq C\left(\varepsilon^{-2\theta}||v||_W^2 + \varepsilon^{2(1-\theta)}||v||_V^2\right) \qquad \forall v \in V \ , \tag{3.15}$$

so that from (3.14) and using the coercivity of $a^m(\cdot, \cdot) + a^b(\cdot, \cdot)$, we get

$$\begin{aligned} ||v||_H^2 &\leq C\left(\varepsilon^{-2\theta}||v||_W^2 + \varepsilon^{2(1-\theta)}||v||_V^2\right) \\ &\leq C\left(\varepsilon^{-2\theta}a^m(v, v) + \varepsilon^{2(1-\theta)}\left(a^b(v, v) + a^m(v, v)\right)\right) \\ &\leq C\left(\varepsilon^{-2\theta}a^m(v, v) + \varepsilon^{2(1-\theta)}a^b(v, v)\right) \\ &\leq C\varepsilon^{-2\theta-1}\left(\varepsilon a^m(v, v) + \varepsilon^3 a^b(v, v)\right) \qquad \forall v \in V \ . \end{aligned} \tag{3.16}$$

From (3.16) we deduce

$$\varepsilon^{2\theta+1} \leq C\,\frac{\varepsilon a^m(v, v) + \varepsilon^3 a^b(v, v)}{||v||_H^2} \qquad \forall v \in V \ . \tag{3.17}$$

Hence, we have

$$\varepsilon^{2\theta+1} \leq C \inf_{v\in V} \frac{\varepsilon a^m(v, v) + \varepsilon^3 a^b(v, v)}{||v||_H^2} = C\lambda_\varepsilon \ , \tag{3.18}$$

which means $\varepsilon^{2\theta+1}\lambda_\varepsilon^{-1} \in L^\infty(0, 1)$. The proof is complete. □

The following corollary is an immediate consequence of Theorem 3.2 and Definition 2.1.

**Corollary 3.1.** *The order* $\alpha$ *of the eigenvalue problem* (2.2) *is given by*

$$\alpha = \inf \Big\{ 2\theta + 1 \ : \ (V, W)_{1-\theta,1} \subseteq H \Big\} \ . \tag{3.19}$$

□

**Remark 3.2.** We remark that for $0 < \theta < 1$, we have

$$\inf \Big\{ 2\theta + 1 \ : \ (V, W)_{1-\theta,1} \subseteq H \Big\} = \inf \Big\{ 2\theta + 1 \ : \ (V, W)_{1-\theta,p} \subseteq H \Big\} \ , \tag{3.20}$$

for every $p \in [1, +\infty]$. The above equality is a consequence of the standard Interpolation inclusions (see Refs. 12, 29)

$$(V, W)_{1-\eta,p} \subseteq (V, W)_{1-\theta,q} \subseteq (V, W)_{1-\sigma,r}$$

when $0 < 1 - \eta < 1 - \theta < 1 - \sigma < 1$ and $p$, $q$, $r \in [1, +\infty]$. This means that, in determining the problem order, one can decide to work with any $p \in [1, +\infty]$.

### 3.2.1. *On the ratio* $R(\varepsilon, u_\varepsilon)$

In order to study the ratio $R(\varepsilon, u_\varepsilon)$ (cf. (2.9)), we first need the following two lemmata.

**Lemma 3.1.** *For every* $\eta \in (0, 1)$*, the function* $\varepsilon \rightarrow \lambda_\varepsilon$ *is Lipschitz continuous in the interval* $[\eta, 1]$*. Therefore, the function* $\varepsilon \rightarrow \lambda_\varepsilon$ *is absolutely continuous in* $[\eta, 1]$*.*

*Proof.* Fix $\tau, \varepsilon \in [\eta, 1]$. For every $u_\tau \in \Lambda_\tau$ and $u_\varepsilon \in \Lambda_\varepsilon$, with $||u_\tau||_H = ||u_\varepsilon||_H = 1$, we have (see (2.3) and (2.5)):

$$\begin{aligned} \lambda_\tau - \lambda_\varepsilon &= \big(\tau a^m(u_\tau, u_\tau) + \tau^3 a^b(u_\tau, u_\tau)\big) - \big(\varepsilon a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^3 a^b(u_\varepsilon, u_\varepsilon)\big) \\ &\le \big(\tau a^m(u_\varepsilon, u_\varepsilon) + \tau^3 a^b(u_\varepsilon, u_\varepsilon)\big) - \big(\varepsilon a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^3 a^b(u_\varepsilon, u_\varepsilon)\big) \ . \end{aligned} \tag{3.21}$$

Therefore, we get

$$\begin{aligned} \lambda_\tau - \lambda_\varepsilon &\le (\tau - \varepsilon) a^m(u_\varepsilon, u_\varepsilon) + (\tau^3 - \varepsilon^3) a^b(u_\varepsilon, u_\varepsilon) \\ &\le |\tau - \varepsilon| \, a^m(u_\varepsilon, u_\varepsilon) + |\tau^3 - \varepsilon^3| \, a^b(u_\varepsilon, u_\varepsilon) \\ &\le \max\big\{|\tau - \varepsilon|, |\tau^3 - \varepsilon^3|\big\} \big(a^m(u_\varepsilon, u_\varepsilon) + a^b(u_\varepsilon, u_\varepsilon)\big) \ . \end{aligned} \tag{3.22}$$

