On the Enhanced Strain Technique for Elasticity Problems

C. Lovadina [a,1] and F. Auricchio [b],

[a] *Dipartimento di Matematica, Università di Pavia, Italy*

[b] *Dipartimento di Meccanica Strutturale, Università di Pavia, Italy*

Dedicated to Professor K.J. Bathe

Abstract

The Enhanced Strain Technique is considered in the context of incompressible elasticity problems. A new triangular element is proposed and proved to be convergent and stable when applied to linear analysis. Moreover, a model problem arising from large deformations is presented. Such a model is used to develop some theoretical considerations about the use of Enhanced Strain methods in nonlinear elasticity.

Key words: Incompressible Elasticity, Displacement/Pressure formulation, Finite Element Methods, Enhanced Strain Technique, Nonconforming Methods, Stability Analysis.

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1 Introduction

The aim of this paper is to report some considerations on the Enhanced Strain Technique for elasticity problems, when applied to incompressible materials. The basic idea of such a technique (cf. [22]) consists in enriching the strains arising from the discrete displacement field by means of suitable local modes. In the context of linear elasticity, the resulting element shows better performances than standard schemes, especially with coarse meshes and in the

[1] Corresponding author.
Address: Dipartimento di Matematica, Università di Pavia.
Via Ferrata 1, I-27100, Italy.
Tel.: ++39-0382-505685. Fax: ++39-0382-505602
E-mail: lovadina@dimat.unipv.it

nearly incompressible regime. Also, some theoretical results about the Enhanced Strain philosophy are nowadays available. We wish to mention here the work of Reddy and Simo [18], where a priori error estimates are provided starting from the Hu-Washizu variational principle. Another interesting contribution is due to Braess [3], who deeply investigated the softening effect of strain enhancements. This analysis has been recently improved in [4], where also a posteriori error estimates have been developed. Pantuso and Bathe (cf. [15]) used the Enhanced Strain technique within the framework of the displacement/pressure formulation (cf. [2]). This in order to propose a finite element scheme satisfying the inf-sup condition (cf. [2] and [6]), a feature which is crucial in the incompressible regime. A rigorous mathematical analysis of Pantuso-Bathe element has been developed in [13], where the strain enhancements are interpreted as a kind of *nonconforming* approximation. Motivated by the success obtained in linear analysis, the Enhanced Strain concept has been extended to large deformation analysis (cf. [21], for instance). Unfortunately, it was soon realized that classical enhanced schemes suffered from undesirable unphysical instabilities, especially when applied to strong compression tests (cf. [23]). Much effort has been done in order to explain and, possibly, to cure the occurrence of such numerical spurious modes; here we quote, in a totally non-exhaustive way, References [1], [14], [20], [19]. What is interesting to note is that even the Pantuso-Bathe element, which is, to our knowledge, the only enhanced strain scheme satisfying the inf-sup condition in linear analysis, shows unrealistic instabilities in some large deformation situations (cf. [16]).

In this paper we will contribute to the analysis of the Enhanced Strain Technique for 2-D problems in two directions: firstly, we will present and analyze a new *triangular* element to be used in linear incompressible (plane) elasticity problems; secondly, we will propose a *simple* model problem which can be of some help in understanding why discrete schemes may fail the convergence in nonlinear analysis. The rest of the paper is organized as follows. In section 2 we briefly recall the linear incompressible elasticity problem, together with its classical displacement/pressure formulation. We then introduce the finite element discretization by means of enhanced strains and we review the results obtained in [13] (subsection 2.1). In subsection 2.2 we describe the new element, and we prove an a priori error estimate by using the techniques of [13]. We remark that our element can be regarded as the triangular version of the Pantuso-Bathe element presented in [15]. Also, our enhanced strain shape functions have been already used in [17] in a different context. Section 3 is devoted to a discussion related to the nonlinear case. Due to the mathematical complexity of the general nonlinear elastic problem, our efforts have been addressed to a very simplified model problem, which nonetheless retains some of the difficulties encountered in the finite element discretization. More precisely, we consider the minimization of a family of simple quadratic functionals (dependent on a real parameter λ) under the nonlinear constraint of incompressibility $\det(I + Du) = 1$, Du being the gradient of the displacement field (cf. (24) and (25)). For each value of λ a trivial solution of the associated Euler equations can be easily found. We then proceed to consider the linearized mixed problem around this trivial solution, and we study its stability. In subsection 3.1 we will show that the linearized *continuous* problem satisfies the coercivity-on-the-kernel condition if the parameter λ is chosen in a suitable range of values. On the other hand, in subsection 3.2 we focus our attention on the discrete linearized problem arising from the Pantuso-Bathe element, and we analytically prove that it "resists" to be coercive only in a smaller range of the parameter. Furthermore, in subsection 3.3 we analyze the classical $\mathcal{Q}_2/\mathcal{P}_1$ element (cf. [2]) and show that its range of stability is quite similar to that of the continuous problem. These considerations seem to be in accordance with the numerical results presented in [16]. In section 4 we draw some conclusions and comments. We also wish to remark that several interesting contributions on the theoretical analysis of finite element methods for nonlinear incompressible problems can be found in literature, for example in [12], [11], [10], [5] and [9]. Finally, in the paper we will follow standard notations for functional spaces and their norms (cf. [6], for instance).

2 The linear incompressible elasticity problem

In this section we will consider the incompressible plane elasticity problem in the framework of the infinitesimal theory (cf. [2]). Let the elastic body occupy a regular region Ω in $\mathbf{R}^2$. We are therefore led to solve a problem of the following type:

Find (u, p) such that

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{1}$$

where $u = (u_1, u_2) : \Omega \to \mathbf{R}^2$ is the displacement field, $p : \Omega \to \mathbf{R}$ is the pressure field and $f = (f_1, f_2) : \Omega \to \mathbf{R}^2$ is the loading term.

Remark 1 *For simplicity, we are considering only homogeneous boundary conditions for the displacement field along the whole $\partial\Omega$, but the results of this section can be easily extended to other more realistic situations.*

A standard variational formulation of problem (1) consists in finding $(u, p) \in$

$V \times P = (H_0^1(\Omega))^2 \times L^2(\Omega)/\mathbf{R}$ which solves the system

$$\begin{cases} \int_\Omega D^s u : D^s v - \int_\Omega p \operatorname{div} v = \int_\Omega f \cdot v & \forall v \in V \\ \\ \int_\Omega q \operatorname{div} u = 0 & \forall q \in P \ , \end{cases} \tag{2}$$

where D^s denotes the usual symmetric gradient operator acting on vector fields. It is well-known that problem (2) is well-posed and it fits into the framework of the classical saddle-point theory, extensively studied in [6], for instance.

