ANALYSIS OF MIXED FINITE ELEMENTS FOR LAMINATED COMPOSITE PLATES

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Abstract

The approximation to the solution of a laminated plate problem is considered. The main result consists in proving that a performant scheme can be obtained by simply coupling a standard membrane element for the in-plane displacements with a robust method for plate bending problem, both elements related to the case of homogeneous material. A rigorous convergence analysis is developed for the case of clamped boundary conditions. Numerical results are presented, showing the accordance between the theoretical results and the actual computations.

1 Introduction

Laminated composite plates are nowadays widely used in engineering practice, especially in space, automobile and civil applications [18]. The main motivation for this growing interest is related to the improved ratio between performances and weight associated with laminated plates with respect

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to the case of homogeneous ones. However, the behavior of laminated plates is more complex, because

- they are anisotropic materials;
- they present significant shear deformation in the thickness direction;
- they exhibit extension-bending coupling.

In literature, many different models have been presented. The simplest one is surely the Classical Laminated Plate Theory (CLPT), which is based on the Kirchhoff hypotheses (cf. e.g. [22] for more details). However, due to the fact that shear deformations are not taken into account, the model approximation is quite poor. Moreover, in the framework of finite element procedures (cf. [7], [15]), CLPT models require C^1-conforming schemes, which are quite expensive from the computational point of view.

According to these considerations, models arising from the Reissner-Mindlin assumptions (called First-order Shear-Deformation Theory: FSDT) are often preferred. The main FSDT features are related to the possibility of using C^0-conforming methods and the ability to capture shear deformations, which makes it possible to consider also the case of moderately thick plates.

In the present paper we develop an analysis of mixed finite element schemes relative to a FSDT model. Our main result is that a performant method can be obtained by simply taking a standard membrane element for the in-plane displacements and a robust plate bending element, both elements related to homogeneous plates (cf. [2], [3] and [10], for example). We remark that this conclusion is not straightforward, since the laminated structure exhibits an extension-bending coupling (i.e. in-plane displacements arise even for vertical loads).

The paper is organized as follows. In Section 2 we briefly introduce the laminated plate model, together with the variational formulation that will be used as the starting point of the discretization procedure. We establish an existence and uniqueness result for the continuous solution, mainly by recognizing that the variational formulation of the problem is well-posed. In Section 3 we develop our error bounds for mixed finite element methods in an abstract framework, as we want to give an analysis covering as many schemes as possible. In Section 4 we present two examples of applications of our theory. Section 5 is devoted to present numerical experiments, showing that the predicted behaviors are in fact met by actual computations.

In what follows, we will use standard notations (cf. [9] and [13]). Moreover, C will be a constant independent of t and h, not necessarily the same in each occurrence. Finally, for the continuous solution and the applied loads, we will suppose the necessary regularity.

2 The laminated composite plate model and the mixed formulation

Let us denote with $A = \Omega \times (-t/2, t/2)$ the region in $\mathbf{R}^3$ occupied by an undeformed elastic plate of thickness $t > 0$ and midplane Ω. We suppose that the plate is formed by n superposed orthotropic layers of equal thickness $\frac{t}{n}$ and with material axes arbitrarily oriented in the x-y plane. Thus, the k-th layer ($1 \leq k \leq n$) is positioned in $\Omega \times (z_{k-1}, z_k)$, where $z_0 = -\frac{t}{2}$ and $z_k = -\frac{t}{2} + k\frac{t}{n}$. Moreover we will work in the framework of the First-Order Shear-Deformation Theory (shortly: FSDT (cf. [22])). Restricting ourselves to the case of a clamped plate subjected to a vertical load f, the functional to be considered reads as follows

$$\begin{aligned} \Pi \;=\; & \frac{1}{2}\left(\int_\Omega \mathcal{A}_n \underline{\underline{\varepsilon}}(\underline{u}^*) : \underline{\underline{\varepsilon}}(\underline{u}^*)\, d\Omega + 2\int_\Omega \mathcal{B}_n \underline{\underline{\varepsilon}}(\underline{u}^*) : \underline{\underline{\varepsilon}}(\underline{\theta}^*)\, d\Omega + \int_\Omega \mathcal{D}_n \underline{\underline{\varepsilon}}(\underline{\theta}^*) : \underline{\underline{\varepsilon}}(\underline{\theta}^*)\, d\Omega\right) + \\ + & \int_\Omega \underline{\mathrm{s}}^* \cdot (\underline{\theta}^* - \underline{\nabla} w^*)\, d\Omega - \frac{1}{2}\int_\Omega \mathcal{H}_n^{-1} \underline{\mathrm{s}}^* \cdot \underline{\mathrm{s}}^*\, d\Omega - \int_\Omega f w^*\, d\Omega\ . \end{aligned} \tag{1}$$

Above, $\underline{u}^*$ are the in-plane displacements, $\underline{\theta}^*$ are the rotations, w^* represents the vertical displacement field, and $\underline{\mathrm{s}}^*$ are the resultant shear stresses. Moreover, $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{D}_n$ are fourth order tensors defined by

$$\mathcal{A}_n \underline{\underline{\tau}} = \sum_{k=1}^{n} (z_k - z_{k-1}) \mathcal{C}_k \underline{\underline{\tau}}\ , \tag{2}$$

$$\mathcal{B}_n \underline{\underline{\tau}} = \frac{1}{2}\sum_{k=1}^{n} (z_k^2 - z_{k-1}^2) \mathcal{C}_k \underline{\underline{\tau}}\ , \tag{3}$$

$$\mathcal{D}_n \underline{\underline{\tau}} = \frac{1}{3}\sum_{k=1}^{n} (z_k^3 - z_{k-1}^3) \mathcal{C}_k \underline{\underline{\tau}}, \tag{4}$$

where $\underline{\underline{\tau}}$ is a second order tensor (which in our case will always be symmetric). Moreover, $\mathcal{C}_k$ is a fourth order symmetric and positive definite tensor.

Finally, $\mathcal{H}_n$ is a second order tensor, defined by

$$\mathcal{H}_n \underline{\tau} = \kappa \sum_{k=1}^{n} (z_k - z_{k-1}) \mathcal{S}_k \underline{\tau}\ , \tag{5}$$

where $\underline{\tau}$ is a given vector and $\mathcal{S}_k$ is a second order symmetric and positive definite tensor.

Remark 2.1 *The coefficient κ in (5) is called shear correction factor, and, in the case of homogeneous plates, it is known a priori and it is taken as $5/6$. For a laminated structure this value is not* *the optimal one, in general, and it is not known a priori. However, in the following we suppose to know the value of the shear correction factor κ, also for the composite plate.*

Remark 2.2 *The functional (1) reveals an interesting feature of laminated composites: the bending behaviour and the membrane one are coupled through the term involving the tensor $\mathcal{B}_n$. Thus, even though the plate is acted upon a vertical load, in-plane displacements can occur.*

2.1 Analysis of the model

We first begin by setting (cf. (2)–(5))

$$
\begin{aligned}
&\mathcal{A}\underline{\underline{\tau}} := t^{-1}\mathcal{A}_n\underline{\underline{\tau}} = t^{-1}\sum_{k=1}^{n}(z_k - z_{k-1})\mathcal{C}_k\underline{\underline{\tau}}\ , \qquad \mathcal{B}\underline{\underline{\tau}} := t^{-2}\mathcal{B}_n\underline{\underline{\tau}} = \frac{t^{-2}}{2}\sum_{k=1}^{n}(z_k^2 - z_{k-1}^2)\mathcal{C}_k\underline{\underline{\tau}} \\
&\mathcal{D}\underline{\underline{\tau}} := t^{-3}\mathcal{D}_n\underline{\underline{\tau}} = \frac{t^{-3}}{3}\sum_{k=1}^{n}(z_k^3 - z_{k-1}^3)\mathcal{C}_k\underline{\underline{\tau}}\ , \qquad \mathcal{H}\underline{\tau} := t^{-1}\mathcal{H}_n\underline{\tau} = t^{-1}\kappa\sum_{k=1}^{n}(z_k - z_{k-1})\mathcal{S}_k\underline{\tau}\ .
\end{aligned}
\tag{6}
$$

Hence, the functional to be considered is easily recognized to be (cf. (1))

$$
\begin{aligned}
\Pi_t \ &= \ \frac{1}{2}\left(t\int_\Omega \mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}^*) : \underline{\underline{\varepsilon}}(\underline{u}^*)\, d\Omega + 2t^2\int_\Omega \mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}^*) : \underline{\underline{\varepsilon}}(\underline{\theta}^*)\, d\Omega + t^3\int_\Omega \mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}^*) : \underline{\underline{\varepsilon}}(\underline{\theta}^*)\, d\Omega\right) + \\
&+ \ \int_\Omega \underline{\mathrm{s}}^* \cdot (\underline{\theta}^* - \underline{\nabla} w^*)\, d\Omega - \frac{t^{-1}}{2}\int_\Omega \mathcal{H}^{-1}\underline{\mathrm{s}}^* \cdot \underline{\mathrm{s}}^*\, d\Omega - \int_\Omega f w^*\, d\Omega\ .
\end{aligned}
\tag{7}
$$