We now notice that

$$\varepsilon^3 \big(a^m(u_\varepsilon, u_\varepsilon) + a^b(u_\varepsilon, u_\varepsilon)\big) \le \varepsilon a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^3 a^b(u_\varepsilon, u_\varepsilon) = \lambda_\varepsilon \le \lambda_1 \ , \tag{3.23}$$

which implies

$$a^m(u_\varepsilon, u_\varepsilon) + a^b(u_\varepsilon, u_\varepsilon) \le \frac{\lambda_1}{\eta^3} \qquad \forall\, \varepsilon \in [\eta, 1] \ . \tag{3.24}$$

Therefore, from (3.22) and (3.24) we get

$$\lambda_\tau - \lambda_\varepsilon \le \max\left\{|\tau-\varepsilon|, |\tau^3-\varepsilon^3|\right\} \frac{\lambda_1}{\eta^3} \ . \tag{3.25}$$

A similar argument shows that it also holds

$$\lambda_\varepsilon - \lambda_\tau \le \max\left\{|\tau-\varepsilon|, |\tau^3-\varepsilon^3|\right\} \frac{\lambda_1}{\eta^3} \ . \tag{3.26}$$

From (3.25) and (3.26) we obtain

$$|\lambda_\varepsilon - \lambda_\tau| \le \max\left\{|\tau-\varepsilon|, |\tau^3-\varepsilon^3|\right\} \frac{\lambda_1}{\eta^3} \ , \tag{3.27}$$

which implies

$$\frac{|\lambda_\tau - \lambda_\varepsilon|}{|\tau-\varepsilon|} \le \max\left\{\frac{|\tau-\varepsilon|}{|\tau-\varepsilon|}, \frac{|\tau^3-\varepsilon^3|}{|\tau-\varepsilon|}\right\} \frac{\lambda_1}{\eta^3} \ . \tag{3.28}$$

Hence, we obtain

$$\frac{|\lambda_\tau - \lambda_\varepsilon|}{|\tau-\varepsilon|} \le \frac{C(\eta)\lambda_1}{\eta^3} \ . \tag{3.29}$$

The proof is complete. □

**Lemma 3.2.** *It holds*

$$\lambda'_\varepsilon = a^m(u_\varepsilon, u_\varepsilon) + 3\varepsilon^2 a^b(u_\varepsilon, u_\varepsilon) \qquad a.e.\ in\ (0, 1] \ . \tag{3.30}$$

*Therefore,* $\varepsilon \to \lambda_\varepsilon$ *is an increasing function in* $(0, 1]$.

*Proof.* From Lemma 3.1, we deduce that $\lambda_\varepsilon$ is differentiable a.e. in $(0, 1]$. Let $\varepsilon \in (0, 1]$ be a point where the derivative $\lambda'_\varepsilon$ exists. Fix $\tau > \varepsilon$. From the first line of (3.22), we have

$$\lambda_\tau - \lambda_\varepsilon \le (\tau - \varepsilon) a^m(u_\varepsilon, u_\varepsilon) + (\tau^3 - \varepsilon^3) a^b(u_\varepsilon, u_\varepsilon) \ . \tag{3.31}$$

Therefore, it follows

$$\frac{\lambda_\tau - \lambda_\varepsilon}{\tau - \varepsilon} \le a^m(u_\varepsilon, u_\varepsilon) + (\tau^2 + \tau\varepsilon + \varepsilon^2) a^b(u_\varepsilon, u_\varepsilon) \ , \tag{3.32}$$

by which we obtain, letting $\tau \to \varepsilon^+$,

$$\lambda'_\varepsilon \leq a^m(u_\varepsilon, u_\varepsilon) + 3\varepsilon^2 a^b(u_\varepsilon, u_\varepsilon) \ . \tag{3.33}$$

Now, take $\tau < \varepsilon$. Similar arguments show that it holds

$$\frac{\lambda_\tau - \lambda_\varepsilon}{\tau - \varepsilon} \geq a^m(u_\varepsilon, u_\varepsilon) + (\tau^2 + \tau\varepsilon + \varepsilon^2) a^b(u_\varepsilon, u_\varepsilon) \ , \tag{3.34}$$

by which, letting $\tau \to \varepsilon^-$,

$$\lambda'_\varepsilon \geq a^m(u_\varepsilon, u_\varepsilon) + 3\varepsilon^2 a^b(u_\varepsilon, u_\varepsilon) \ . \tag{3.35}$$

