A LAGRANGE MULTIPLIER METHOD FOR ELLIPTIC INTERFACE PROBLEMS USING NON-MATCHING MESHES

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In this Note we consider the Lagrange multiplier method proposed in [9] for the finite element solution of multi-domain elliptic equations using non-matching meshes. The ideas are illustrated with the Poisson's problem as a model, and some numerical results are presented.

1. Introduction

When considering multi-domain problems with non-matching meshes using Lagrange multiplier techniques, two basic problems occur. First and foremost, the discrete spaces for the discretization of the primal variable and the multipliers have to fulfill the *inf-sup* condition (see [7]) in order the resulting numerical scheme to be stable; it turns out that for many natural choices of approximations this is not the case. Fortunately, this problem can be alleviated by using stabilized multiplier methods (cf. [14, 17, 1, 4], for instance) or by using mesh-dependent penalty methods (see e.g. [2, 3]).

The second problem is that products of traces of the primal variable and the multipliers have to be integrated on the interfaces. For methods known

to fulfill the *inf-sup* condition, such as the mortar element method (see, e.g., [5, 6] and [18]), as well as most stabilized methods, this will mean integrating products of piecewise polynomials on unrelated meshes. This might be difficult, in practice, for problems in $\mathbb{R}^3$ (see, however, [15]). To mitigate these two problems, a stabilization method has been proposed in [9]. Stability is achieved under the very mild assumption that the approximation space for the interface multiplier contains the constants. Furthermore, the method avoids the cumbersome integration of products of unrelated mesh functions by using, as already done in [12], global polynomial multipliers on the interfaces: only products of global polynomials and local polynomials have to be considered and this makes the integration problem much simpler in many cases.

The aim of this Note is to report on the results proved in [9]; we present our method in Section 2, while in Section 3 we give the stability and convergence estimates, together with some comments. Finally, Section 4 is devoted to few numerical tests, showing the performance of the method.

2. Formulation of the method

In this section we introduce an interface Lagrange multiplier method for the finite element discretization of elliptic problems on non-matching grids. Before doing that, we make precise the model problem we will be working on, together with some notation.

2.1. *Model problem and notation*

Let Ω be a bounded domain in $\mathbb{R}^2$, with boundary $\partial\Omega$. (The extension to $\mathbb{R}^3$ is straightforward.) As a model problem, we consider a stationary heat conduction problem in the case where there is a piecewise straight internal boundary Γ dividing Ω into two subdomains Ω_1 and Ω_2. Thus, we want to solve for u the problem

$$\begin{aligned} -\nabla\cdot(\kappa_i\nabla u_i) &= f \quad \text{in} \quad \Omega_i, \\ u_i &= 0 \quad \text{on} \quad \partial\Omega_i\cap\partial\Omega, \\ u_1 - u_2 &= 0 \quad \text{on} \quad \Gamma, \\ \boldsymbol{n}_1\cdot\kappa_1\nabla u_1 + \boldsymbol{n}_2\cdot\kappa_2\nabla u_2 &= 0 \quad \text{on} \quad \Gamma, \end{aligned} \tag{1}$$

for $i = 1, 2$, where we have denoted by u_i the restriction of u to Ω_i. Here f is a given function, κ_i, which is assumed to be positive and smooth in Ω_i, is the conductivity, and $\boldsymbol{n}_i$ is the outward pointing normal to Ω_i at Γ,

$i = 1, 2$. Define

$$V = \{v : v_i \in H^1(\Omega_i),\ v_i = 0 \text{ on } \partial\Omega_i \setminus \Gamma,\ i = 1, 2\}$$

and

$$\Lambda = \left(H^{1/2}_{00}(\Gamma)\right)',$$

the dual space of $H^{1/2}_{00}(\Gamma)$ (see, e.g., [13]).