In order to study the behaviour of the solution when t is small, we decide to scale the unknowns and the load in a proper way. We thus set

$$
\underline{u} = t^{-1}\underline{u}^*\ , \qquad \underline{\theta} = \underline{\theta}^*\ , \qquad w = w^*\ , \qquad \underline{\mathrm{s}} = t^{-3}\underline{\mathrm{s}}^*\ , \tag{8}
$$

and we choose the load as

$$
f = t^3 g\ . \tag{9}
$$

Inserting these expressions in (7) and dividing by t^3, we finally obtain the functional

$$
\begin{aligned}
\Pi_t(\underline{u}, \underline{\theta}, w, \underline{\mathrm{s}}) \ &= \ \frac{1}{2}\left(\int_\Omega \mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{u})\, d\Omega + 2\int_\Omega \mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{\theta})\, d\Omega + \int_\Omega \mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}) : \underline{\underline{\varepsilon}}(\underline{\theta})\, d\Omega\right) + \\
&+ \ \int_\Omega \underline{\mathrm{s}} \cdot (\underline{\theta} - \underline{\nabla} w)\, d\Omega - \frac{t^2}{2}\int_\Omega \mathcal{H}^{-1}\underline{\mathrm{s}} \cdot \underline{\mathrm{s}}\, d\Omega - \int_\Omega g w\, d\Omega\ .
\end{aligned}
\tag{10}
$$

Setting $U = H_0^1(\Omega)^2$, $\Theta = H_0^1(\Omega)^2$, $W = H_0^1(\Omega)$, and $S = L^2(\Omega)^2$, we are thus led to consider the variational problem

Problem 2.1 *Find* $(\underline{u}, \underline{\theta}, w; \underline{\mathrm{s}}) \in U \times \Theta \times W \times S$*, solution of the following system*

$$\begin{cases} (\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{v})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{v}), \underline{\underline{\varepsilon}}(\underline{\theta})) = 0 & \forall \underline{v} \in U \\ (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\underline{\mathrm{s}}, \underline{\eta} - \underline{\nabla}\zeta) = (g, \zeta) & \forall (\underline{\eta}, \zeta) \in \Theta \times W \\ (\underline{\mathrm{r}}, \underline{\theta} - \underline{\nabla} w) - t^2(\mathcal{H}^{-1}\underline{\mathrm{s}}, \underline{\mathrm{r}}) = 0 & \forall \underline{\mathrm{r}} \in S \end{cases} . \tag{11}$$

We will show that Problem 2.1 admits a unique solution in $U \times \Theta \times W \times S$. To this end, we need to establish the following lemma.

Lemma 2.1 *There exist positive constants C_1 and C_2, independent of t, such that*

$$C_1 \left(||\underline{\underline{\tau}}||_0^2 + ||\underline{\underline{\sigma}}||_0^2 \right) \le (\mathcal{A}\underline{\underline{\tau}}, \underline{\underline{\tau}}) + 2(\mathcal{B}\underline{\underline{\tau}}, \underline{\underline{\sigma}}) + (\mathcal{D}\underline{\underline{\sigma}}, \underline{\underline{\sigma}}) \le C_2 \left(||\underline{\underline{\tau}}||_0^2 + ||\underline{\underline{\sigma}}||_0^2 \right) , \tag{12}$$

for every $\underline{\underline{\tau}}, \underline{\underline{\sigma}} \in L^2(\Omega)^4_s$, $L^2(\Omega)^4_s$ denoting the space of symmetric tensors with components in $L^2(\Omega)$.

Proof: We first introduce the linear operator

$$\mathcal{L} \, : \, L^2(\Omega)^4_s \times L^2(\Omega)^4_s \longrightarrow L^2(\Omega)^4_s \times L^2(\Omega)^4_s$$

defined by

$$\mathcal{L}[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] = [\mathcal{A}\underline{\underline{\tau}} + \mathcal{B}\underline{\underline{\sigma}}, \mathcal{B}\underline{\underline{\tau}} + \mathcal{D}\underline{\underline{\sigma}}] \ , \tag{13}$$

for each $[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] \in L^2(\Omega)^4_s \times L^2(\Omega)^4_s$. Notice that we have

$$(\mathcal{L}[\underline{\underline{\tau}}, \underline{\underline{\sigma}}], [\underline{\underline{\tau}}, \underline{\underline{\sigma}}]) = (\mathcal{A}\underline{\underline{\tau}}, \underline{\underline{\tau}}) + 2(\mathcal{B}\underline{\underline{\tau}}, \underline{\underline{\sigma}}) + (\mathcal{D}\underline{\underline{\sigma}}, \underline{\underline{\sigma}}) \ , \tag{14}$$

because of the symmetry of the operator $\mathcal{B}$. Moreover, if one defines, for $1 \le k \le n$,

$$\mathcal{L}_k[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] = [t^{-1}(z_k - z_{k-1})\mathcal{C}_k\underline{\underline{\tau}} + \frac{t^{-2}}{2}(z_k^2 - z_{k-1}^2)\mathcal{C}_k\underline{\underline{\sigma}}, \frac{t^{-2}}{2}(z_k^2 - z_{k-1}^2)\mathcal{C}_k\underline{\underline{\tau}} + \frac{t^{-3}}{3}(z_k^3 - z_{k-1}^3)\mathcal{C}_k\underline{\underline{\sigma}}] \ , \tag{15}$$

then it is easily recognized that it holds (cf. (6))

$$\mathcal{L}[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] = \sum_{k=1}^{n} \mathcal{L}_k[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] \ , \tag{16}$$

for each $[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] \in L^2(\Omega)^4_s \times L^2(\Omega)^4_s$. We will show that each $\mathcal{L}_k$ is indeed a continuous and coercive linear operator with continuity and coerciveness constants independent of t. Hence, (12) will be a straightforward consequence of (14) and (16). We proceed by splitting $\mathcal{L}_k$ as

$$\mathcal{L}_k = L_k^{(2)} \, L_k^{(1)} \; , \tag{17}$$

where

$$\mathcal{L}_k^{(1)}[\underline{\underline{\tau}}, \underline{\underline{\sigma}}] := [\mathcal{C}_k \underline{\underline{\tau}}, \mathcal{C}_k \underline{\underline{\sigma}}] \qquad \forall [\underline{\underline{\tau}}, \underline{\underline{\sigma}}] \in L^2(\Omega)^4_s \times L^2(\Omega)^4_s \; , \tag{18}$$

and

$$\mathcal{L}_k^{(2)}[\underline{\underline{\tau}}', \underline{\underline{\sigma}}'] := [t^{-1}(z_k - z_{k-1})\underline{\underline{\tau}}' + \frac{t^{-2}}{2}(z_k^2 - z_{k-1}^2)\underline{\underline{\sigma}}', \frac{t^{-2}}{2}(z_k^2 - z_{k-1}^2)\underline{\underline{\tau}}' + \frac{t^{-3}}{3}(z_k^3 - z_{k-1}^3)\underline{\underline{\sigma}}'] \; , \tag{19}$$

for each $[\underline{\underline{\tau}}', \underline{\underline{\sigma}}'] \in L^2(\Omega)^4_s \times L^2(\Omega)^4_s$. It is straightforward to check that both $\mathcal{L}_k^{(1)}$ and $\mathcal{L}_k^{(2)}$ are symmetric operators and they commute. Moreover, $\mathcal{L}_k^{(1)}$ is clearly continuous and coercive with continuity and coerciveness constants independent of t. To study the features of $\mathcal{L}_k^{(2)}$ we only need to consider the real matrix $A = A(k)$ (cf. (19)), where $A(k)$ is defined by

$$A(k) = \begin{bmatrix} t^{-1}(z_k - z_{k-1}) & \frac{t^{-2}(z_k^2 - z_{k-1}^2)}{2} \\ \frac{t^{-2}(z_k^2 - z_{k-1}^2)}{2} & \frac{t^{-3}}{3}(z_k^3 - z_{k-1}^3) \end{bmatrix} . \tag{20}$$