Estimates (3.33) and (3.35) imply

$$\lambda'_\varepsilon = a^m(u_\varepsilon, u_\varepsilon) + 3\varepsilon^2 a^b(u_\varepsilon, u_\varepsilon) \ , \tag{3.36}$$

for each point $\varepsilon$ where $\lambda_\varepsilon$ is differentiable. □

**Remark 3.3.** The monotone character of $\lambda_\varepsilon$ can also be deduced by the following argument. Setting $\psi_v(\varepsilon) = \varepsilon a^m(v, v) + \varepsilon^3 a^b(v, v)$ and recalling that

$$\lambda_\varepsilon = \inf_{v \in V} \frac{\varepsilon a^m(v, v) + \varepsilon^3 a^b(v, v)}{||v||_H^2} \ , \tag{3.37}$$

we obtain that

$$\lambda_\varepsilon = \inf_{||v||_H = 1} \psi_v(\varepsilon) \ . \tag{3.38}$$

Since each $\varepsilon \to \psi_v(\varepsilon)$ is an increasing function, it follows from (3.38) that $\varepsilon \to \lambda_\varepsilon$ is monotone non-decreasing.

A straightforward consequence of Lemmata 3.1 and 3.2 is the following corollary.

**Corollary 3.2.** *For every $\eta \in (0, 1)$ and every $\alpha \in \mathbb{R}$, the function $\varepsilon \to \varepsilon^{-\alpha}\lambda_\varepsilon$ is Lipschitz continuous, hence absolutely continuous, in the interval $[\eta, 1]$. Moreover, it holds*

$$\left(\varepsilon^{-\alpha}\lambda_\varepsilon\right)' = (1 - \alpha)\varepsilon^{-\alpha} a^m(u_\varepsilon, u_\varepsilon) + (3 - \alpha)\varepsilon^{2-\alpha} a^b(u_\varepsilon, u_\varepsilon) \ . \tag{3.39}$$

□

We are now ready to prove the following result.

**Proposition 3.1.** *Let the eigenvalue problem (2.2) be of order $\alpha$. Suppose also that there exist*

$$\lim_{\varepsilon\to 0^+}\big(\varepsilon^{-\alpha}\lambda_\varepsilon\big) = l_0 > 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0^+} R(\varepsilon, u_\varepsilon) \geq 0 \ , \tag{3.40}$$

*where* $u_\varepsilon \in \Lambda_\varepsilon$. *Then it holds*

$$\lim_{\varepsilon\to 0^+} R(\varepsilon, u_\varepsilon) = \frac{\alpha-1}{2} \ . \tag{3.41}$$

*Proof.* From (3.39) we have a.e. in $(0,1]$:

$$\begin{aligned}\big(\varepsilon^{-\alpha}\lambda_\varepsilon\big)' &= (1-\alpha)\varepsilon^{-\alpha}a^m(u_\varepsilon,u_\varepsilon) + (3-\alpha)\varepsilon^{2-\alpha}a^b(u_\varepsilon,u_\varepsilon) \\ &= (1-\alpha)\varepsilon^{-\alpha-1}\lambda_\varepsilon + 2\varepsilon^{2-\alpha}a^b(u_\varepsilon,u_\varepsilon) \ .\end{aligned} \tag{3.42}$$

Therefore, we get, a.e.in $(0,1]$:

$$\frac{(\varepsilon^{-\alpha}\lambda_\varepsilon)'}{\varepsilon^{-\alpha}\lambda_\varepsilon} = \frac{1}{\varepsilon}\left(1-\alpha+2\frac{\varepsilon^3 a^b(u_\varepsilon,u_\varepsilon)}{\lambda_\varepsilon}\right) = \frac{1}{\varepsilon}\Big(1-\alpha+2R(\varepsilon,u_\varepsilon)\Big) \ , \tag{3.43}$$

by which

$$\big(\log(\varepsilon^{-\alpha}\lambda_\varepsilon)\big)' = \frac{1}{\varepsilon}\Big(1-\alpha+2R(\varepsilon,u_\varepsilon)\Big) \qquad \text{a.e. in } (0,1] \ . \tag{3.44}$$

Fix now $\eta \in (0,1)$. Noting that $\log(\varepsilon^{-\alpha}\lambda_\varepsilon)$ is absolutely continuous in $[\eta,1]$, since $\varepsilon^{-\alpha}\lambda_\varepsilon$ is so (see Corollary 3.2), we obtain

$$\int_\eta^1 \frac{1}{\varepsilon}\Big(1-\alpha+2R(\varepsilon,u_\varepsilon)\Big)\mathrm{d}\,\varepsilon = \int_\eta^1 \big(\log(\varepsilon^{-\alpha}\lambda_\varepsilon)\big)'\,\mathrm{d}\,\varepsilon = \log\lambda_1 - \log\big(\eta^{-\alpha}\lambda_\eta\big) \ . \tag{3.45}$$

By (3.40) it follows that

$$\lim_{\eta\to 0^+}\log\big(\eta^{-\alpha}\lambda_\eta\big) = \log\, l_0 \in \mathbb{R} \ . \tag{3.46}$$