2.1 Enhanced strain discretization

The finite element analysis of problem (2), as it is now well-established (cf. [2], [6]), requires a suitable choice of the discretization spaces, in order to properly deal with the divergence-free constraint. Several methods have been proposed, analyzed and proved to be efficient in actual computations.

Among them, there are the ones based on the so-called *Enhanced Strain Technique*, whose basic idea is briefly recalled below. As usual, given a regular (triangular or quadrilateral) mesh $\mathcal{T}_h$ of Ω, h being the mesh-size, we choose a finite element space $V_h \subset V$ for the approximation of the displacements, and a finite element space $P_h \subset P$ for the pressure field. A *conforming* mixed method is thus given by the discrete problem

Find $(u_h, p_h) \in V_h \times P_h$ such that

$$\begin{cases} \int_\Omega D^s u_h : D^s v - \int_\Omega p_h \operatorname{div} v = \int_\Omega f \cdot v & \forall v \in V_h \\ \\ \int_\Omega q \operatorname{div} u_h = 0 & \forall q \in P_h \end{cases} \tag{3}$$

An *Enhanced Strain Method* in this mixed context can be seen as a *nonconforming* scheme (cf. [13]), for which the strains arising from the displacements are "enriched" by means of some additional modes. Therefore, we are led to introduce a further finite element space S_h of symmetric tensors, and we solve the problem

Find $(u_h, H_h; p_h) \in (V_h \times S_h) \times P_h$ such that

$$\begin{cases} \int_\Omega (D^s u_h + H_h) : (D^s v + R) - \int_\Omega p_h(\operatorname{div} v + \mathrm{tr} R) = \int_\Omega f \cdot v \\ \\ \int_\Omega q(\operatorname{div} u_h + \mathrm{tr} H_h) = 0 \end{cases} \tag{4}$$

for every $(v, R) \in V_h \times S_h$ and for every $q \in P_h$. Above and in the sequel, we denote with "tr" the trace operator acting on tensors. Typically, the space S_h of strain enhancement consists of functions for which *no continuity* is required across the elements of the mesh. The advantage of introducing these additional degrees of freedom is clearly seen: enlarging the space of symmetric gradients of the displacement results in a relaxation of the constraint (cf. second equation of (4)), possibly improving the stability feature of the method at hand. Moreover, since the enhanced strains are discontinuous across the elements, a static condensation procedure can be performed element by element, so that the computational efforts are not (generally) dramatically increased. Of course, not every choice of V_h, S_h and P_h leads to an efficient method: introducing the notation

$$b_h(v_h, R_h; q_h) = -\int_\Omega q_h(\operatorname{div} v_h + \mathrm{tr} R_h) \ , \tag{5}$$

for every $(v_h, R_h; q_h) \in (V_h \times S_h) \times P_h$, it has been proved in [13] that, in order to have a reliable scheme, one should choose the discretization spaces satisfying the following three conditions.

- *The discrete inf-sup condition*: there exists a positive constant β, independent of h, such that

$$\inf_{q_h \in P_h} \sup_{(v_h, R_h) \in V_h \times S_h} \frac{b_h(v_h, R_h; q_h)}{||(v_h, R_h)|| \, ||q_h||_P} \geq \beta \ ; \tag{6}$$

  Above, we have set

$$||(v_h, R_h)|| = \left(|D^s v_h|_0^2 + |R_h|_0^2 \right)^{1/2} \ . \tag{7}$$

- *The minimum angle condition*: there exists a constant $\theta < 1$, independent of h, such that

$$\sup_{(v_h, R_h) \in V_h \times S_h} \frac{\int_\Omega D^s v_h : R_h}{|D^s v_h|_0 \, |R_h|_0} \leq \theta \ ; \tag{8}$$

This condition implies, in particular, that $D^s(V_h) \cap S_h = (0)$. Moreover, the spaces $D^s(V_h)$ and S_h "stays far from being parallel" uniformly in h. We remark that condition (8) has been recognized to be crucial for the analysis of enhanced strain methods also in [3].

- *The consistency condition*: the enhanced strain modes should have zero mean value on each element, i.e. for every $K \in \mathcal{T}_h$

$$\int_K R_h = 0 \qquad \forall R_h \in S_h \ . \tag{9}$$

This condition allows to control the error arising from the introduction of the *nonconforming* space S_h.

2.2 A triangular version of Pantuso-Bathe element

Aim of this subsection is to introduce and analyze a new finite element method based on the Enhanced Strain Technique described in subsection 2.1. Our scheme can be seen as the *triangular* version of a quadrilateral element introduced by Pantuso-Bathe in [15], the latter theoretically proved to be efficient in [13]. We thus suppose to have a regular triangular mesh $\mathcal{T}_h$ of Ω. In Ω, let us define a global Cartesian system of coordinates (x, y). For each triangle $T \in \mathcal{T}_h$, let (x_T, y_T) be the coordinates of its barycenter, and define on T a local Cartesian system of coordinates by simply setting

$$\begin{cases} \bar{x} = x - x_T \\ \\ \bar{y} = y - y_T \ . \end{cases} \tag{10}$$

We are now ready to introduce the finite element spaces for the discretization of problem (4): for the displacements we choose

$$V_h = \Big\{ v_h \in V \ : \ v_{h|T} \in \mathcal{P}_1(T) \qquad \forall T \in \mathcal{T}_h \Big\} \ , \tag{11}$$

where $\mathcal{P}_1(T)$ is the space of linear functions defined on T. For the pressure interpolation, we set

$$P_h = \Big\{ q_h \in H^1(\Omega) \ : \ q_{h|T} \in \mathcal{P}_1(T) \qquad \forall T \in \mathcal{T}_h \Big\} \ . \tag{12}$$