In fact, an easy computation shows that $\mathcal{L}_k^{(2)}$ is coercive on $L^2(\Omega)^4_s \times L^2(\Omega)^4_s$ if and only if $A(k)$ is positive-definite. Thus, the determinant $\Delta(k)$ of $A(k)$ is

$$\Delta(k) = \frac{t^{-4}}{3}(z_k^3 - z_{k-1}^3)(z_k - z_{k-1}) - \frac{t^{-4}}{4}(z_k^2 - z_{k-1}^2)^2 = \frac{t^{-4}}{12}(z_k - z_{k-1})^4 = \frac{1}{12n^4} > 0 \; , \tag{21}$$

since $z_k - z_{k-1} = \frac{t}{n}$. Moreover, its trace is

$$Tr(k) = \frac{1}{n} + \frac{t^{-2}}{3n}(z_k^2 + z_k z_{k-1} + z_{k-1}^2) \; . \tag{22}$$

It follows that the eigenvalues are given by

$$\lambda_m(k) = \frac{Tr(k)}{2}\left(1 - \sqrt{1 - \frac{4\Delta(k)}{Tr(k)^2}}\right)$$

$$\lambda_M(k) = \frac{Tr(k)}{2}\left(1 + \sqrt{1 - \frac{4\Delta(k)}{Tr(k)^2}}\right) \; . \tag{23}$$

Noting that (cf. (22))

$$\frac{1}{n} \leq Tr(k) \leq \frac{5}{4n} \ , \tag{24}$$

and recalling (21), we get

$$\lambda_m(k) \geq C_1(k)$$

$$\lambda_M(k) \leq C_2(k) \ , \tag{25}$$

for some positive constants $C_1(k)$ and $C_2(k)$, independent of t.

Hence, $\mathcal{L}_k^{(2)}$ is coercive and continuous on $L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$, with continuity and coerciveness constants independent of t. It follows that the composition $\mathcal{L}_k = \mathcal{L}_k^{(2)}\mathcal{L}_k^{(1)}$ is coercive and continuous on $L^2(\Omega)_s^4 \times L^2(\Omega)_s^4$, with continuity and coerciveness constants independent of t. Recalling also (14) and (16) we finally obtain

$$C_1 \left(||\underline{\underline{\tau}}||_0^2 + ||\underline{\underline{\sigma}}||_0^2 \right) \leq (\mathcal{A}\underline{\underline{\tau}}, \underline{\underline{\tau}}) + 2(\mathcal{B}\underline{\underline{\tau}}, \underline{\underline{\sigma}}) + (\mathcal{D}\underline{\underline{\sigma}}, \underline{\underline{\sigma}}) \leq C_2 \left(||\underline{\underline{\tau}}||_0^2 + ||\underline{\underline{\sigma}}||_0^2 \right) \ , \tag{26}$$

for every $\underline{\underline{\tau}}, \underline{\underline{\sigma}} \in L^2(\Omega)_s^4$.

□

Remark 2.3 *We remark that the constants C_1 and C_2 above, even though they do not depend on t, do depend on the number n of layers in the laminated plate. The analysis of the model behavior at fixed t and $n \to +\infty$, although interesting, will not be treated in this paper.*

We are now ready to prove the following Proposition.

Proposition 2.1 *For each $t > 0$ fixed, variational Problem 2.1 has a unique solution $(\underline{u}, \underline{\theta}, w; \underline{\mathrm{s}})$ in $U \times \Theta \times W \times S$.*

Proof: We proceed in two steps.

1– Uniqueness. Suppose $f = 0$. Choosing in (11) $\underline{v} = \underline{u}$, $\underline{\eta} = \underline{\theta}$, $\zeta = w$ and $\underline{\mathrm{r}} = -\underline{\mathrm{s}}$, we get, summing the equations of (11)

$$(\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{u})) + 2(\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}), \underline{\underline{\varepsilon}}(\underline{\theta})) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{\varepsilon}}(\underline{\theta})) + t^2(\mathcal{H}^{-1}\underline{\mathrm{s}}, \underline{\mathrm{s}}) = 0 \ . \tag{27}$$

We first remark that there exists $C > 0$ such that (cf. (6))

$$t^2(\mathcal{H}^{-1}\underline{\mathrm{s}}, \underline{\mathrm{s}}) \geq Ct^2||\underline{\mathrm{s}}||_0^2 \ . \tag{28}$$

To continue, by Lemma 2.1 and Korn's inequality, we have

$$(\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}),\underline{\underline{\varepsilon}}(\underline{u})) + 2(\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}),\underline{\underline{\varepsilon}}(\underline{\theta})) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}),\underline{\underline{\varepsilon}}(\underline{\theta})) \geq C\left(||\underline{u}||_1^2 + ||\underline{\theta}||_1^2\right) \ . \tag{29}$$

From (27)–(29) we infer

$$\underline{u} = 0 \ , \qquad \underline{\theta} = 0 \ , \qquad \underline{\mathrm{s}} = 0 \ . \tag{30}$$

Finally, using (30), from the third equation of (11) we get

$$(\underline{\mathrm{r}}, \underline{\nabla} w) = 0 \qquad \forall \underline{\mathrm{r}} \in S \ , \tag{31}$$

which means $w = 0$, due to Poincaré's inequality. We conclude that for variational Problem 2.1 the solution, if it exists, is unique.

2– Existence. As far as existence is concerned, consider

Problem 2.2 *Find $(\underline{u}, \underline{\theta}, w) \in U \times \Theta \times W$, solution of the following system*

$$\begin{cases} (\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}),\underline{\underline{\varepsilon}}(\underline{v})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{v}),\underline{\underline{\varepsilon}}(\underline{\theta})) = 0 & \forall \underline{v} \in U \\ (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}),\underline{\underline{\varepsilon}}(\underline{\eta})) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}),\underline{\underline{\varepsilon}}(\underline{\eta})) + t^{-2}(\mathcal{H}(\underline{\theta} - \underline{\nabla} w), \underline{\eta} - \underline{\nabla}\zeta) = (g, \zeta) & \forall (\underline{\eta}, \zeta) \in \Theta \times W \end{cases} . \tag{32}$$

By (29) and a little algebra, we can apply Lax-Milgram Lemma to conclude that Problem 2.2 has a unique solution $(\underline{u}, \underline{\theta}, w) \in U \times \Theta \times W$. Setting

$$\underline{\mathrm{s}} = t^{-2}\mathcal{H}(\underline{\theta} - \underline{\nabla} w) \in S \ , \tag{33}$$

we easily see that $(\underline{u}, \underline{\theta}, w, \underline{\mathrm{s}}) \in U \times \Theta \times W \times S$ is a solution of Problem 2.1. The proof is now complete.

□

Furthermore, for Problem 2.1 we have, using the technique of [12] and Lemma 2.1

Proposition 2.2 *There exist positive constants C_1 and C_2, independent of t, such that*

$$C_1 \leq ||\underline{u}||_1 + ||\underline{\theta}||_1 + ||w||_1 + ||\underline{\mathrm{s}}||_{H^{-1}(\mathrm{div})} \leq C_2 \ , \tag{34}$$

where

$$H^{-1}(\mathrm{div}) = \left\{ \underline{\mathrm{r}} : \ \underline{\mathrm{r}} \in H^{-1}(\Omega)^2 \ , \mathrm{div}\, \underline{\mathrm{r}} \in H^{-1}(\Omega) \right\} \ ,$$

equipped with the obvious norm.

Moreover, as t tends to zero, $(\underline{u}, w)$ converges to $(\underline{u}_0, w_0) \in H_0^1(\Omega)^2 \times H_0^2(\Omega)$, solution of

$$\begin{cases} (\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}_0), \underline{\underline{\varepsilon}}(\underline{v})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{v}), \underline{\underline{\varepsilon}}(\underline{\nabla} w_0)) = 0 & \forall \underline{v} \in H_0^1(\Omega)^2 \\ (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}_0), \underline{\underline{\varepsilon}}(\underline{\nabla}\zeta)) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\nabla} w_0), \underline{\underline{\varepsilon}}(\underline{\nabla}\zeta)) = (g, \zeta) & \forall \zeta \in H_0^2(\Omega) \end{cases} . \tag{35}$$

□

Remark 2.4 *Notice that Problem (35) is a Kirchhoff-type laminated composite plate problem (CLPT)- cf. [22]. In a distributional sense, $(\underline{u}_0, w_0)$ is the solution of the system*

$$\begin{cases} \underline{\mathrm{div}}\left(\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}_0) + \mathcal{B}\underline{\underline{\varepsilon}}(\underline{\nabla} w_0)\right) = 0 & \text{in } \Omega \\ \mathrm{div}\,\underline{\mathrm{div}}\left(\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}_0) + \mathcal{D}\underline{\underline{\varepsilon}}(\underline{\nabla} w_0)\right) = g & \text{in } \Omega \end{cases} . \tag{36}$$

Remark 2.5 *In the sequel, we will use the variational formulation (11) as the starting point for our discretization procedure by means of finite elements.*

3 The discretized problem and error analysis

The aim of this section is essentially to show that a performant discretization procedure can be obtained by coupling a good element for homogeneous plate problem with a good membrane element.