Therefore, from (3.45) and (3.46), we deduce that there exists the *improper* integral

$$\int_0^1 \frac{1}{\varepsilon}\Big(1-\alpha+2R(\varepsilon,u_\varepsilon)\Big)\mathrm{d}\,\varepsilon := \lim_{\eta\to 0^+}\int_\eta^1 \frac{1}{\varepsilon}\Big(1-\alpha+2R(\varepsilon,u_\varepsilon)\Big)\mathrm{d}\,\varepsilon = \log\left(\frac{\lambda_1}{l_0}\right) \ . \tag{3.47}$$

Now, from (3.47) and (3.40) we infer (3.41). □

**Remark 3.4.** We remark that, in general, the function $\varepsilon \to \lambda_\varepsilon$ is not differentiable at *every* $\varepsilon \in (0,1]$. Here below, we give an example where the set of non-differentiable points for $\lambda_\varepsilon$ is an infinite sequence. Let us consider the following eigenvalue problem:

$$\begin{cases} \text{Find } (u_\varepsilon, \lambda_\varepsilon) \text{ such that:} \\ -\varepsilon\Delta^{-1}u_\varepsilon(x) - \varepsilon^3 {u_\varepsilon}''(x) = \lambda_\varepsilon u_\varepsilon(x) \qquad x \in (0,\pi) \\ u_\varepsilon(0) = u_\varepsilon(\pi) = 0 \ , \end{cases} \tag{3.48}$$

where $\Delta^{-1} : H^{-1}(0,\pi) \to H^1_0(0,\pi)$ denotes the inverse of the second derivative operator. Even though Problem (3.48) does not arise from a shell problem, it fits the abstract framework of Section 2 with the following choices.

- The Hilbert spaces are $V = H^1_0(0,\pi)$ and $H = L^2(0,\pi)$.
- The bilinear forms are given by:

$$a^m(u,v) = -\langle \Delta^{-1}u, v\rangle \ , \qquad a^b(u,v) = \int_0^\pi u'v'\mathrm{d}x \ , \tag{3.49}$$

where $\langle\cdot,\cdot\rangle$ is the duality product between $H^{-1}(0,\pi)$ and $H^1_0(0,\pi)$.

By setting $w_\varepsilon(x) := \Delta^{-1}u_\varepsilon(x)$, we get that Problem (3.48) may be written in the equivalent form

$$\begin{cases} \text{Find } (w_\varepsilon, \lambda_\varepsilon) \text{ such that:} \\ -\varepsilon w_\varepsilon(x) - \varepsilon^3 w_\varepsilon^{(iv)}(x) = \lambda_\varepsilon w''_\varepsilon(x) \qquad x \in (0,\pi) \\ w_\varepsilon(0) = w''_\varepsilon(0) = 0 \\ w_\varepsilon(\pi) = w''_\varepsilon(\pi) = 0 \ . \end{cases} \tag{3.50}$$

A direct computation shows that, for each $\varepsilon \in (0,1]$, we have the eigenvalue sequence:

$$\lambda_\varepsilon(n) = \varepsilon\left(\varepsilon^2 n^2 + \frac{1}{n^2}\right) \ . \tag{3.51}$$

Therefore, the function $\varepsilon \to \lambda_\varepsilon$ returning the *smallest* eigenvalue is easily seen to be:

$$\lambda_\varepsilon = \varepsilon\left(\varepsilon^2 n^2 + \frac{1}{n^2}\right) \qquad \forall\, \varepsilon \in (\varepsilon_n, \varepsilon_{n-1}] \ , \tag{3.52}$$

where $\varepsilon_0 = 1$ and, for $n \geq 1$, $\varepsilon_n = \dfrac{1}{n(n+1)}$. Obviously, $\lambda_\varepsilon$ is differentiable in the open intervals $(\varepsilon_n, \varepsilon_{n-1})$. However, at each $\varepsilon_n$ with $n \geq 1$, the left and right derivatives of $\lambda_\varepsilon$ are given by the two *different* values:

$$D^-\lambda_\varepsilon(\varepsilon_n) = \frac{4n^2+6n+3}{n^2(n+1)^2} \ , \qquad D^+\lambda_\varepsilon(\varepsilon_n) = \frac{4n^2+2n+1}{n^2(n+1)^2} \ . \tag{3.53}$$

It follows that the sequence $\{\varepsilon_n\}_{n\geq 1}$ is the set of non-differentiable points for $\lambda_\varepsilon$.