We notice that, as for the Pantuso-Bathe element described in [15], the discrete pressure field is *continuous*. Finally, the enhanced strain modes are chosen to be

$$S_h = \Big\{ R_h \in (L^2(\Omega))^4_s \ : \ R_{h|T} \in E_4(T) \qquad \forall T \in \mathcal{T}_h \Big\} \ , \tag{13}$$

where $E_4(T)$ is the space of tensor-valued functions defined on T, spanned by the following shape functions (cf. (10))

$$\begin{bmatrix} \alpha_1\bar{x} + \alpha_2\bar{y} \ ; & (\alpha_2 - \alpha_4)\bar{x} + (\alpha_3 - \alpha_1)\bar{y} \\ \text{symm.} \quad ; & \alpha_3\bar{x} + \alpha_4\bar{y} \end{bmatrix} \qquad \text{with } \alpha_i \in \mathbf{R} \ . \tag{14}$$

Remark 2 *We remark that the enhanced strain modes described in (14) have already been used in [17]. Moreover, another set of enhanced modes, still with four degrees of freedom per element, can be chosen: namely, we could take* $\tilde{E}_4(T)$ *as the space spanned by*

$$\begin{bmatrix} \alpha_1\bar{x} & ; \alpha_2\bar{x} + \alpha_3\bar{y} \\ \textit{symm.} \ ; & \alpha_4\bar{y} \end{bmatrix} \qquad \textit{with } \alpha_i \in \mathbf{R} \ . \tag{15}$$

It is easily seen that the subsequent analysis remains valid also for $\tilde{E}_4(T)$*. However, this element is not frame invariant. A strategy to make the results at least independent of the user's input data is detailed in [17].*

Remark 3 *It has been shown in [18] that the strain enhancement is degenerate for triangles. However, the considerations of [18] apply for displacement-based methods. The displacement/pressure formulation used in this paper opens the possibility to design enhanced strain schemes also for triangular elements. A brief discussion about this point is provided in appendix.*

We can now prove the following proposition.

Proposition 4 *The choice (11)–(14) satisfies conditions (6)–(9).*

Proof. First of all, notice that by construction condition (9) is satisfied. Moreover, take $v_h \in V_h$ and $R_h \in S_h$. The symmetric gradient $D^s v_h$ is constant on each $T \in \mathcal{T}_h$, so that

$$\int_\Omega D^s v_h : R_h = 0 \ . \tag{16}$$

It turns out that condition (8) holds, with $\theta = 0$. It remains to verify the inf-sup condition (6). To this end, we will use Fortin's criterion. In our context, we thus want to build a linear operator $\Pi_h : V \to V_h \times S_h$ such that

$$||\Pi_h v|| \leq C||v||_V \qquad \forall v \in V \tag{17}$$

and

$$\int_\Omega q_h \,\mathrm{div}\, v = b_h(\Pi_h v; q_h) \qquad \forall v \in V \ , \ \forall q_h \in P_h \ . \tag{18}$$

We first define $\Pi_1 : V \to V_h$ as the standard Clément's operator (cf. [8]). Next, we search for a linear operator $\Pi_2 : V \to V_h \times S_h$ such that

$$\int_\Omega q_h \,\mathrm{div}\, v = b_h(\Pi_2 v; q_h) \qquad \forall v \in V \ , \ \forall q_h \in P_h \ . \tag{19}$$

For our purposes, it suffices to set $\Pi_2 v = (0, L_h) \in V_h \times S_h$, i.e. the operator Π_2 is essentially valued only on S_h. In fact, since the pressure interpolation is continuous, from (19) we see that we need to solve:

Given $v \in V$, find $L_h \in S_h$ such that

$$\int_\Omega q_h \,\mathrm{tr} L_h = -\int_\Omega \nabla\, q_h \cdot v \qquad \forall q_h \in P_h \ . \tag{20}$$

This can be seen as a finite dimensional linear system, whose unknown is L_h. Thus, problem (20) is solvable (for all $v \in V$) if and only if the adjoint problem is injective. Hence, we have to show that if $q_h \in P_h$ satisfies

$$\int_\Omega q_h \,\mathrm{tr} R_h = 0 \qquad \forall R_h \in S_h \ , \tag{21}$$

then $q_h = 0$ in P_h.

Arguing locally on each $T \in \mathcal{T}_h$, we find that if $q_h \in P_h$ satisfies (21), then $q_{h|T}$ is constant in T (cf. (12) and (14)). Since the pressure interpolation is *continuous* across the elements, it follows that q_h is indeed a *global* constant on Ω, i.e. $q_h = 0$ in P_h (because $P_h \subset P = L^2(\Omega)/\mathbf{R}$). As a consequence, problem (20) is solvable. Taking the solution of minimum norm to problem (20), a standard scaling argument shows that

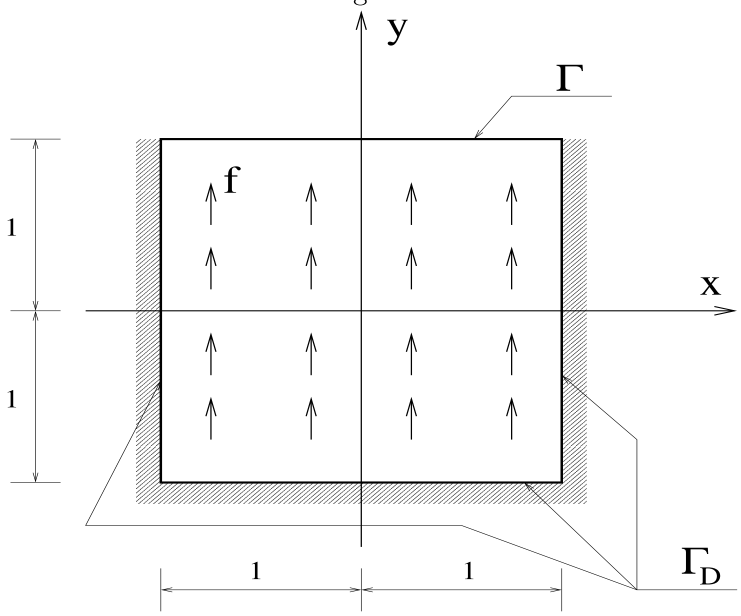
$$||\Pi_2 v|| = |L_h|_0 \leq C h^{-1} |v|_0 \ . \tag{22}$$

Finally, if we set

$$\Pi_h v = \Pi_2(v - \Pi_1 v) \qquad \forall v \in V \ ,$$

it is easily seen that Π_h verifies (17) and (18). The proof is then complete. $\square$

Fig. 1.