3.1 The discretized scheme

To begin, we select finite element spaces $U_h \subset U$, $\Theta_h \subset \Theta$, $W_h \subset W$ and $S_h \subset S$ (cf. [13]), $h > 0$ denoting the meshsize. We then consider the discretized problem

Problem 3.1 *Find $(\underline{u}_h, \underline{\theta}_h, w_h; \underline{\mathrm{s}}_h) \in U_h \times \Theta_h \times W_h \times S_h$, solution of the following system*

$$\begin{cases} (\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}_h), \underline{\underline{\varepsilon}}(\underline{v})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{v}), \underline{\underline{\varepsilon}}(\underline{\theta}_h)) = 0 & \forall \underline{v} \in U_h \\ (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}_h), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}_h), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\underline{\mathrm{s}}_h, R_h\underline{\eta} - \underline{\nabla}\zeta) = G_h(\underline{\eta}, \zeta) & \forall (\underline{\eta}, \zeta) \in \Theta_h \times W_h \\ (\underline{\mathrm{r}}, R_h\underline{\theta}_h - \underline{\nabla} w_h) - t^2(\mathcal{H}^{-1}\underline{\mathrm{s}}_h, \underline{\mathrm{r}}) = 0 & \forall \underline{\mathrm{r}} \in S_h \end{cases} . \tag{37}$$

Above, R_h is an interpolation or projection operator acting on a suitable space and valued in S_h.

Remark 3.1 *The loading term $G_h(\underline{\eta},\zeta)$ appearing in (37) should be a proper approximation of $G(\underline{\eta},\zeta) =: (g,\zeta)$ (cf. (10)). In many cases, but not always, it holds $G_h(\underline{\eta},\zeta) = G(\underline{\eta},\zeta) \quad \forall(\underline{\eta},\zeta) \in \Theta_h \times W_h$. Moreover, we will suppose that $(G - G_h)(\underline{\eta},\zeta) = (G - G_h)(\underline{\eta})$, i.e. the consistency error depends only on the rotational field; this is the case of schemes based on the Linked Interpolation Technique.*

In the sequel, we suppose that $U_h \subset U$ is a good finite element space for standard second-order elliptic problems, i.e. for every $\underline{u} \in U$ it holds

$$\inf_{\underline{v} \in U_h} ||\underline{u} - \underline{v}||_1 = O(h) \ . \tag{38}$$

Moreover, we assume that Θ_h, W_h and S_h, together with the reduction operator R_h, provide a good scheme for homogeneous plate problem. This means that we assume the following

HYPOTHESIS

Consider the problem

Problem 3.2 *Find $(\underline{\theta}^1, w^1; \underline{\mathrm{s}}^1) \in \Theta \times W \times S$, solution of the following system*

$$\begin{cases} (\mathcal{D}_1 \underline{\underline{\varepsilon}}(\underline{\theta}^1), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\underline{\mathrm{s}}^1, \underline{\eta} - \underline{\nabla}\,\zeta) = (\underline{m}, \underline{\eta}) + (g,\zeta) & \forall(\underline{\eta},\zeta) \in \Theta \times W \\ (\underline{\mathrm{r}}, \underline{\theta}^1 - \underline{\nabla} w^1) - t^2(\mathcal{H}_1^{-1}\underline{\mathrm{s}}^1, \underline{\mathrm{r}}) = 0 & \forall \underline{\mathrm{r}} \in S \end{cases} , \tag{39}$$

where $\mathcal{D}_1$ and $\mathcal{H}_1$ are defined by (4)–(5).

We suppose that for the discretized problem

Problem 3.3 *Find $(\underline{\theta}_h^1, w_h^1; \underline{\mathrm{s}}_h^1) \in \Theta_h \times W_h \times S_h$, solution of the following system*

$$\begin{cases} (\mathcal{D}_1 \underline{\underline{\varepsilon}}(\underline{\theta}_h^1), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\underline{\mathrm{s}}_h^1, R_h\underline{\eta} - \underline{\nabla}\,\zeta) = (\underline{m}, \underline{\eta}) + G_h(\underline{\eta},\zeta) & \forall(\underline{\eta},\zeta) \in \Theta_h \times W_h \\ (\underline{\mathrm{r}}, R_h\underline{\theta}_h^1 - \underline{\nabla} w_h^1) - t^2(\mathcal{H}_1^{-1}\underline{\mathrm{s}}_h^1, \underline{\mathrm{r}}) = 0 & \forall \underline{\mathrm{r}} \in S_h \end{cases} , \tag{40}$$

we have the error estimate

$$||\underline{\theta}^1 - \underline{\theta}_h^1||_1 + ||w^1 - w_h^1||_1 + t||\underline{\mathrm{s}}^1 - \underline{\mathrm{s}}_h^1||_0 \leq Ch \ . \tag{41}$$

□

3.2 Error estimates

We begin by setting, for notational simplicity,

$$\mathcal{E}(\underline{u},\underline{\theta};\underline{v},\underline{\eta}) =: (\mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}),\underline{\underline{\varepsilon}}(\underline{v})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{v}),\underline{\underline{\varepsilon}}(\underline{\theta})) + (\mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}),\underline{\underline{\varepsilon}}(\underline{\eta})) + (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}),\underline{\underline{\varepsilon}}(\underline{\eta})) \ . \tag{42}$$

We remark that, by Lemma 2.1 and Korn's inequality, it holds

$$\mathcal{E}(\underline{v},\underline{\eta};\underline{v},\underline{\eta}) \geq C\left(||\underline{v}||_1^2 + ||\underline{\eta}||_1^2\right) \qquad \forall(\underline{v},\underline{\eta}) \in U \times \Theta \ , \tag{43}$$

with C independent of t.

We have the following error estimate, whose proof is very similar to that developed in [14].

Proposition 3.1 *Suppose that* $\underline{\nabla}\, W_h \subset S_h$ *and that*

$$||R_h\underline{\eta} - \underline{\eta}||_0 \leq Ch||\underline{\eta}||_1 \qquad \forall \underline{\eta} \in \Theta \ . \tag{44}$$

Let $\underline{u}_I \in U_h$, $\underline{\theta}_I \in \Theta_h$ *and* $w_I \in W_h$. *Set*

$$\underline{\mathrm{s}}_I = t^{-2}\mathcal{H}(R_h\underline{\theta}_I - \underline{\nabla}\, w_I) \ . \tag{45}$$

Then it holds

$$\begin{aligned} ||\underline{u} - \underline{u}_h||_1 \quad &+ \quad ||\underline{\theta} - \underline{\theta}_h||_1 + t||\underline{\mathrm{s}} - \underline{\mathrm{s}}_h||_0 \\ &\leq \quad C\Big(||\underline{u}_I - \underline{u}||_1 + ||\underline{\theta}_I - \underline{\theta}||_1 + t||\underline{\mathrm{s}}_I - \underline{\mathrm{s}}||_0 + h||\underline{\mathrm{s}}||_0 + ||G - G_h||_{-1}\Big) \ . \end{aligned} \tag{46}$$

Proof: Subtracting the first two equations of (37) from the first two of (11) we have the error equation

$$\mathcal{E}(\underline{u} - \underline{u}_h, \underline{\theta} - \underline{\theta}_h; \underline{v}, \underline{\eta}) + (\underline{\mathrm{s}} - \underline{\mathrm{s}}_h, R_h\underline{\eta} - \underline{\nabla}\,\zeta) = (G - G_h)(\underline{\eta},\zeta) + (\underline{\mathrm{s}}, R_h\underline{\eta} - \underline{\eta}) \ , \tag{47}$$

for every $\underline{v} \in U_h$, $\underline{\eta} \in \Theta_h$ and $\zeta \in W_h$.