## 4. An application to cylindrical shells

We consider the eigenvalue problem of a clamped cylindrical shell of thickness $\varepsilon$, radius 1 and length $L$. This problem has been analyzed also in Ref. 11, by means of a Fourier expansion technique. Among the results of that paper, it has been proved and numerically tested that

$$\lambda_\varepsilon \sim \varepsilon^2 \ , \quad \lim_{\varepsilon \to 0^+} R(\varepsilon, u_\varepsilon) = \frac{1}{2} \ . \tag{4.1}$$

In the terminology of the present paper, Equations (4.1) correspond to the choice $\alpha = 2$, a result which we now prove as an application of the theory developed in Section 3. We first describe the cylindrical shell mid-surface by using the axial coordinate $0 \le x \le L$, and the radial coordinate $0 \le y \le 2\pi$ (see Figure 1). Accordingly, the coordinate domain is $\Omega = [0, L] \times [0, 2\pi]$. The displacement field will be denoted by $u = (u_1, u_2, u_3)$, where $u_1$, $u_2$, and $u_3$ are the axial, angular and normal component, respectively. A similar notation will be used for other displacement fields.

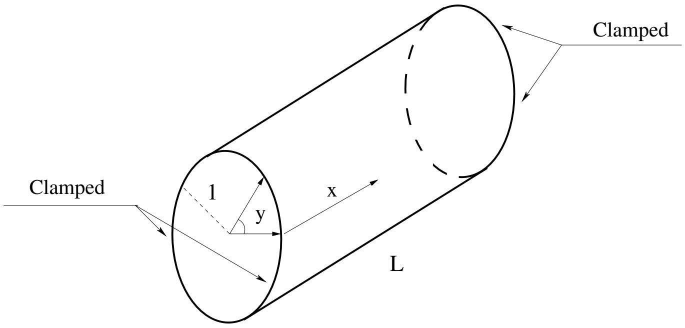

Fig. 1. The Cylindrical shell

We will adopt the Koiter shell model (see Refs. 14, 21, 22, 23, for example), which fits our abstract framework with the following choices (in the case of a clamped cylindrical shell of homogeneous and isotropic material).

- The Hilbert spaces are given by

$$\begin{aligned} V &= H^1_{0,P}(\Omega) \times H^1_{0,P}(\Omega) \times H^2_{0,P}(\Omega) \\ H &= L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \ , \end{aligned} \tag{4.2}$$

  where $H^k_{0,P}(\Omega)$ denotes the space of functions in $H^k(\Omega)$, periodic in the $y$ variable, and vanishing (up to the $k-1$ order derivatives) on the edges $\{0\} \times [0, 2\pi]$ and $\{L\} \times [0, 2\pi]$.
- The bilinear forms are

$$\begin{aligned} a^m(u,v) := \ & k \int_\Omega \Big[ \beta_{11}(u)\beta_{11}(v) + \beta_{22}(u)\beta_{22}(v) + \nu\big(\beta_{11}(u)\beta_{22}(v) \\ & + \beta_{22}(u)\beta_{11}(v)\big) + 2(1-\nu)\beta_{12}(u)\beta_{12}(v)\Big] \mathrm{d}x \ , \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} a^b(u,v) := \ & \frac{k}{12} \int_\Omega \Big[ k_{11}(u)k_{11}(v) + k_{22}(u)k_{22}(v) + \nu\big(k_{11}(u)k_{22}(v) \\ & + k_{22}(u)k_{11}(v)\big) + +2(1-\nu)k_{12}(u)k_{12}(v)\Big] \mathrm{d}x \ . \end{aligned} \tag{4.4}$$

Here, $k = E/[(1+\nu)(1-\nu)]$, where $E > 0$ is the Young modulus, and $0 < \nu < 1/2$ is the Poisson's ratio.
The membrane operators in (4.3) are given by

$$\begin{cases} \beta_{11}(u) = u_{1,x} \ , \\ \beta_{12}(u) = \dfrac{1}{2}(u_{1,y} + u_{2,x}) \ , \\ \beta_{22}(u) = u_{2,y} + u_3 \ , \end{cases} \tag{4.5}$$

where the comma in subscripts indicates a derivative with respect to the variables that follow.
The bending operators in (4.4) are defined by

$$\begin{cases} k_{11}(u) = u_{3,xx} \ , \\ k_{12}(u) = u_{3,xy} - u_{2,x} \ , \\ k_{22}(u) = u_{3,yy} - 2u_{2,y} - u_3 \ . \end{cases} \tag{4.6}$$

- Due to the imposed boundary conditions, it is easily seen that $K = \{0\}$, where $K$ is defined in (2.4). Therefore, we are in the framework of Section 3.2. In the space $W$ (i.e the completion of $V$ with respect to the norm $a^m(\cdot,\cdot)$), we will use the following *equivalent* norm, still denoted by $||\cdot||_W$, with a little abuse of notation.

$$\|u\|_W^2 := \|u_{1,x}\|^2_{L^2(\Omega)} + \|u_{1,y} + u_{2,x}\|^2_{L^2(\Omega)} + \|u_{2,y} + u_3\|^2_{L^2(\Omega)} \simeq a^m(u,u) \ . \tag{4.7}$$

We can now prove the following result (see Definition 2.1).