As a consequence, applying the theory detailed in [13], we obtain

Proposition 5 *For the scheme detailed by (11)–(14) and provided that the solution of the continuous problem is sufficiently regular, we have the error estimate*

$$||u - u_h||_V + ||p - p_h||_P \leq Ch \ . \tag{23}$$

3 A model problem for nonlinear incompressible elasticity

In this section we present a *simple* model problem which nonetheless shows some of the difficulties arising in general nonlinear elastic problems for incompressible materials. In the sequel we consider the square $\Omega = (-1, 1) \times (-1, 1)$ endowed with the usual Cartesian coordinates (x, y), and we denote with $\Gamma = [-1, 1] \times \{1\}$ the upper part of its boundary, while the remaining part of $\partial\Omega$ is denoted with Γ_D (cf. Fig. 1).

Our starting point is to find the stationary points of the following functional:

$$E(u) =: \frac{1}{2} \int_\Omega |Du|^2 - \lambda \int_\Omega f \cdot u \ , \tag{24}$$

where D is the tensorial gradient operator and the vector field f will have the particular shape $f = (0, 1)$ (i.e. we are considering vertical loads). Above, λ is a real parameter and the critical points have to be searched for in the set

$$\tilde{V}_I = \left\{ v \in \tilde{V} \ : \ \det(I + Du) = 1 \right\} \ , \tag{25}$$

where $\tilde{V}$ is a space of sufficiently regular functions v vanishing on Γ_D. We point out that, since the determinant is a nonlinear function, V_I is not a vector space, but it could be regarded as a sub-manifold of $\tilde{V}$. Moreover, in the rest of the paper we will denote with $C[F]$ the tensor of the cofactors of F, for any generic tensor F. Introducing a Lagrange multiplier $\hat{p}$ (in a suitable space $\tilde{P}$) for the incompressibility constraint and taking variations, we are led to solve the following system:

Find $(\hat{u}, \hat{p}) \in \tilde{V} \times \tilde{P}$ such that:

$$\begin{cases} \displaystyle\int_\Omega D\hat{u} : Dv - \int_\Omega \hat{p}\, C[I + D\hat{u}] : Dv = \lambda \int_\Omega f \cdot v & \forall v \in \tilde{U} \\ \displaystyle\int_\Omega q \left(\det(I + D\hat{u}) - 1\right) = 0 & \forall q \in \tilde{P} \ , \end{cases} \tag{26}$$

where $\tilde{U}$ is an appropriate space of variations for the displacement field. It is not our intention to rigorously specify the regularity needed for the space involved in variational formulation (26), and we refer to [12] and [11] for details on such a point. Instead, we wish to notice that system (26) constitutes a set of *nonlinear* equations for which a trivial solution can be easily found *for every* $\lambda \in \mathbf{R}$, i.e. $(\hat{u}, \hat{p}) = (0, \lambda(y - 1))$.

Remark 6 *We are not claiming that, for each $\lambda \in \mathbf{R}$, $(\hat{u}, \hat{p}) = (0, \lambda(y - 1))$ is the only solution of the system.*

Since any numerical approximation of problem (26) requires a linearization of the system itself, we fix our attention on the linearized problem around any solution $(\hat{u}, \hat{p})$. We are thus led to consider the problem

Find $(u, p) \in \tilde{V} \times \tilde{P}$ such that:

$$\begin{cases} \displaystyle\int_\Omega Du : Dv - \int_\Omega \hat{p}\, C[Du] : Dv - \int_\Omega p\, C[I + D\hat{u}] : Dv = \mathcal{F}(v) \\ \displaystyle\int_\Omega qC[I + D\hat{u}] : Du = 0 \ . \end{cases} \tag{27}$$

for every $v \in \tilde{U}$ and every $q \in \tilde{P}$. If the solution $(\hat{u}, \hat{p})$ is chosen to be the trivial one, system (27) simplifies to the parameter-depending problem:

Find $(u,p) \in \tilde{V} \times \tilde{P}$ such that:

$$\begin{cases} \int\limits_\Omega Du : Dv - \lambda \int\limits_\Omega r\, C[Du] : Dv - \int\limits_\Omega p \ \mathrm{div}\, v = \mathcal{F}(v) \qquad \forall v \in \tilde{U} \\ \\ \int\limits_\Omega q\, \mathrm{div}\, u = 0 \qquad\qquad \forall q \in \tilde{P} \ , \end{cases} \tag{28}$$

where we have set $r = r(x,y) = y - 1$. Letting

$$V = \tilde{U} = \tilde{V} = \left\{ v \in H^1(\Omega)^2 \ : \ v_{|\Gamma_D} = 0 \right\} \ ; \qquad P = \tilde{P} = L^2(\Omega) \ , \tag{29}$$

we consider system (28) as a *model problem* for our subsequent considerations. Problem (28) has a clear structure: if one introduces the forms

$$a_\lambda(F,G) =: \int\limits_\Omega F : G - \lambda \int\limits_\Omega r\, C[F] : G \tag{30}$$

for tensor fields $F, G \in L^2(\Omega)^4$, and

$$b(v,q) =: -\int\limits_\Omega q \ \mathrm{div}\, v \ , \tag{31}$$

then problem (28) can be written as:

Find $(u,p) \in V \times P$ such that:

$$\begin{cases} a_\lambda(Du, Dv) + b(v,p) = \mathcal{F}(v) \qquad \forall v \in V \\ \\ b(u,q) = 0 \qquad\qquad \forall q \in P \ . \end{cases} \tag{32}$$

We notice that, since $C[F] : G = F : C[G]$, the form $a_\lambda(\cdot,\cdot)$ results to be symmetric. Furthermore, we are clearly facing a typical (parameter-depending) saddle-point problem. As it is well-established (cf. [6]), the crucial properties for the well-posedness are, together with continuity:

- *the inf-sup condition*: there exists a positive constant β such that

$$\inf_{q \in P} \sup_{v \in V} \frac{b(v,q)}{||v||_V ||q||_P} \geq \beta \ ; \tag{33}$$

- *the invertibility on the kernel condition*: there exists a positive constant $\alpha(\lambda)$ such that

$$\inf_{v_0 \in Ker\,B} \sup_{u_0 \in Ker\,B} \frac{a_\lambda(Du_0, Dv_0)}{||u_0||_V ||v_0||_V} \geq \alpha(\lambda) \ , \tag{34}$$

where

$$Ker\,B = \{v_0 \in V \ : \ b(v_0, q) = 0 \qquad \forall q \in P\} \ . \tag{35}$$

As far as the inf-sup condition is concerned, it is a classical result that it holds for the divergence operator. We therefore focus our attention on condition (34). In particular, we will show that the form $a_\lambda(\cdot,\cdot)$ is *coercive* on $Ker\,B$ whenever λ stays in a suitable range of values. We thus expect that within such choices of the parameter, for the *continuous* problem the trivial solution $(\hat{u}, \hat{p}) = (0, \lambda r)$ is unique and stable. On the other hand, we will show that for a discretization based on an Enhanced Strain Technique, the range of stability is smaller (even though the scheme at hand satisfies the inf-sup condition).