A weak form of (1) using the Lagrange multiplier approach is as follows: Find $(u, \lambda) \in V \times \Lambda$ such that

$$\begin{aligned} &\sum_i \int_{\Omega_i} \kappa_i \nabla u_i \cdot \nabla v_i \, dx + \int_\Gamma \lambda \, [v] \, ds = \sum_i \int_{\Omega_i} f \, v_i \, dx \quad \forall v \in V, \\ &\int_\Gamma [u] \, \mu \, ds = 0 \quad \forall \mu \in \Lambda, \end{aligned} \tag{2}$$

where $[v] := (v_1 - v_2)|_\Gamma$ is the jump of v across Γ. Notice that

$$\lambda = -\kappa_1 \nabla u_1 \cdot \boldsymbol{n}_1 = \kappa_2 \nabla u_2 \cdot \boldsymbol{n}_2 \quad \text{on } \Gamma.$$

We now introduce the necessary notation for the definition of the method we are going to present, focusing, for simplicity, on the case of triangular elements. Therefore, we assume that we are given a triangular mesh $\mathcal{T}_i^h$ of the domain Ω_i, $i = 1, 2$. We denote by h_i the mesh-size of $\mathcal{T}_i^h$. Obviously, $\mathcal{T}^h = \mathcal{T}_1^h \cup \mathcal{T}_2^h$ provides a mesh for Ω, whose mesh-size is $h = \max\{h_1, h_2\}$. We introduce the (family of) finite element space

$$V^h = \{v \in V :\ v|_K \in P^k(K),\ \ \forall K \in \mathcal{T}^h\},$$

where $P^k(K)$ denotes the space of polynomials of degree at most k on K, with $k \geq 1$. On Γ we introduce a family of spaces Λ^p of discrete multipliers as follows: the interface Γ is decomposed as the union $\Gamma = \bigcup \Gamma_j$ of n_Γ straight lines Γ_j of length ℓ_j; we associate with each Γ_j the non-negative integer p_j and define $\underline{p} := [p_1, \ldots, p_{n_\Gamma}]$; then our choice is

$$\Lambda^p = \{\mu \in \Lambda :\ \mu|_{\Gamma_j} \in P^{p_j}(\Gamma_j),\ j = 1, \ldots, n_\Gamma\}, \tag{3}$$

with $P^{p_j}(\Gamma_j)$ denoting the space of polynomials of degree at most p_j on Γ_j with respect to a local coordinate. In this case the elements of Λ^p are *global* polynomials on each Γ_j, and they can be discontinuous at the endpoints of the Γ_j's. We also note that it holds

$$P^0(\Gamma) \subset \Lambda^p, \tag{4}$$

$P^0(\Gamma)$ being the space of constants on the whole interface Γ.

2.2. *Formulation of the method: A Nitsche-type interface condition*

The method we propose takes advantage of a Nitsche-type interface condition in connection with the use of the interjacent multiplier space. We first define $\boldsymbol{n} := \boldsymbol{n}_1 = -\boldsymbol{n}_2$ on Γ and

$$\{\boldsymbol{n} \cdot \boldsymbol{w}\} := \alpha\, \boldsymbol{n} \cdot \boldsymbol{w}_1 + (1-\alpha)\boldsymbol{n} \cdot \boldsymbol{w}_2,$$

where $0 \leq \alpha \leq 1$. We then consider an unsymmetric method and its symmetric variant, the latter one only for the choices $\alpha = 0$ and $\alpha = 1$.

2.2.1. *The unsymmetric method*

For any choice of $0 \leq \alpha \leq 1$, the unsymmetric method reads as follows: Find $(u^h, \lambda^p) \in V^h \times \Lambda^p$ such that

$$\begin{aligned} &\sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \lambda^p \, [v] \, ds = \sum_i \int_{\Omega_i} f \, v_i \, dx \quad \forall v \in V^h, \\ &\int_\Gamma [u^h] \, \mu \, ds - \int_\Gamma \gamma \left\{ \boldsymbol{n} \cdot \kappa \nabla u^h \right\} \mu \, ds - \int_\Gamma \gamma \, \lambda^p \, \mu \, ds = 0 \quad \forall \mu \in \Lambda^p, \end{aligned} \tag{5}$$

where γ is the function of $L^\infty(\Gamma)$ defined as follows. Denote by $\mathcal{N}_\Gamma$ the set of nodes of $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ lying on Γ. Fix a point $\boldsymbol{x}$ on $\Gamma \setminus \mathcal{N}_\Gamma$ and let K_1 and K_2 be the two elements of $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$, respectively, such that the interior of $\partial K_1 \cap \partial K_2 \cap \Gamma$ is non empty and $\boldsymbol{x} \in \partial K_1 \cap \partial K_2 \cap \Gamma$. Denote by h_{K_1} and h_{K_2} the diameters of K_1 and K_2, respectively. For $\boldsymbol{x} \in \Gamma \setminus \mathcal{N}_\Gamma$, we define

$$\gamma(\boldsymbol{x}) = \gamma_0 \, \min \left\{ \alpha \frac{h_{K_1}}{\kappa_1(\boldsymbol{x})}, (1-\alpha) \frac{h_{K_2}}{\kappa_2(\boldsymbol{x})} \right\}, \tag{6}$$

with γ_0 a constant independent on the mesh-size and the material properties. Notice that γ is defined almost everywhere on Γ.