Hence we get, recalling also that $(G - G_h)(\underline{\eta},\zeta) = (G - G_h)(\underline{\eta})$ (cf. Remark 3.1)

$$\begin{aligned} \mathcal{E}(\underline{u}_I - \underline{u}_h, \underline{\theta}_I - \underline{\theta}_h; \underline{v}, \underline{\eta}) \quad &+ \quad (\underline{\mathrm{s}}_I - \underline{\mathrm{s}}_h, R_h\underline{\eta} - \underline{\nabla}\,\zeta) = \mathcal{E}(\underline{u}_I - \underline{u}, \underline{\theta}_I - \underline{\theta}; \underline{v}, \underline{\eta}) \\ &+ \quad (\underline{\mathrm{s}}_I - \underline{\mathrm{s}}, R_h\underline{\eta} - \underline{\nabla}\,\zeta) + (G - G_h)(\underline{\eta}) + (\underline{\mathrm{s}}, R_h\underline{\eta} - \underline{\eta}) \ . \end{aligned} \tag{48}$$

Choosing $\underline{v} = \underline{u}_I - \underline{u}_h$, $\underline{\eta} = \underline{\theta}_I - \underline{\theta}_h$ and $\zeta = w_I - w_h$, we have

$$R_h\underline{\eta} - \underline{\nabla}\,\zeta = t^2\mathcal{H}^{-1}(\underline{\mathrm{s}}_I - \underline{\mathrm{s}}_h)\ . \tag{49}$$

Inserting (49) in (48), using coercivity (cf. (43)) we get

$$\begin{aligned} ||\underline{u}_I - \underline{u}_h||_1^2 + ||\underline{\theta}_I - \underline{\theta}_h||_1^2 + t^2||\underline{\mathrm{s}}_I - \underline{\mathrm{s}}_h||_0^2 \leq C\Big(\mathcal{E}(\underline{u}_I - \underline{u}, \underline{\theta}_I - \underline{\theta}; \underline{u}_I - \underline{u}_h, \underline{\theta}_I - \underline{\theta}_h) \\ + \ t^2(\underline{\mathrm{s}}_I - \underline{\mathrm{s}}, \mathcal{H}^{-1}(\underline{\mathrm{s}}_I - \underline{\mathrm{s}}_h)) + (G - G_h)(\underline{\theta}_I - \underline{\theta}_h) + (\underline{\mathrm{s}}, R_h[\underline{\theta}_I - \underline{\theta}_h] - [\underline{\theta}_I - \underline{\theta}_h])\Big)\ . \end{aligned} \tag{50}$$

We thus easily obtain

$$\begin{aligned} ||\underline{u}_I - \underline{u}_h||_1 \ &+ \ ||\underline{\theta}_I - \underline{\theta}_h||_1 + t||\underline{\mathrm{s}}_I - \underline{\mathrm{s}}_h||_0 \\ &\leq \ C\Big(||\underline{u}_I - \underline{u}||_1 + ||\underline{\theta}_I - \underline{\theta}||_1 + t||\underline{\mathrm{s}}_I - \underline{\mathrm{s}}||_0 + h||\underline{\mathrm{s}}||_0 + ||G - G_h||_{-1}\Big)\ . \end{aligned} \tag{51}$$

The triangle inequality now leads to the following estimate

$$\begin{aligned} ||\underline{u} - \underline{u}_h||_1 \ &+ \ ||\underline{\theta} - \underline{\theta}_h||_1 + t||\underline{\mathrm{s}} - \underline{\mathrm{s}}_h||_0 \\ &\leq \ C\Big(||\underline{u}_I - \underline{u}||_1 + ||\underline{\theta}_I - \underline{\theta}||_1 + t||\underline{\mathrm{s}}_I - \underline{\mathrm{s}}||_0 + h||\underline{\mathrm{s}}||_0 + ||G - G_h||_{-1}\Big)\ . \end{aligned} \tag{52}$$

The proof is complete.

□

We now choose suitable interpolants $\underline{u}_I \in U_h$, $\underline{\theta}_I \in \Theta_h$ and $w_I \in W_h$. For $\underline{u}_I$ we take the usual Clemént interpolant of $\underline{u}$. For the other variables, we take advantage from that fact that we have selected a performant scheme for homogeneous plate problems. More precisely, fix $(\underline{\theta}, w; \underline{\mathrm{s}}) \in \Theta \times W \times S$, part of the solution of Problem 2.1. In particular, we have

$$\underline{\mathrm{s}} = t^{-2}\mathcal{H}(\underline{\theta} - \underline{\nabla}\,w)\ . \tag{53}$$

Consider now the following homogeneous-type problem.

Problem 3.4 *Find $(\hat{\underline{\theta}}, \hat{w}; \hat{\underline{\mathrm{s}}}) \in \Theta \times W \times S$, solution of the following system*

$$\begin{cases} (\mathcal{D}\underline{\underline{\varepsilon}}(\hat{\underline{\theta}}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\hat{\underline{\mathrm{s}}}, \underline{\eta} - \underline{\nabla}\zeta) = (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\underline{\mathrm{s}}, \underline{\eta} - \underline{\nabla}\zeta) & \forall(\underline{\eta}, \zeta) \in \Theta \times W \\ (\underline{\mathrm{r}}, \hat{\underline{\theta}} - \underline{\nabla}\,\hat{w}) - t^2(\mathcal{H}^{-1}\hat{\underline{\mathrm{s}}}, \underline{\mathrm{r}}) = 0 & \forall \underline{\mathrm{r}} \in S \end{cases}\ . \tag{54}$$

Clearly, Problem 3.4 has the unique solution $(\hat{\underline{\theta}}, \hat{w}; \hat{\underline{\mathrm{s}}}) = (\underline{\theta}, w; \underline{\mathrm{s}})$. For the discretized problem

Problem 3.5 *Find* $(\hat{\underline{\theta}}_I, \hat{w}_I; \hat{\underline{\mathrm{s}}}_I) \in \Theta_h \times W_h \times S_h$*, solution of the following system*

$$\begin{cases} (\mathcal{D}\underline{\underline{\varepsilon}}(\hat{\underline{\theta}}_I), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\hat{\underline{\mathrm{s}}}_I, R_h\underline{\eta} - \underline{\nabla}\zeta) = (\mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}), \underline{\underline{\varepsilon}}(\underline{\eta})) + (\underline{\mathrm{s}}, \underline{\eta} - \underline{\nabla}\zeta) & \forall(\underline{\eta}, \zeta) \in \Theta_h \times W_h \\ (\underline{\mathrm{r}}, R_h\hat{\underline{\theta}}_I - \underline{\nabla}\, \hat{w}_I) - t^2(\mathcal{H}^{-1}\hat{\underline{\mathrm{s}}}_I, \underline{\mathrm{r}}) = 0 & \forall \underline{\mathrm{r}} \in S_h \end{cases} , \qquad (55)$$

we thus have the error estimate (cf. (41))

$$||\underline{\theta} - \hat{\underline{\theta}}_I||_1 + ||w - \hat{w}_I||_1 + t||\underline{\mathrm{s}} - \hat{\underline{\mathrm{s}}}_I||_0 \leq Ch \ . \qquad (56)$$

If we now choose $\underline{\theta}_I = \hat{\underline{\theta}}_I$ and $w_I = \hat{w}_I$, we see that (cf. (45))

$$\underline{\mathrm{s}}_I = t^{-2}\mathcal{H}(R_h\underline{\theta}_I - \underline{\nabla}\, w_I) = \hat{\underline{\mathrm{s}}}_I \ . \qquad (57)$$

Hence, from Proposition 3.1, (56), (57) and standard interpolation estimates (cf. [13]), we get

$$||\underline{u} - \underline{u}_h||_1 + ||\underline{\theta} - \underline{\theta}_h||_1 + t||\underline{\mathrm{s}} - \underline{\mathrm{s}}_h||_0 \leq C\Big(h + ||G - G_h||_{-1,h}\Big) \ . \qquad (58)$$

If the consistency error $||G - G_h||_{-1}$ is $O(h)$, we obtain

$$||\underline{u} - \underline{u}_h||_1 + ||\underline{\theta} - \underline{\theta}_h||_1 + t||\underline{\mathrm{s}} - \underline{\mathrm{s}}_h||_0 \leq Ch \ . \qquad (59)$$

The estimate above provides an error bound for all the variables, but the vertical displacement fields. We notice that

$$\underline{\mathrm{s}} = t^{-2}\mathcal{H}(\underline{\theta} - \underline{\nabla}\, w) \ , \qquad \underline{\mathrm{s}}_h = t^{-2}\mathcal{H}(R_h\underline{\theta}_h - \underline{\nabla}\, w_h) \ , \qquad (60)$$

so that

$$\underline{\nabla}\,(w - w_h) = t^2\mathcal{H}^{-1}(\underline{\mathrm{s}}_h - \underline{\mathrm{s}}) + (\underline{\theta} - R_h\underline{\theta}_h) \ . \qquad (61)$$

Hence we obtain

$$||\underline{\nabla}\,(w - w_h)||_0 \leq C\left(t^2||\underline{\mathrm{s}}_h - \underline{\mathrm{s}}||_0 + ||\underline{\theta} - R_h\underline{\theta}||_0 + ||R_h(\underline{\theta} - \underline{\theta}_h)||_0\right) \ . \qquad (62)$$