**Proposition 4.1.**
*The order of the eigenvalue problem for the clamped cylinder shell under consideration is $\alpha = 2$.*

*Proof.* According to Corollary 3.1, we need to show that

$$(V,W)_{1-\theta,1} = (W,V)_{\theta,1} \nsubseteq H \quad \text{if} \quad \theta < 1/2 \ , \tag{4.8}$$

$$(V,W)_{1-\theta,1} = (W,V)_{\theta,1} \subseteq H \quad \text{if} \quad \theta > 1/2 \ . \tag{4.9}$$

*Step 1.* We first prove (4.8). It is sufficient to find a sequence $\{v^{(n)}\}_{n\in\mathbb{N}}$ in V such that

$$\|v^{(n)}\|_H \geq C \qquad\qquad \text{with } C > 0 \tag{4.10}$$

$$\|v^{(n)}\|_{(W,V)_{\theta,1}} \longrightarrow 0 \qquad \text{as } n \to \infty \tag{4.11}$$

for $0 < \theta < 1/2$. Now, let $g \neq 0$ be any fixed function in $C_0^\infty(0,L)$. We then define $v^{(n)} = (v_1^{(n)}, v_2^{(n)}, v_3^{(n)}) \in V$ as:

$$v_1^{(n)} = n^{-2} g'(x) \cos(ny) \tag{4.12}$$

$$v_2^{(n)} = n^{-1} g(x) \sin(ny) \tag{4.13}$$

$$v_3^{(n)} = -g(x) \cos(ny) \ . \tag{4.14}$$

It is easy to check that:

$$\begin{aligned}
\|v^{(n)}\|_H &= \Big( \sum_{i=1,3} \|v_i^{(n)}\|^2_{L^2(\Omega)} \Big)^{1/2} \longrightarrow \|g\|_{L^2(0,L)}\sqrt{\pi} \qquad \text{as } n \to \infty \\
\|v^{(n)}\|_W &= \|v_{1,x}^{(n)}\|_{L^2(\Omega)} = n^{-2}\|g''\|_{L^2(0,L)}\|\cos(ny)\|_{L^2(0,\pi)} \sim n^{-2} \\
\|v^{(n)}\|_V &\sim \|v_{3,yy}^{(n)}\|_{L^2(\Omega)} = n^2\|g\|_{L^2(0,L)}\|\cos(ny)\|_{L^2(0,\pi)} \sim n^2 \ .
\end{aligned} \tag{4.15}$$

Since it holds (see for example Ref. 12):

$$\|v^{(n)}\|_{(W,V)_{\theta,1}} \leq c\|v^{(n)}\|_W^{1-\theta}\|v^{(n)}\|_V^{\theta} \ , \tag{4.16}$$

from (4.15) we infer (4.10) and (4.11), for all $0 < \theta < 1/2$.

*Step 2.* We now prove (4.9). We first show that it holds:

$$W \subseteq L^2(\Omega) \times H^{-1}(\Omega) \times H^{-2}(\Omega) \ . \tag{4.17}$$

Let $v = (v_1, v_2, v_3) \in V$, and recall that (see (4.7))

$$\|v\|_W^2 = \|v_{1,x}\|^2_{L^2(\Omega)} + \|v_{1,y} + v_{2,x}\|^2_{L^2(\Omega)} + \|v_{2,y} + v_3\|^2_{L^2(\Omega)} \ . \tag{4.18}$$

Due to the considered boundary conditions, it is immediate to show that

$$\|v_1\|_{L^2(\Omega)} \leq C\|v_{1,x}\|_{L^2(\Omega)} \leq C\|v\|_W \ . \tag{4.19}$$

Next, we bound $||v_2||_{H^{-1}(\Omega)}$. Consistently with a previous notation, we use the subscript $P$ to denote spaces of $y$-periodic functions. Integrating by parts and using the Cauchy-Schwarz inequality, we get

$$\|v_{1,y}\|_{H^1_P(\Omega)'} = \sup_{\varphi\in C^\infty_P(\Omega)} \frac{\displaystyle\int_\Omega v_{1,y}\,\varphi}{\|\varphi\|_{H^1(\Omega)}} = \sup_{\varphi\in C^\infty_P(\Omega)} \frac{\displaystyle -\int_\Omega v_1\,\varphi_{,y}}{\|\varphi\|_{H^1(\Omega)}} \le \|v_1\|_{L^2(\Omega)} \ . \tag{4.20}$$

Using the triangle inequality and (4.18), from (4.19) and (4.20) we obtain

$$\|v_{2,x}\|_{H^1_P(\Omega)'} \le \|v_{2,x} + v_{1,y}\|_{H^1_P(\Omega)'} + \|v_{1,y}\|_{H^1_P(\Omega)'} \le C\|v\|_W \ . \tag{4.21}$$

Now, for a generic function $\varphi \in C^\infty_P(\Omega)$, let us define $I(\varphi)(x,y) \in C^\infty_P(\Omega)$ as

$$I(\varphi)(x,y) := \int_0^x \varphi(s,y)\,\mathrm{d}s \ . \tag{4.22}$$

We have

$$\begin{cases} I(\varphi)_{,x} = \varphi \ , \\ \|I(\varphi)\|_{H^1(\Omega)} \le C\|I(\varphi)_{,x}\|_{L^2(\Omega)} \le C\|\varphi\|_{H^1(\Omega)} \qquad \forall\varphi \in C^\infty_P(\Omega) \ . \end{cases} \tag{4.23}$$