2.2.2. *The symmetric variant*

In the cases of $\alpha = 0$ and $\alpha = 1$, the method (5) can be symmetrized without introducing integration of cross terms across the interface; this would not be possible for $0 < \alpha < 1$. The symmetric formulation reads as follows:

Find $(u^h, \lambda^p) \in V^h \times \Lambda^p$ such that

$$\begin{aligned}
&\sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \lambda^p \, [v] \, ds - \int_\Gamma \gamma \lambda^p (\boldsymbol{n} \cdot \kappa_j \nabla v_j) \, ds \\
&\qquad - \int_\Gamma \gamma (\boldsymbol{n} \cdot \kappa_j \nabla u_j^h)(\boldsymbol{n} \cdot \kappa_j \nabla v_j) \, ds = \sum_i \int_{\Omega_i} f \, v_i \, dx \quad \forall v \in V^h, \\
&\int_\Gamma [u^h] \, \mu \, ds - \int_\Gamma \gamma (\boldsymbol{n} \cdot \kappa_j \nabla u_j^h) \mu \, ds - \int_\Gamma \gamma \, \lambda^p \, \mu \, ds = 0 \quad \forall \mu \in \Lambda^p,
\end{aligned} \tag{7}$$

with subscript $j = 2$ if $\alpha = 0$, and $j = 1$ if $\alpha = 1$. Again, the function γ is as in (6). One reason of using the symmetric formulation is its adjoint consistency, that is useful whenever duality arguments need to be applied. We also notice that both the formulations (5) and (7) are consistent.

We may rewrite the formulations (5) and (7) in a unified way:
Find $(u^h, \lambda^p) \in V^h \times \Lambda^p$ such that

$$\begin{aligned}
&\sum_i \int_{\Omega_i} \kappa_i \nabla u_i^h \cdot \nabla v_i \, dx + \int_\Gamma \lambda^p \, [v] \, ds - S \int_\Gamma \gamma \lambda^p \Big\{ \boldsymbol{n} \cdot \kappa \nabla v \Big\} \, ds \\
&\qquad - S \int_\Gamma \gamma \left\{ \boldsymbol{n} \cdot \kappa \nabla u^h \right\} \Big\{ \boldsymbol{n} \cdot \kappa \nabla v \Big\} \, ds = \sum_i \int_{\Omega_i} f \, v_i \, dx \quad \forall v \in V^h, \\
&\int_\Gamma [u^h] \, \mu \, ds - \int_\Gamma \gamma \left\{ \boldsymbol{n} \cdot \kappa \nabla u^h \right\} \mu \, ds - \int_\Gamma \gamma \, \lambda^p \, \mu \, ds = 0 \quad \forall \mu \in \Lambda^p,
\end{aligned} \tag{8}$$

either with $S = 0$ and any $0 \le \alpha \le 1$ (to get the unsymmetric formulation (5)), or with $S = 1$ and $\alpha = 0$ or $\alpha = 1$ (to get the symmetric formulation (7)). Introducing

$$\begin{aligned}
\mathcal{B}^h(w, \nu; v, \mu) \; := &\sum_i \int_{\Omega_i} \kappa_i \nabla w_i \cdot \nabla v_i \, dx + \int_\Gamma \nu \, [v] \, ds \\
&- S \int_\Gamma \gamma \nu \left\{ \boldsymbol{n} \cdot \kappa \nabla v \right\} \, ds - S \int_\Gamma \gamma \left\{ \boldsymbol{n} \cdot \kappa \nabla w \right\} \left\{ \boldsymbol{n} \cdot \kappa \nabla v \right\} \, ds \\
&- \int_\Gamma [w] \, \mu \, ds + \int_\Gamma \gamma \left\{ \boldsymbol{n} \cdot \kappa \nabla w \right\} \mu \, ds + \int_\Gamma \gamma \, \nu \, \mu \, ds,
\end{aligned}$$

and

$$\mathcal{F}^h(v, \mu) = \sum_i \int_{\Omega_i} f \, v_i \, dx,$$

the formulation (8) can be written as follows:
Find $(u^h, \lambda^p) \in V^h \times \Lambda^p$ such that

$$\mathcal{B}^h(u^h, \lambda^p; v, \mu) = \mathcal{F}^h(v, \mu) \qquad \forall (v, \mu) \in V^h \times \Lambda^p. \tag{9}$$