3.1 The stability range

We now investigate on the coercivity on $Ker\,B$ of $a_\lambda(\cdot,\cdot)$. More precisely, we would like to find conditions on λ assuring that there exists a $c(\lambda) > 0$ such that:

$$\int_\Omega Dv : Dv - \lambda \int_\Omega r\, C[Dv] : Dv \geq c(\lambda) |Dv|_0^2 \qquad \forall v \in Ker\,B \ . \tag{36}$$

Using Piola's identity (cf. [7]), an integration by parts gives

$$\int_\Omega Dv : Dv - \lambda \int_\Omega r\, C[Dv] : Dv = \int_\Omega Dv : Dv + \lambda \int_\Omega \nabla r \cdot C[Dv]^T v \ . \tag{37}$$

Above, the boundary integral arising from integration by parts disappears because of the boundary conditions on v, and because on Γ the function $r = y - 1$ vanishes. To continue, we notice that, since v is divergence-free, it holds $C[Dv]^T = -Dv$. Hence, a further integration by parts and using $\operatorname{div} v = 0$, leads to

$$\lambda \int_\Omega \nabla r \cdot C[Dv]^T v = \lambda \left(\int_\Omega H(r) v \cdot v - \int_\Gamma (v \cdot \nabla r)(v \cdot n) \right) \ , \tag{38}$$

where n is the outward normal vector and $H(r)$ is the Hessian matrix of the function r. But $r = y - 1$ is linear, hence $H(r) = 0$. On the other hand, on the boundary Γ we have $\nabla r = n$, so that we deduce that

$$\lambda \int_\Omega \nabla r \cdot C[Dv]^T v = -\lambda \int_\Gamma (v \cdot n)^2 \ . \tag{39}$$

From (36), (37) and (39) we conclude that our form $a_\lambda(\cdot,\cdot)$ will be coercive on $Ker\, B$ if there exists a $c(\lambda) > 0$ such that:

$$\int_\Omega Dv : Dv - \lambda \int_\Gamma (v \cdot n)^2 \geq c(\lambda)|Dv|_0^2 \qquad \forall v \in Ker\, B \ . \tag{40}$$

By (40) we first infer that for $\lambda \leq 0$ we can simply take $c(\lambda) = 1$. Furthermore, setting

$$\alpha_M = \sup_{v \in Ker\, B} \frac{\int_\Gamma (v \cdot n)^2}{|Dv|_0^2} > 0 \ , \tag{41}$$

we see that condition (40) still holds whenever

$$\lambda < \frac{1}{\alpha_M} \ . \tag{42}$$

We give now an estimate of α_M by establishing the following

Proposition 7 *Suppose that $\Omega = (-1,1) \times (-1,1)$. Then $\alpha_M \leq 1$.*

Proof. First take $v = (v_1, v_2) \in (Ker\, B) \cap C^1(\bar{\Omega})$. Writing

$$v_2(x,1) = \int_{-1}^{1} v_{2,y}(x,y)dy \ ,$$

(above we used that $v_2(x,-1) = 0$), Cauchy-Schwarz inequality gives

$$\int_\Gamma |v_2|^2 \leq 2 \int_\Omega |v_{2,y}|^2 \ . \tag{43}$$

Since $\operatorname{div} v = 0$, we get

$$\int_\Gamma |v_2|^2 \leq \int_\Omega |v_{2,y}|^2 + \int_\Omega |v_{1,x}|^2 \leq |Dv|_0^2 \ . \tag{44}$$

Noting that on Γ we have $v \cdot n = v_2$, we obtain

$$\int_\Gamma |v \cdot n|^2 \leq |Dv|_0^2 \ , \tag{45}$$

for all $v \in (Ker\, B) \cap C^1(\bar{\Omega})$. By density the same estimate holds in $Ker\, B$, so that

$$\sup_{v \in Ker\, B} \frac{\int_\Gamma (v \cdot n)^2}{|Dv|_0^2} \leq 1 \ , \tag{46}$$

which concludes the proof. □

Remark 8 *We remark that the above estimate of α_M is not sharp, and it could be improved. However, it is sufficient for our subsequent considerations.*

To summarize, our analysis shows that the linearized *continuous* problem (28) is well-posed and *positive-definite* on the relevant kernel $Ker\, B$ if

$$\lambda \in \Big(-\infty, \frac{1}{\alpha_M} \Big) \supseteq (-\infty, 1) \ . \tag{47}$$

3.2 Enhanced Strain discretization

We now consider the discretized counterpart of problem (28), using an Enhanced Strain Technique. This means that we wish to solve:

Find $(u_h, H_h; p_h) \in (V_h \times S_h) \times P_h$ such that:

$$\begin{cases} \displaystyle\int_\Omega (Du_h + H_h) : (Dv + R) - \lambda \int_\Omega r\, C[Du_h + H_h] : (Dv + R) \\ \qquad\qquad \displaystyle - \int_\Omega p_h\, (\mathrm{div}\, v + \mathrm{tr} R) = \mathcal{F}(v) \qquad \forall (v, R) \in V_h \times S_h \\ \displaystyle\int_\Omega q(\mathrm{div}\, u_h + \mathrm{tr} H_h) = 0 \qquad\qquad \forall q \in P_h \ , \end{cases} \tag{48}$$

where, still, $r(x, y) = y - 1$. It is clear that an optimal method should be able to reproduce the "stability" features of the continuous problem. More precisely, it is natural to ask that if λ stays in the range given by (47), the following two conditions should hold (cf. (33), (34) and also (36)):