By Poicaré's inequality, Proposition 3.1 and (59), we get

$$||w - w_h||_1 \leq Ch \ . \qquad (63)$$

We can summarize what we have developed so far in the following

Proposition 3.2 *Let $(\underline{u}, \underline{\theta}, w; \underline{s})$ be the solution of Problem 2.1, and $(\underline{u}_h, \underline{\theta}_h, w_h; \underline{s}_h)$ be the solution of Problem 3.1. If*

- *U_h is chosen so that*

$$\inf_{\underline{v} \in U_h} ||\underline{u} - \underline{v}||_1 = O(h) \ ; \tag{64}$$

- *Θ_h, W_h, S_h and R_h provides a good aproximation scheme for homogeneous plate bending problem (in particular $\underline{\nabla} W_h \subset S_h$, (41) and (44) hold);*
- *the consistency error $||G - G_h||_{-1}$ is $O(h)$,*

then the following error bound holds true:

$$||\underline{u} - \underline{u}_h||_1 + ||\underline{\theta} - \underline{\theta}_h||_1 + ||w - w_h||_1 + t||\underline{s} - \underline{s}_h||_0 \leq Ch \ . \tag{65}$$

□

4 Applications

In this section we will present two examples of laminated composite plate element, showing that they meet the analysis developed above. We will suppose that our computational domain Ω is a rectangle, and we will partition it into uniform meshes $\mathcal{T}_h$ of rectangles K, whose diameters are bounded by h. For the in-plane displacement field, we choose standard bilinear and continuous functions for both the schemes, i.e. we set

$$U_h = \{\underline{u}_h \in \Theta \, : \, \underline{u}_{h|K} \in Q_1(K)^2 \quad \forall K \in \mathcal{T}_h\} \ , \tag{66}$$

where $Q_1(T)$ is the space of the functions defined on K and bilinear with respect to the usual local coordinates ξ and η. Hence, from standard approximation theory we have

$$\inf_{\underline{v} \in U_h} ||\underline{u} - \underline{v}||_1 = O(h) \ . \tag{67}$$

4.1 A Linked Intepolation Scheme

Our first example arises from an element based on the Linked Interpolation Technique, which has already been analysed in the framework of homogeneous plate problems (cf. [6]). We first set

$$S_h = \{\underline{s}_h \in S \,:\, \underline{s}_{h|K} = (a + b\eta, c + d\xi) \quad \forall K \in \mathcal{T}_h\} \;. \tag{68}$$

We then select

$$\Theta_h = \{\underline{\theta}_h \in \Theta \,:\, \underline{\theta}_{h|K} \in Q_1(K)^2 \oplus S_h b_4 \quad \forall K \in \mathcal{T}_h\} \;, \tag{69}$$

where $b_4 = (1 - \xi^2)(1 - \eta^2)$. Finally, we take

$$W_h = \{w_h \in W \,:\, w_{h|K} \in Q_1(K) \quad \forall K \in \mathcal{T}_h\} \;. \tag{70}$$

Notice that in this case $\underline{\nabla}\, W_h \subset S_h$. The reduction operator R_h is defined through the relation

$$R_h \underline{\eta} = P_h(Id - \underline{\nabla}\, L)\underline{\eta} \;, \tag{71}$$

where P_h is the L^2-projection operator on S_h, Id is the identity operator and L is the so-called Linking operator, which we define below. Let us first introduce for each $K \in \mathcal{T}_h$ the functions

$$\varphi_i = \lambda_j \lambda_k \lambda_m \,. \tag{72}$$

In (72) $\{\lambda_i\}_{1 \le i \le 4}$ are the equations of the sides of K and the indices (i, j, k, m) form a permutation of the set $(1, 2, 3, 4)$. The function φ_i is an edge bubble relatively to the edge e_i of K. Let us now set

$$EB(K) = \mathrm{Span}\, \{\varphi_i\}_{1 \le i \le 4} \,. \tag{73}$$

The operator L is locally defined as

$$L_{|K} \underline{\eta}_h = \sum_{i=1}^{4} \alpha_i \varphi_i \in EB(K), \tag{74}$$

by requiring that

$$(\underline{\eta}_h - \underline{\nabla}\, L \underline{\eta}_h) \cdot \underline{t}_i = \quad \text{constant along each } e_i, \tag{75}$$

where $\underline{t}_i$ is the tangent vector along the side e_i. Notice that with the definition above, L is defined only on Θ_h, but a natural extension to Θ is provided by setting

$$L\underline{\eta} := L\underline{\eta}_P \qquad \forall \underline{\eta} \in \Theta \;, \tag{76}$$

where $\underline{\eta}_P$ is the H^1-projection of $\underline{\eta}$ over Θ_h. Thus, also R_h is defined on the whole Θ. Next, we define G_h by

$$G_h(\underline{\eta}, \zeta) = (g, \zeta + L\underline{\eta}) \ , \tag{77}$$

so that

$$(G - G_h)(\underline{\eta}, \zeta) = -(g, L\underline{\eta}) \ . \tag{78}$$

It has been proved that for the operator L it holds (cf. [6])

$$||L\underline{\eta}||_1 \leq Ch||\underline{\eta}||_1 \qquad \forall \underline{\eta} \in \Theta_h \ , \tag{79}$$

so that (cf. (76))

$$||L\underline{\eta}||_1 = ||L\underline{\eta}_P||_1 \leq Ch||\underline{\eta}_P||_1 \leq Ch||\underline{\eta}||_1 \qquad \forall \underline{\eta} \in \Theta \ . \tag{80}$$

It follows that

- $||G - G_h||_{-1} \leq Ch$ (cf. (78));
- for the operator R_h we have (cf. (71))

$$||\underline{\eta} - R_h\underline{\eta}||_0 = ||\underline{\eta} - P_h\underline{\eta} + P_h\underline{\nabla}\, L\underline{\eta}||_0 \leq ||\underline{\eta} - P_h\underline{\eta}||_0 + ||P_h\underline{\nabla}\, L\underline{\eta}||_0 \leq Ch||\underline{\eta}||_1 \qquad \forall \underline{\eta} \in \Theta \ . \tag{81}$$

Moreover, the scheme under consideration is a robust and first order convergent when applied to homogeneous plate bending problems (cf. [6]). We thus conclude that our method is first order convergent also when used in the case of a laminated composite structure (cf. Proposition 3.2).

4.2 A Mixed Interpolated with Tensorial Components Scheme

The second example we provide comes from the well-known MITC4 plate element (cf. [8] and [10]), which has been successfully used for plate bending problems in the case of homogeneous materials. For this scheme we set

$$S_h = \{\underline{s}_h \in H_0(\mathrm{rot}; \Omega) \ : \ \underline{s}_{h|K} = (a + b\eta, c + d\xi) \quad \forall K \in \mathcal{T}_h\} \ , \tag{82}$$

where (cf. [9], for instance)

$$H_0(\mathrm{rot}; \Omega) = \left\{ \underline{r} \in L^2(\Omega)^2 : \mathrm{rot}\, \underline{r} \in L^2(\Omega) \ , \underline{r} \cdot \underline{t} = 0 \quad \text{on} \quad \partial\Omega \right\} \ .$$

For the approximated space of rotations we select

$$\Theta_h = \{\underline{\theta}_h \in \Theta \, : \, \underline{\theta}_{h|K} \in Q_1(K)^2 \quad \forall K \in \mathcal{T}_h\} \ . \tag{83}$$

Finally, we take

$$W_h = \{w_h \in W \, : \, w_{h|K} \in Q_1(K) \quad \forall K \in \mathcal{T}_h\} \ . \tag{84}$$

Notice that also in this case $\underline{\nabla}\, W_h \subset S_h$. The reduction operator R_h is defined through the relation (cf. [10])

$$R_h \underline{\eta} \in S_h \ , \qquad \int_e R_h \underline{\eta} \cdot \underline{t}_e = \int_e \underline{\eta} \cdot \underline{t}_e \ , \tag{85}$$

for $\underline{\eta}$ smooth enough, and for each side e of the rectangles in the mesh.

As for G_h, we simply choose

$$G_h(\underline{\eta}, \zeta) = (g, \zeta) \ , \tag{86}$$

so that

$$(G - G_h)(\underline{\eta}, \zeta) = 0 \ . \tag{87}$$

It is well-known (cf. [8] and [14]) that if the refinement of the mesh is obtained by splitting each rectangle into sixteen equal subrectangles, such a scheme meets all the features required by Proposition 3.2 (in particular, estimate (41) holds). Hence, we can conclude that in this case our method is first order convergent when used for a laminated composite structure. We remark that some numerical tests relative to this element have been already presented in [1].