An integration by parts and (4.23) lead to

$$\begin{aligned} \|v_2\|_{H^{-1}(\Omega)} &\le \sup_{\varphi\in C^\infty_P(\Omega)} \frac{\displaystyle\int_\Omega v_2\,\varphi}{\|\varphi\|_{H^1(\Omega)}} \le C \sup_{\varphi\in C^\infty_P(\Omega)} \frac{\displaystyle\int_\Omega v_2\,I(\varphi)_{,x}}{\|I(\varphi)_{,x}\|_{H^1(\Omega)}} \\ &\le C \sup_{\varphi\in C^\infty_P(\Omega)} \frac{\displaystyle -\int_\Omega v_{2,x}\,I(\varphi)}{\|I(\varphi)\|_{H^1(\Omega)}} \le C\|v_{2,x}\|_{H^1_P(\Omega)'} \ . \end{aligned} \tag{4.24}$$

Therefore, from (4.21) and (4.24) we obtain

$$\|v_2\|_{H^{-1}(\Omega)} \le C||v||_W \ . \tag{4.25}$$

We now bound $||v_3||_{H^{-2}(\Omega)}$. To this aim, we first notice that the same argument as in (4.20) gives

$$\|v_{2,y}\|_{H^{-2}(\Omega)} \le C\|v_2\|_{H^{-1}(\Omega)} \ . \tag{4.26}$$

From (4.25) and (4.26) we get

$$\|v_{2,y}\|_{H^{-2}(\Omega)} \le C\|v\|_W \ . \tag{4.27}$$

Hence, recalling (4.18) and using the triangle inequality, from (4.21) we deduce

$$\|v_3\|_{H^{-2}(\Omega)} \leq \|v_3 + v_{2,y}\|_{H^{-2}(\Omega)} + \|v_{2,y}\|_{H^{-2}(\Omega)} \leq C\|v\|_W \ . \tag{4.28}$$

From estimates (4.19), (4.25), and (4.28) we get

$$\|v_1\|_{L^2(\Omega)} + \|v_2\|_{H^{-1}(\Omega)} + \|v_3\|_{H^{-2}(\Omega)} \leq C\|v\|_W \quad \forall v \in V \ . \tag{4.29}$$

A density argument implies (4.17), i.e.:

$$W \subseteq L^2(\Omega) \times H^{-1}(\Omega) \times H^{-2}(\Omega) \tag{4.30}$$

From (4.30) we immediately get (see for instance Ref. 12):

$$(W, V)_{\theta,1} \subseteq (W, V)_{\theta,2} \subseteq (L^2(\Omega) \times H^{-1}(\Omega) \times H^{-2}(\Omega), V)_{\theta,2} \ . \tag{4.31}$$

Inclusion (4.31), applying well-known results on the interpolation of Sobolev spaces (see for example Ref. 35), gives

$$(W, V)_{\theta,1} \subseteq H^{\theta}(\Omega) \times H^{2\theta-1}(\Omega) \times H^{4\theta-2}(\Omega) \qquad \forall\, \theta \in (0, 1). \tag{4.32}$$

Recalling that $H = L^2(\Omega) \times H = L^2(\Omega) \times H = L^2(\Omega)$, inclusion (4.9) is then a straightforward consequence of (4.32). □

**Acknowledgements.** The authors are grateful to F. Brezzi and G. Gilardi (University of Pavia) for the very useful discussions about the subject of the manuscript. This work has been partly supported by IMATI–CNR of Pavia, Italy.

## References

1. D.N. Arnold and F. Brezzi, Locking-free finite element methods for shells, *Math. Comp.* **66** (1997), 1–14.
2. F. Auricchio, L. Beirão da Veiga and C. Lovadina, Remarks on the asymptotic behaviour of Koiter shells, *Comput. & Structures* **80** (2002) 735–745.
3. C. Baiocchi and C. Lovadina, A shell classification by Interpolation, *Math. Models and Methods Appl. Sci.* **12** (2002) 1359–1380.
4. K.J. Bathe, A. Iosilevich and D. Chapelle, An evaluation of the MITC shell elements, *Comput. & Structures* **75** (2000) 1–30.
5. K.J. Bathe, A. Iosilevich and D. Chapelle, An inf-sup test for shell finite elements, *Comput. & Structures* **75** (2000) 439–456.
6. K.J. Bathe, P.S. Lee and D. Chapelle, A shell problem 'highly sensitive' to thickness changes *Internat. J. Numer. Methods Engrg.* **57** (2003) 1039–1052.
7. L. Beirão da Veiga, Asymptotic energy behavior of two classical intermediate benchmark shell problems, *Math. Models and Methods Appl. Sci.* **13** (2003) 1279–1302.
8. L. Beirão da Veiga, Uniform error estimates for a class of intermediate cylindrical shell problems *Numer. Math.* **96** (2004) 661–689.