- *The discrete inf-sup condition*: there exists a positive constant $\tilde{\beta}$, independent of h, such that

$$\inf_{q_h \in P_h} \sup_{(v_h, R_h) \in V_h \times S_h} \frac{b_h(v_h, R_h; q_h)}{||(v_h, R_h)|| \, ||q_h||_P} \geq \tilde{\beta} \tag{49}$$

where the form $b_h(\cdot, \cdot; \cdot)$ is defined in (5) and the norm $||(\cdot, \cdot)||$ is given by (cf. (7))

$$||(v_h, R_h)|| = \left(|Dv_h|_0^2 + |R_h|_0^2 \right)^{1/2} . \tag{50}$$

- *The discrete coercivity on the kernel condition*: there exists a positive constant $\alpha(\lambda)$, independent of h, such that

$$a_\lambda(Dv_h + R_h, Dv_h + R_h) \geq \alpha(\lambda) ||(v_h, R_h)||^2 , \tag{51}$$

for every $(v_h, R_h) \in K_h$ where

$$K_h = \{(v_h, R_h) \in V_h \times S_h \ : \ b_h(v_h, R_h; q_h) = 0 \qquad \forall q_h \in P_h\} . \tag{52}$$

In the sequel, we focus our attention on the quadrilateral method proposed by Pantuso-Bathe (cf. [15]). As already mentioned, the scheme optimally performs in linear analysis, it verifies the inf-sup condition and the minimum angle condition, as theoretically proved in [13]. However, in [16] it is shown by some numerical experiments that such an element may fail the convergence when applied to large strain analysis. In particular, unrealistic instabilities occur and dramatically compromise the reliability of the results in compression tests. We stress that this bad behavior is *not* typical of Pantuso-Bathe element, but it is a common drawback of other methods based on the Enhanced Strain philosophy (cf. [23]). Moreover, Pantuso and Bathe conjectured that the lack of stability should be addressed to an "unphysical" loss of discrete ellipticity. We now give an analytical proof of such a conjecture by means of our simple model problem. For our purposes, it suffices to consider a sequence $\mathcal{T}_h$ of uniform meshes; each mesh is formed by N^2 equal square elements, denoted by K, whose side length is h (hence $h = 2/N$). The Pantuso-Bathe element is then described by the following choice of spaces. For the displacement field we set

$$V_h = \left\{ v_h \in V \ : \ v_{h|K} \in \mathcal{Q}_1(K) \qquad \forall K \in \mathcal{T}_h \right\} , \tag{53}$$

where $\mathcal{Q}_1(K)$ is the standard space of bilinear functions. For the pressure discretization, we take

$$P_h = \left\{ q_h \in H^1(\Omega) \ : \ q_{h|K} \in \mathcal{Q}_1(K) \qquad \forall K \in \mathcal{T}_h \right\} . \tag{54}$$

Finally, the strain enhancement space is given by

$$S_h = \Big\{ R_h \in (L^2(\Omega))^4 \ : \ R_{h|K} \in E_6(K) \qquad \forall K \in \mathcal{T}_h \Big\} \ . \tag{55}$$

Above, $E_6(K)$ is the space of tensor-valued functions defined on K, spanned by the following shape functions

$$\begin{bmatrix} \alpha_1\xi + \alpha_2\xi\eta \ ; & \alpha_3\xi \\ \alpha_4\eta & ; \ \alpha_5\eta + \alpha_6\xi\eta \end{bmatrix} \qquad \text{with } \alpha_i \in \mathbf{R} \ , \tag{56}$$

where (ξ, η) denotes the standard local coordinates on K.

We are now ready to prove

Proposition 9 *For the choice (53)–(56), the discrete coercivity on the kernel condition does not hold, whenever* $\lambda > 1/2$.

Proof. Let us define the set $F(\lambda) \subset \Omega$ by

$$F(\lambda) = \{(x,y) \in \Omega \ : \ 1 + \lambda r(x,y) \leq 0\} \ . \tag{57}$$

Recalling that $r = r(x,y) = y - 1$, the set $F(\lambda)$ has positive area for $\lambda > 1/2$. It follows that, for sufficiently small h, there exists $\tilde{K} \in \mathcal{T}_h$ completely contained in $F(\lambda)$. Now take $(v_h, R_h) \in V_h \times S_h$ by choosing $v_h = 0$ and R_h vanishing outside $\tilde{K}$. Moreover, in $\tilde{K}$ choose (cf. (56))

$$R_{h|\tilde{K}} = \begin{bmatrix} \alpha\xi\eta \ ; & 0 \\ 0 & ; \ -\alpha\xi\eta \end{bmatrix} \qquad \text{with } \alpha \neq 0 \ . \tag{58}$$

Notice that $\mathrm{tr} R_h = 0$; hence $(0,0) \neq (0, R_h) \in K_h$ (cf. (52)). On the other hand, a direct computation shows that

$$a_\lambda(R_h, R_h) = \int_{\tilde{K}} (1 + \lambda r)|R_h|^2 \ . \tag{59}$$

Since $\tilde{K} \subseteq F(\lambda)$, we have $(1 + \lambda r) \leq 0$, so that (59) implies

$$a_\lambda(R_h, R_h) \leq 0 \ . \tag{60}$$

As a consequence the form $a_\lambda(\cdot,\cdot)$ can not be coercive on K_h for $\lambda > 1/2$, uniformly in h. $\square$

From proposition 9 we see that the coercivity on the kernel breaks down, for the enhanced scheme, when $\lambda > 1/2$, even though the continuous problem "resists" at least until $\lambda = 1$ (cf. (47)).

3.3 The $\mathcal{Q}_2/\mathcal{P}_1$ element

We now consider the well-known "biquadratic displacements/(discontinuous) linear pressures" element, which satisfies the inf-sup condition and it has been reported in [16] to be very efficient when applied to large deformation analysis. Given a quadrilateral mesh $\mathcal{T}_h$ we thus discretize the displacement field by means of

$$V_h = \Big\{ v_h \in V \;:\; v_{h|K} \in \mathcal{Q}_2(K) \qquad \forall K \in \mathcal{T}_h \Big\} \;, \tag{61}$$

where $\mathcal{Q}_2(K)$ is the usual space of biquadratic functions. For the pressure field, we take

$$P_h = \Big\{ q_h \in L^2(\Omega) \;:\; q_{h|K} \in \mathcal{P}_1(K) \qquad \forall K \in \mathcal{T}_h \Big\} \;. \tag{62}$$

We remark that for this scheme we do not use any kind of strain enhancement. Hence, we are interested in studying the features of the following problem with respect to the real parameter λ:

Find $(u_h, p_h) \in V_h \times P_h$ such that:

$$\begin{cases} \displaystyle\int_\Omega Du_h : Dv - \lambda \int_\Omega r\, C[Du_h] : Dv - \int_\Omega p_h \operatorname{div} v = \mathcal{F}(v) \qquad \forall v \in V_h \\[2ex] \displaystyle\int_\Omega q \operatorname{div} u_h = 0 \qquad\qquad \forall q \in P_h \;, \end{cases} \tag{63}$$

For this element the relevant kernel K_h is defined by

$$K_h = \left\{ v_h \in V_h \;:\; \int_\Omega q_h \operatorname{div} v_h = 0 \qquad \forall q_h \in P_h \right\} \;. \tag{64}$$

We notice that the functions in K_h are not *exactly* divergence-free, but they satisfy the constraint only in a weak sense. As a consequence, given $v_h \in K_h$

we cannot infer that it holds (cf. subsection 3.1)

$$a_\lambda(Dv_h, Dv_h) = \int_\Omega Dv_h : Dv_h - \lambda \int_\Gamma (v_h \cdot n)^2 \ , \tag{65}$$

so that the coercivity on the kernel is not *automatically* assured for $\lambda < 0$, contrary to what happens for the continuous problem. However, we are able to prove the following proposition.

Proposition 10 *Given $\lambda < 1$, the element $\mathcal{Q}_2/\mathcal{P}_1$ satisfies the discrete coercivity on the kernel condition, for h sufficiently small.*

Before giving the proof of the above proposition, we need the lemma:

Lemma 11 *Let Ω be the square $(-1,1) \times (-1,1)$, $\Gamma = [-1,1] \times \{1\}$ and $\Gamma_D = \partial\Omega - \Gamma$. For every $\varphi \in H^1(\Omega)$ such that $\varphi_{|\Gamma_D} = 0$ it holds*

$$\int_\Omega |\varphi|^2 \leq \int_\Omega |\nabla \varphi|^2 \ . \tag{66}$$

Proof. For regular functions we have, taking into account the boundary conditions on $[-1,1] \times \{-1\}$

$$\varphi(x,y) = \int_{-1}^{y} \varphi_{,y}(x,t)dt \ , \tag{67}$$

by which

$$\varphi(x,y)^2 \leq (1+y) \int_{-1}^{y} |\varphi_{,y}(x,t)|^2 dt \leq (1+y) \int_{-1}^{1} |\varphi_{,y}(x,y)|^2 dy \ . \tag{68}$$

Integrating on Ω we obtain

$$\int_\Omega |\varphi|^2 \leq 2 \int_\Omega |\varphi_{,y}|^2 \ . \tag{69}$$

On the other hand, from a similar argument and using the boundary conditions on $\{-1\} \times [-1,1]$ we get

$$\int_\Omega |\varphi|^2 \leq 2 \int_\Omega |\varphi_{,x}|^2 \ . \tag{70}$$

Summing (69) and (70) we obtain

$$\int_\Omega |\varphi|^2 \leq \int_\Omega |\nabla \varphi|^2 \ . \tag{71}$$

By density, the above estimate still holds for functions in H^1 having the same boundary conditions. $\square$

Proof of proposition 10. We have to prove the coercivity on K_h of

$$a_\lambda(Dv_h, Dv_h) = \int_\Omega Dv_h : Dv_h - \lambda \int_\Omega r\, C[Dv_h] : Dv_h \ , \tag{72}$$

for $\lambda < 1$ and h sufficiently small. We split the proof into two parts. First, consider $\lambda \leq 0$. As already mentioned, for $v_h \in K_h$ we do not have

$$a_\lambda(Dv_h, Dv_h) = \int_\Omega Dv_h : Dv_h - \lambda \int_\Gamma (v_h \cdot n)^2 \ . \tag{73}$$

However, it holds

$$a_\lambda(Dv_h, Dv_h) = \int_\Omega |Dv_h|^2 + \lambda \left(2 \int_\Omega (\operatorname{div} v_h) v_h \cdot \nabla r - \int_\Gamma (v_h \cdot n)^2 \right) \ . \tag{74}$$

If $\lambda \leq 0$ is fixed, we thus have

$$a_\lambda(Dv_h, Dv_h) \geq \int_\Omega |Dv_h|^2 + 2\lambda \int_\Omega (\operatorname{div} v_h) v_h \cdot \nabla r \ . \tag{75}$$

Let us denote with $\pi_h(v_h \cdot \nabla r)$ the L^2-projection of $v_h \cdot \nabla r$ over the space P_h. Since $v_h \in K_h$, we get

$$2\lambda \int_\Omega (\operatorname{div} v_h) v_h \cdot \nabla r = 2\lambda \int_\Omega (\operatorname{div} v_h)(v_h \cdot \nabla r - \pi_h(v_h \cdot \nabla r)) \ . \tag{76}$$

But

$$|v_h \cdot \nabla r - \pi_h(v_h \cdot \nabla r)|_0 \leq c_1 h |Dv_h|_0 \ , \tag{77}$$

so that

$$2 \int_\Omega (\operatorname{div} v_h)(v_h \cdot \nabla r - \pi_h(v_h \cdot \nabla r)) \leq 2c_1 h |Dv_h|_0^2 \ . \tag{78}$$

From (78), (75), (76) and recalling that λ is negative, we obtain

$$a_\lambda(Dv_h, Dv_h) \geq (1 + 2\lambda c_1 h)|Dv_h|_0^2 \ . \tag{79}$$

For a fixed λ, we can take h sufficiently small to get $(1 + 2\lambda c_1 h) > 0$, and coercivity follows. We now consider the case $0 < \lambda < 1$. Take $v_h = (v_1, v_2) \in V_h$, not necessarily in K_h; if one takes into account the boundary conditions for v_h and $r = y - 1$, it is easily seen that (74) can be written as

$$a_\lambda(Dv_h, Dv_h) = \int_\Omega |Dv_h|^2 + 2\lambda \int_\Omega v_{1,x} v_2 \ . \tag{80}$$

Since

$$2 \int_\Omega v_{1,x} v_2 \geq - \int_\Omega |v_{1,x}|^2 - \int_\Omega |v_2|^2 \ , \tag{81}$$

we can apply lemma 11 to v_2 and we obtain

$$2 \int_\Omega v_{1,x} v_2 \geq - \int_\Omega |v_{1,x}|^2 - \int_\Omega (|v_{2,x}|^2 + |v_{2,y}|^2) \geq - \int_\Omega |Dv_h|^2 \ . \tag{82}$$

From (80) and (82) we infer

$$a_\lambda(Dv_h, Dv_h) \geq (1 - \lambda) \int_\Omega |Dv_h|^2 \ . \tag{83}$$

Hence the coercivity condition is fulfilled also for $0 < \lambda < 1$. $\square$

Compared to the enhanced element we have considered in this paper, we see (cf. proposition 9) that the $\mathcal{Q}_2/\mathcal{P}_1$ scheme retains the coercivity in a range which is closer to that of the continuous problem.