Remark 4.1 *The analysis of [8] and [14] can be easily extended to the case of a refinement obtained by splitting each rectangle into four equal subrectangles (cf. also the results in [16] and [20], concerning the analysis of the $Q_1 - P_0$ element for the Stokes problem, to which the $MITC4$ element is strictly linked).*

5 Numerical examples

The aim of this Section is to investigate the numerical performances of the interpolating schemes previously described. To reach this goal the schemes have been implemented into the Finite Element Analysis Program (FEAP) [26] and the convergence rate of the proposed schemes is checked on a model problem, for which the exact solution can be computed.

We consider a composite made with two layers of an high modulus graphite/epoxy material, whose properties are set as follows

$$E_L = 25 \text{ MPa} \ , \quad E_T = 1 \text{ MPa} \ , \quad \nu_{TT} = 0.25 \ , \quad G_{LT} = 0.5 \text{ MPa} \ , \quad G_{TT} = 0.2 \text{ MPa}$$

where the indeces L and T indicate the longitudinal and the transversal directions, while E indicates a Young's modulus, ν a Poisson ratio, G a shear modulus. In particular, we observe that

$$\frac{E_L}{E_T} = 25 \; , \quad \frac{G_{LT}}{E_T} = 0.5 \; , \quad \frac{G_{TT}}{E_T} = 0.2 \; ,$$

We study the case of an unsymmetric 0/90 cross-ply laminate. Accordingly, recalling equations (2)–(5) and expressing all the entries in MPa, we have

$$\mathcal{C}_1 = \begin{bmatrix} 250.627 & .250627 & 0 \\ .250627 & 1.00251 & 0 \\ 0 & 0 & .5 \end{bmatrix} \quad , \quad \mathcal{C}_2 = \begin{bmatrix} 1.00251 & .250627 & 0 \\ .250627 & 250.627 & 0 \\ 0 & 0 & .5 \end{bmatrix}$$

$$\mathcal{S}_1 = \begin{bmatrix} .5 & 0 \\ 0 & .2 \end{bmatrix} \quad , \quad \mathcal{S}_2 = \begin{bmatrix} .2 & 0 \\ 0 & .5 \end{bmatrix}$$

The plate midplane is assumed to be the square $\Omega = [0, a] \times [0, a]$ and subjected to the following sinusoidal load

$$f(x, y) = t^3 g(x, y) = t^3 q_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \tag{88}$$

with q_0 measuring the maximum load intensity. Finally, we consider the following simply supported boundary conditions

$$\begin{array}{llll} \text{at } x = 0 & \text{and} \quad x = a & \Rightarrow & u_2 = 0, w = 0, \theta_1 = 0 \\ \text{at } y = 0 & \text{and} \quad y = a & \Rightarrow & u_1 = 0, w = 0, \theta_2 = 0 \end{array}$$

and the shear factor is set equal to 5/6.

Following References [21] and [22], the exact solution has the form

$$u_1(x, y) = u_{1,0} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \tag{89}$$

$$u_2(x, y) = u_{2,0} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{a}\right) \tag{90}$$

$$w(x, y) = w_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \tag{91}$$

$$\theta_1(x, y) = \theta_{1,0} \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{a}\right) \tag{92}$$

$$\theta_2(x, y) = \theta_{2,0} \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \tag{93}$$

where $u_{1,0}$, $u_{2,0}$, w_0, $\theta_{1,0}$ and $\theta_{2,0}$ are factors that can be easily computed. To do so, we start from functional (10), we take the variation with respect to $\underline{s}$ and we require a strong satisfaction of the corresponding equation. It follows that $(\underline{u}, \underline{\theta}, w)$ minimizes the potential energy functional

$$\tilde{\Pi}_t = \frac{1}{2}\left(\int_\Omega \mathcal{A}\underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{u})\, d\Omega + 2\int_\Omega \mathcal{B}\underline{\underline{\varepsilon}}(\underline{u}) : \underline{\underline{\varepsilon}}(\underline{\theta})\, d\Omega + \int_\Omega \mathcal{D}\underline{\underline{\varepsilon}}(\underline{\theta}) : \underline{\underline{\varepsilon}}(\underline{\theta})\, d\Omega\right) +$$

$$+ \frac{t^{-2}}{2}\int_\Omega \mathcal{H}(\underline{\theta} - \underline{\nabla} w)\cdot(\underline{\theta} - \underline{\nabla} w)\, d\Omega - \int_\Omega g w\, d\Omega\ . \quad (94)$$

Hence, using positions (88)-(93) and requiring the potential energy stationarity, we obtain a system of five equations, which can be solved in terms of the five quantities $u_{1,0}$, $u_{2,0}$, w_0, $\theta_{1,0}$ and $\theta_{2,0}$. The solution clearly depends on the specific form of the fourth order tensors $\mathcal{A}, \mathcal{B}, \mathcal{D}, \mathcal{H}$, hence on the specific lamination sequence as well as on the plate geometry.

Although the numerical computations have been performed for a simply supported boundary condition (while the theoretical analysis has been developed for the case of a clampled plate), we nonetheless believe that the presented tests are significant of the actual performances of the schemes.

The error of a discrete solution is measured through the relative errors E_u, E_w, E_θ and E_{tot}, defined as

$$E_u^2 = \frac{\sum_{N_i}\left[(u_{h1}(N_i) - u_1(N_i))^2 + (u_{h2}(N_i) - u_2(N_i))^2\right]}{\sum_{N_i}\left[(u_1(N_i))^2 + (u_2(N_i))^2\right]} \quad (95)$$

$$E_w^2 = \frac{\sum_{N_i}(w_h(N_i) - w(N_i))^2}{\sum_{N_i} w(N_i)^2}, \quad (96)$$

$$E_\theta^2 = \frac{\sum_{N_i}\left[(\theta_{h1}(N_i) - \theta_1(N_i))^2 + (\theta_{h2}(N_i) - \theta_2(N_i))^2\right]}{\sum_{N_i}\left[(\theta_1(N_i))^2 + (\theta_2(N_i))^2\right]} \quad (97)$$

$$E_{tot}^2 = E_u^2 + E_w^2 + E_\theta^2 \quad (98)$$

For simplicity, the summations are performed on all the nodes N_i relative to global interpolation parameters (that is, the internal parameters associated with bubble functions are neglected). The above error measures can also be seen as discrete L^2-type errors and a h^2 convergence rate in L^2 norm actually means a h convergence rate in the H^1 energy-type norm.

The analyses are performed using uniform meshes and discretizing only one quarter of the plate, due to symmetry considerations. Moreover, the refinement procedure is obtained by splitting each square of a mesh into four equal subsquares. The plate side length is set equal to 1, while two different values of thickness are considered, i.e. $t \in \{10^{-1}, 10^{-3}\}$, indicated in the following as thick and thin case, respectively.

Figures 1-4 show the relative errors versus the number of nodes per side for the interpolation schemes considered.

It is interesting to observe that:

- Both the methods show the appropriate convergence rate for both rotations and vertical displacements.

- Both the methods are almost insensitive to the thickness, in such a way that the error graphs for two different choices of t are very close to each other. As a consequence, all the proposed elements are actually locking-free and they can be used for both thick and thin plate problems.

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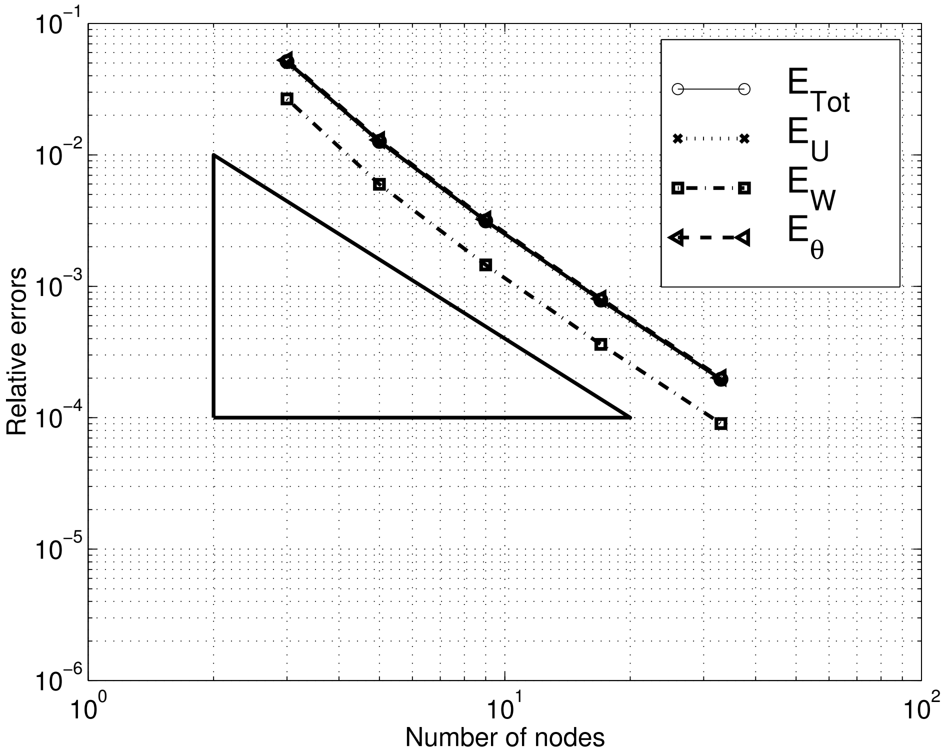

Figure 1: Linked interpolation scheme: thick case (t = 10^{-1}). Relative errors (E_{tot}, E_u, E_w, E_θ) versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^2 convergence rate in the L^2 error norm, corresponding to a h convergence rate in the H^1 energy-type norm.