9. L. Beirão da Veiga, Asymptotic study of the solution for pinched cylindrical shells, *Comput. Methods Appl. Mech. Engrg.* **194** (2005) 1113–1139.
10. L. Beirão da Veiga and C. Chinosi, Numerical evaluation of the asymptotic energy behavior of intermediate shells with application to two classical benchmark tests, *Computer & Structures* **82** (2004) 525–534.
11. L. Beirão da Veiga, H. Hakula and J. Pitkäranta, An analysis of the eigenvalue problem of clamped cylindrical shells, in preparation.
12. J. Bergh and J. Löfstrom, *Interpolation Spaces: an introduction* (Springer-Verlag, 1976).
13. M. Bernadou and P.G. Ciarlet, Sur l'ellipticité du modèle linéaire de W. T. Koiter, in *Computing Methods and Engineering*, eds. R. Glowinski and J.-L. Lions (Springer-Verlag, 1976).
14. A. Blouza and H. Le Dret, Existence and uniqueness for the linear Koiter model for shells with little regularity, *Quarterly of Applied Mathematics* **57** (1999) 317–337.
15. D. Chapelle and K.J. Bathe, *The Finite Element Analysis of Shells* (Springer–Verlag, 2003).
16. D. Chapelle and K.J. Bathe, Fundamental considerations for the finite element analysis of shell structures, *Comput. & Structures* **66** (1998) 19–36.
17. D. Chapelle and K.J. Bathe, Optimal consistency errors for general shell elements. *C. R. Acad. Sci. Paris Sr. I Math.* **332** (2001) 771–776.
18. D. Chapelle, D.L. Oliveira and M.L. Bucalem, MITC elements for a classical shell model, *Comput. & Structures* **81** (2003) 523–533.
19. D. Chapelle and R. Stenberg, Stabilized finite element formulations for shells in a bending dominated state, *SIAM J. Numer. Anal.* **36** (1999) 32–73.
20. D. Chenais and J.C. Paumier, On the locking phenomenon for a class of elliptic problems, *Numer. Math.* **67** (1994) 427–440.
21. P.G. Ciarlet, *Introduction to Linear Shell Theory* Series in Applied Mathematics (Gauthier-Villars, 1998).
22. P.G. Ciarlet and L. Gratie, A new approach to linear shell theory, *Math. Models and Methods Appl. Sci.* **15** (2005) 1181–1202.
23. Ph. Destuynder, *Modélisation des coques minces élastiques* (Masson, 1990).
24. A.L. Gold'enveizer, V.B. Lidskii and P.E. Tovstik, *Free vibrations of thin elastic shells* (Nauka, Moscow, 1979).
25. V. Havu and J. Pitkřanta, Analysis of a bilinear finite element for shallow shells. I. Approximation of inextensional deformations. *Math. Comp.* **71** (2002) 923–943.
26. V. Havu and J. Pitkřanta, Analysis of a bilinear finite element for shallow shells. II. Consistency error. *Math. Comp.* **72** (2003), 1635–1653.
27. M. Malinen and J. Pitkäranta, A benchmark study of reduced-strain shell finite elements: quadratic schemes, *Internat. J. Numer. Methods Engrg.* **48** (2000) 1637–1671.
28. Y. Leino and J. Pitkäranta, On the membrane locking of *h-p* finite elements in a cylindrical shell problem, *Internat. J. Numer. Methods Engrg.* **37** (1994) 1053–1070.
29. Lions J.-L., Peetre J., *Sur une classe d'espaces d'interpolation, Pubbl. I.H.E.S., 1964, 19, 5-68.*
30. *J. Pitkäranta, The problem of membrane locking in finite element analysis of cylindrical shells. Numer. Math.* ***61*** *(1992) 523–542.*
31. *J. Pitkäranta, Y. Leino, O. Ovaskainen and J. Piila, Shell deformation states and the finite element method: a benchmark study of cylindrical shells. Comput. Methods Appl. Mech. Engrg.* ***128*** *(1995), 81–121.*
32. *E. Sanchez-Palencia, Statique et dynamique des coques minces. I. Cas de flexion pure*

*non inhibée, C. R. Acad. Sci. Paris, Série I* ***309*** *(1989) 411–417.*

33. *E. Sanchez-Palencia, Statique et dynamique des coques minces. II. Cas de flexion pure inhibée - Approximation membranaire, C. R. Acad. Sci. Paris, série I,* ***309*** *(1989) 531–537.*
34. *E. Sanchez-Palencia, Asymptotic and spectral properties of a class of singular-stiff problems, J. Math. Pures Appl.* ***71*** *(1992) 379–406.*
35. *H. Triebel, Theory of Function Spaces II (Birkhäuser, Basel, 1992).*
36. *K. Yosida, Functional Analysis, 6th Edition, (Springer-Verlag, 1980).*