3. Stability and convergence of the method

In this section we report on the stability and convergence of the methods (5) and (7). Defining the (weighted) broken H^1–norm

$$\|w\|_V = \Big(\sum_i \|\kappa_i^{1/2} \nabla w_i\|^2_{L_2(\Omega_i)} + \|aw\|^2_{L_2(\Omega)} \Big)^{1/2},$$

with $a = \kappa^{1/2}/\mathrm{diam}(\Omega)$, we introduce the norm

$$|||(w,\nu)||| := \Big(\|w\|^2_V + \|\gamma^{1/2}\nu\|^2_{L_2(\Gamma)} \Big)^{1/2}.$$

Define the constant $\mathtt{k}$ as the mean value of the function κ on Ω and $b := \big(\mathtt{k}/\mathrm{diam}(\Omega)\big)^{1/2}$. On the interface Γ, we will use the norm

$$[\![\varphi]\!]_{1/2,\Gamma} := \Big(\|b\varphi\|^2_{L_2(\Gamma)} + |\mathtt{k}^{1/2}\varphi|^2_{H^{1/2}_{00}(\Gamma)} \Big)^{1/2},$$

together with its dual denoted by $[\![\cdot]\!]_{-1/2,\Gamma}$.

Remark 3.1. The norm $[\![\cdot]\!]_{1/2,\Gamma}$ is the natural norm for the traces on Γ of functions belonging to V, when V is endowed with the $\|\cdot\|_V$–norm.

We proceed by stating a stability property of the form $\mathcal{B}^h$. In the sequel C denotes a generic *strictly positive* constant independent of the mesh-size and of Λ^p. We have

Proposition 3.1. *Provided that γ_0 is small enough, for all $(w,\nu) \in V^h \times \Lambda^p$ there is $(v,\mu) \in V^h \times \Lambda^p$ such that*

$$\begin{aligned} &|||(v,\mu)||| \le C_1 |||(w,\nu)|||, \\ &\mathcal{B}^h(w,\nu;v,\mu) \ge C_2 |||(w,\nu)|||^2, \end{aligned}$$

The following result, proved in [9], provides an error estimate for the method.

Theorem 3.1. *Assume Λ^p given by (3), $u_i \in H^{s+1}(\Omega_i)$, $i = 1,2$, with $s > 1/2$. Denote by T_i^h, $i = 1,2$, the union of the elements contained in Ω_i and having one side on Γ and, for any $K \in T_1^h \cup T_2^h$, define $\gamma_K := \min_{\boldsymbol{x} \in \partial K \cap \Gamma} \gamma$.*

We have

$$
\begin{aligned}
|||(u-u^h,\lambda-\lambda^p)||| \le C\Big(&\mathtt{k}\sum_{K\in\mathcal{T}^h} h_K^{2\min\{k,s\}}|u|^2_{H^{s+1}(K)}\\
&+\sum_{K\in T_1^h\cup T_2^h}\gamma_K^{-1}h_K^{2\min\{k,s\}+1}|u|^2_{H^{s+1}(K)}\\
&+\sum_j\big(\mathtt{k}^{-1}\ell_j^{2s}p_j^{-2s}+\sup_{\boldsymbol{x}\in\Gamma_j}\gamma\,\ell_j^{2s-1}p_j^{-(2s-1)}\big)|\lambda|^2_{H^{s-1/2}(\Gamma_j)}\Big)^{1/2},
\end{aligned}
\tag{10}
$$

with a positive constant C independent of the mesh-size and of $\underline{p}$.

We briefly comment on Theorem 3.1.