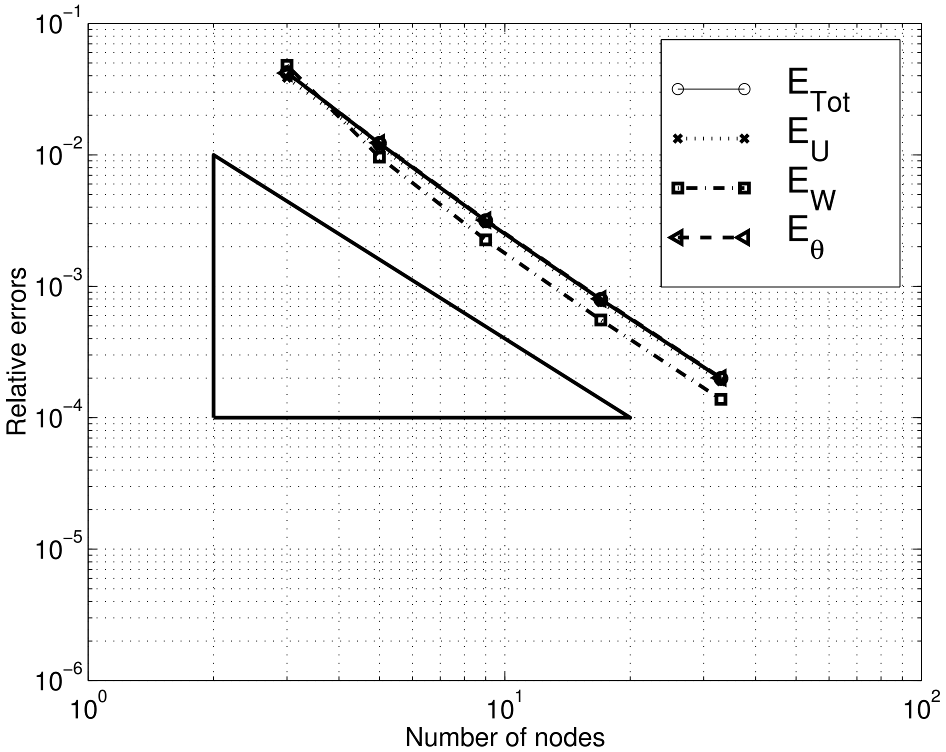

Figure 2: Linked interpolation scheme: thin case (t = 10^{-3}). Relative errors (E_{tot}, E_u, E_w, E_θ) versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^2 convergence rate in the L^2 error norm, corresponding to a h convergence rate in the H^1 energy-type norm.

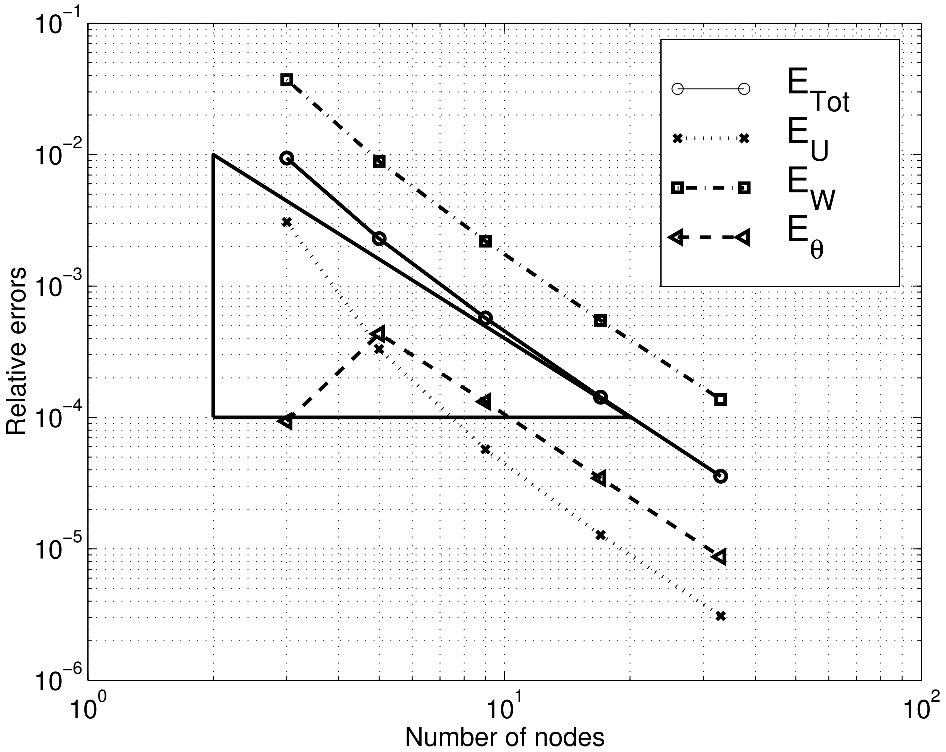

Figure 3: MITC4 interpolation scheme: thick case (t = 10^{-1}). Relative errors (E_{tot}, E_u, E_w, E_θ) versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^2 convergence rate in the L^2 error norm, corresponding to a h convergence rate in the H^1 energy-type norm.

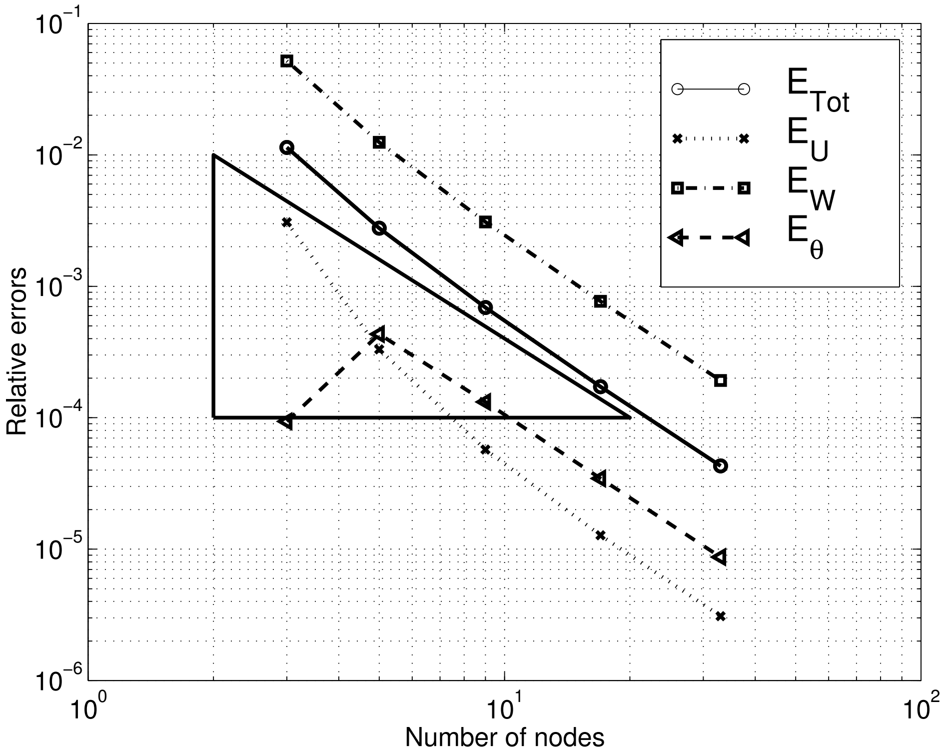

Figure 4: MITC4 interpolation scheme: thin case (t = 10^{-3}). Relative errors (E_{tot}, E_u, E_w, E_θ) versus number of nodes per side for different values of thickness. It can be observed the attainment of the h^2 convergence rate in the L^2 error norm, corresponding to a h convergence rate in the H^1 energy-type norm.