4 Conclusions

In this paper we have proposed a new scheme based on the Enhanced Strain Technique and we have shown its convergence and stability when applied to linear analysis. Furthermore, we have presented a simple model problem which can be used to investigate the properties of finite element discretizations in the framework of large deformation analysis. By means of our model, we have given a further contribution to the theoretical analysis of mixed Enhanced Strain

Methods. However, we wish to stress that much deeper studies are needed. For example, a clear mathematical theory, capable to identify the relevant features the strain enhancements should satisfy, is still missing in the context of nonlinear elasticity. This will be the topic of future communications.

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A Appendix

The aim of this appendix is to briefly discuss the effect of the strain enhancement for *triangular* elements based on different variational formulations. In particular, we will show that:

- For a *displacement-based* (low-order) triangular element, the enhanced strain modes do not provide any kind of improvement. This fact has been already pointed out in [18].
- For a triangular method based on the *displacement/pressure* formulation, the situation is different, and effective enhanced strain modes can be selected.

In what follows, we restrict to consider the case of linear homogeneous and isotropic elasticity. Moreover, the spaces V_h, S_h and P_h will be the ones chosen in subsection 2.2. Hence, both the displacement field and the pressure field are approximated by means of piecewise linear and *globally continuous* functions. Finally, the enhanced strain modes have zero mean-value in each triangle.

Displacement-based formulation.
When the usual displacement-based variational formulation is considered, the enhanced strain problem reads as follows.

Find $(u_h, H_h) \in V_h \times S_h$ such that

$$\int_\Omega 2\,\mu\,(D^s u_h + H_h) : (D^s v + R) + \int_\Omega \lambda\,(\mathrm{div}\,u_h + \mathrm{tr}H_h)\,(\mathrm{div}\,v + \mathrm{tr}R) = \int_\Omega f\cdot v \tag{A.1}$$

for every $(v, R) \in V_h \times S_h$. In particular, taking variations with respect to the enhanced strains, one has

$$\int_\Omega 2\,\mu\,(D^s u_h + H_h) : R + \int_\Omega \lambda\,(\mathrm{div}\,u_h + \mathrm{tr}H_h)\,\mathrm{tr}R = 0 \tag{A.2}$$

for every $R \in S_h$. Since the discrete displacements are piecewise linear functions, their derivatives are locally constants, so that the zero mean-value property of the enhanced strains implies

$$\int_\Omega 2\,\mu\,H_h : R + \int_\Omega \lambda\,\mathrm{tr}H_h\,\mathrm{tr}R = 0 \tag{A.3}$$

for every $R \in S_h$. It follows that the enhanced strain solution H_h is zero, independently of the value of λ. This actually means that the strain enhancement is useless (for both compressible and nearly-incompressible regime), although it has been inserted in the formulation. Therefore, the enhanced strain method collapses into the corresponding standard scheme (cf. [18]).

Mixed displacement/pressure formulation.
When one considers a mixed displacement/pressure formulation, the above considerations, concerning a displacement-based method, do not necessarily apply. In the sequel, we discuss the case of incompressible regime, but the same arguments can be extended to compressible and nearly-incompressible regime. Also, we set $2\,\mu = 1$, for the sake of simplicity and without loss of generality. A standard scheme for the *mixed u/p* formulation reads as follows.

Find $(u_h; p_h) \in V_h \times P_h$ such that

$$\begin{cases} \int\limits_\Omega D^s u_h : D^s v - \int\limits_\Omega p_h \operatorname{div} v = \int\limits_\Omega f \cdot v \\ \\ \int\limits_\Omega q \operatorname{div} u_h = 0 \end{cases} \tag{A.4}$$

for every $v \in V_h$ and for every $q \in P_h$. On the contrary, an *enhanced strain scheme* consists in solving:

Find $(\tilde{u}_h, H_h; \tilde{p}_h) \in (V_h \times S_h) \times P_h$ such that

$$\begin{cases} \int\limits_\Omega (D^s \tilde{u}_h + H_h) : (D^s v + R) - \int\limits_\Omega \tilde{p}_h (\operatorname{div} v + \mathrm{tr} R) = \int\limits_\Omega f \cdot v \\ \\ \int\limits_\Omega q (\operatorname{div} \tilde{u}_h + \mathrm{tr} H_h) = 0 \end{cases} \tag{A.5}$$

for every $(v, R) \in V_h \times S_h$ and for every $q \in P_h$. It is clear that problem (A.5) collapses into problem (A.4) if and only if $(u_h, p_h) = (\tilde{u}_h, \tilde{p}_h)$. Since the displacement gradients are L^2-orthogonal to the enhanced strain modes, problem (A.5) can be written as

$$\begin{cases} \int\limits_\Omega D^s \tilde{u}_h : D^s v - \int\limits_\Omega \tilde{p}_h \operatorname{div} v = \int\limits_\Omega f \cdot v \\ \\ \int\limits_\Omega q \operatorname{div} \tilde{u}_h = - \int\limits_\Omega q \, \mathrm{tr} H_h \\ \\ \int\limits_\Omega H_h : R - \int\limits_\Omega \tilde{p}_h \, \mathrm{tr} R = 0 \end{cases} \tag{A.6}$$

for every $v \in V_h$, $q \in P_h$ and $R \in S_h$. Comparing (A.4) with the first two equations of (A.6) it follows that the enhanced strain scheme collapses into the standard scheme if and only if

$$\int\limits_\Omega q \, \mathrm{tr} H_h = 0 \qquad \forall q \in P_h \ . \tag{A.7}$$

However, for our choice of spaces, the traces of the enhanced strains are *not* L^2-orthogonal to the piecewise linear continuous functions, so that condition (A.7) is not generally satisfied. As a consequence, the enhanced strain (mixed) method is not equivalent to the corresponding standard (mixed) scheme.