• Assume that $\mathcal{T}^h$ is quasi-uniform and denote by h its characteristic mesh-size. Then, from (10), the optimal choice of the polynomial approximation seems to be $p_j \approx \ell_j/h$, $j=1,\dots,n_\Gamma$, when $k\ge s$, or $p_j\approx \ell_j/h^{k/s}$, $j=1,\dots,n_\Gamma$, when $k<s<+\infty$. Estimate (10) then becomes

$$|||(u-u^h,\lambda-\lambda^p)||| \le C\mathtt{k}^{1/2}h^{\min\{k,s\}}\sum_i |u_i|_{H^{s+1}(\Omega_i)}.$$

For analytic solutions, from Theorem 3.20 in [16], we have that the approximation error for λ depends on p exponentially. Therefore, p_j can be chosen as $p_j\approx|\log h|$, $j=1,\dots,n_\Gamma$, which allows for a saving in the number of degrees of freedom for the approximation of λ (see Section 4).

• Assume that $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ are quasi-uniform and denote by h_1 and h_2 their characteristic mesh-sizes (here, we are not assuming any bound of h_1 and h_2 in terms of each other). Then, from the definition of the parameter γ, it is clear that, if h_1 and h_2 are very different in size, estimate (10) significantly depends on α. If $h_1\ll h_2$ (resp. $h_2\ll h_1$), the best result is given for $\alpha=0$ (resp. $\alpha=1$), which gives the symmetric formulation (7). We point out that the same strategy was adopted in [10, 11] for an interior penalty treatment of the interface between non-matching meshes.

If we assume $s=1$, for the sake of simplicity, and the optimal choice of the polynomial approximation orders, namely $p_j\approx\ell_j/\max\{h_1,h_2\}$, $j=1,\dots,n_\Gamma$, estimate (10) becomes

$$|||(u-u^h,\lambda-\lambda^p)||| \le C\mathtt{k}^{1/2}\sum_i h_i|u_i|_{H^2(\Omega_i)}.$$

4. Numerical examples

In the examples below, unless otherwise stated, we doubled the polynomial degree for the approximation of λ, for every doubling of the number of nodes

on the interface, starting with $p_j = 1$ on the coarsest mesh. The parameter γ was chosen as $\gamma|_{\Gamma_j} = h_{\min}/\sqrt{3}$ where $h_{\min}$ denotes the smallest element size along Γ_j.

4.1. *Interior domain*

We considered the domain $\Omega := (0,3) \times (0,3)$ divided into one interior domain $\Omega_1 := (1,2) \times (1/2, 3/2)$ and one exterior $\Omega_2 := \Omega \setminus \Omega_1$. We set $\kappa_1 = \kappa_2 = 1$,

$$f = \frac{2\,\pi^2}{9}\,\sin(\pi x/3)\,\sin(\pi y/3),$$

and boundary conditions such that the analytical solution is

$$u = \sin(\pi x/3)\,\sin(\pi y/3).$$

The initial and final meshes, with the elevation of the corresponding approximate solution, are shown in Fig. 1, and in Fig. 2 we show the corresponding convergence behavior, second order convergence in L_2–norm and first order convergence in broken energy norm.

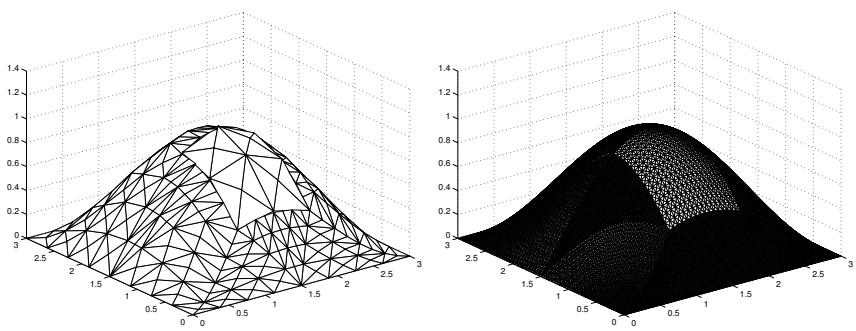

Figure 1. First and last mesh in the sequence corresponding to Fig. 2.

We remark that, in more general cases (e.g., when $\kappa_1 \neq \kappa_2$ and Γ is only Lipschitz continuous), the model problem (1) lacks of elliptic regularity. Typically, full elliptic regularity can only be expected if Γ is sufficiently smooth, cf. [8]. Therefore, in general, we cannot expect optimal convergence in L_2–norm.

Finally, we show that the polynomial degree of the multiplier can grow much slower than the number of degrees of freedom on the interface without

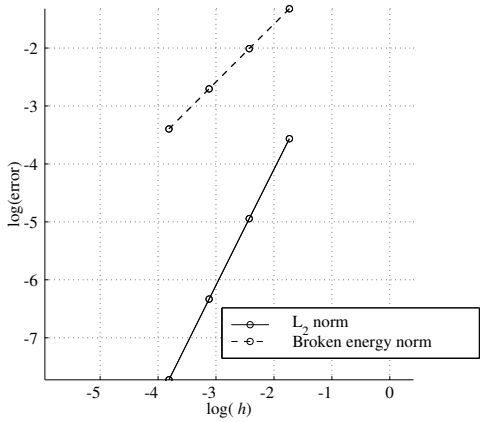

Figure 2. Convergence in L_2 and in broken energy norm.

loss of convergence. In Fig. 3 we give the convergence resulting from letting $p_j = |\log h|$, in accordance with the discussion after Theorem 3.1, which gives $p_j = 4$ on the last mesh in the sequence, much less than the number of nodes on each segment in the final mesh (cf. Fig. 2). Nevertheless, comparable errors are obtained.

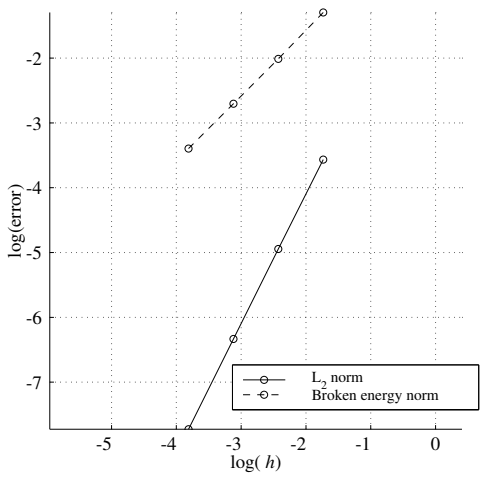

Figure 3. Convergence in L_2 and in broken energy norm using $p_j = |\log h|$.

4.2. *The case* $\kappa_1 \neq \kappa_2$

Consider solutions to the ordinary differential equation

$$-\sum_i \frac{d}{dx}\left(\kappa_i \frac{du_i}{dx}\right) = 1;\ [u(1/2)] = 0;\ \kappa_1 \frac{du_1}{dx}(1/2) = \kappa_2 \frac{du_2}{dx}(1/2).$$

The domain is $(0,1)$, with an interface at $x = 1/2$. While this is a one-dimensional problem, we solved it numerically in 2D on the domain $(0,1)\times(0,1)$, with zero Neumann boundary conditions at $y = 0$ and $y = 1$. The equation has a closed-form solution that, for homogeneous Dirichlet boundary conditions at $x = 0$ and $x = 1$, is given by

$$u_1(x) = \frac{(3\,\kappa_1 + \kappa_2)\ x}{4\,\kappa_1^2 + 4\,\kappa_1\,\kappa_2} - \frac{x^2}{2\,\kappa_1}, \quad u_2(x) = \frac{\kappa_2 - \kappa_1 + (3\,\kappa_1 + \kappa_2)\ x}{4\,\kappa_2^2 + 4\,\kappa_1\,\kappa_2} - \frac{x^2}{2\,\kappa_2}.$$

We chose $\kappa_1 = 1/2$, $\kappa_2 = 3$. In Fig. 4 we show the corresponding convergence behavior, again with second order convergence in L_2–norm and first order convergence in broken energy norm.

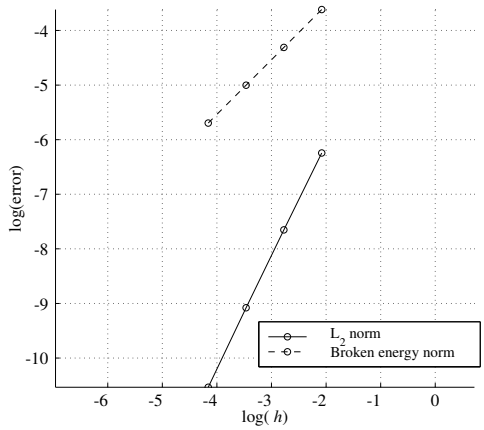

Figure 4. Convergence in L_2 and in broken energy norm.

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