ANALYSIS OF A MIXED FINITE ELEMENT METHOD FOR THE REISSNER-MINDLIN PLATE PROBLEM

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Abstract

An analysis of a triangular mixed finite element method, proposed by Taylor and Auricchio [13] is presented. The method is based on a linked interpolation between deflections and rotations in order to avoid the locking phenomenon (cf. [15]). The analysis shows that the approximated deflections and rotations are first order convergent to the exact solution, uniformly in the thickness.

1 Introduction

The Reissner-Mindlin model is widely used by engineers to describe the behaviour of an elastic plate loaded by a transverse force. The main feature of this model is that the shear deformations are taken into account, thus allowing to consider both thin and moderately thin plates (cf. [10]). Unfortunately, standard low-order finite elements usually fail the approximation, when the plate thickness is numerically small. The reason why this lack of convergence, called *the shear locking phenomenon*, occurs is well-understood (cf. e.g. [10]). Roughly speaking, as the thickness becomes smaller, the shear energy term degenerates to impose, in the limit $t = 0$, the Kirchhoff constraint, which is too severe for low-order elements. In order to overcome this problem a lot of alternative methods have been proposed and studied in recent years. Most of them are based on a suitable mixed formulation of the problem, capable of reducing the influence of the shear energy at the discrete level (cf. e.g. [1], [2], [6], [11] and [12]). In [15] Zienkiewicz et al. have presented a general technique to design finite element schemes which have the hope to perform well. The idea consists in improving the approximated deflection space by means of the rotational degrees of freedom. The discrete vertical displacement is thus appropriately linked to the discrete rotation. An analysis of a method following this philosophy has been developed in [9]. The aim of this paper is to present an error analysis for another method of this type, namely that computationally studied in [13]. This triangular method uses linear continuous functions, augmented by cubic bubbles, to approximate the rotational field. For the vertical displacements, linear continuous functions improved by "edge" bubbles arising from the linking operator are involved. Finally the shear stress field is approximated by means of piecewise constant functions.

The paper is organized as follows. In Section 2 the Reissner-Mindlin plate problem is briefly presented, along with its mixed variational formulation. We also recall a regularity result (Proposition 2.2) established in [3] and another one (Proposition 2.3) proved in [7]. Section 3 is the core of the paper. The discretized problem is introduced and a stability result is proved (Proposition 3.1). The technique used follows the guideline of that developed in [7]. Lemma 3.3 and Corollary 3.1 show that the linking operator has good approximation features. Finally, Proposition 3.2 establishes our error bound, which shows the convergence of the discrete rotations and deflections to the exact solution with order h, uniformly in the thickness. In Section 4 a brief comparison between the method under consideration and another one presented in [6] is given.

Throughout the paper, C will denote a constant independent of h and t, not necessarily the same in each occurrence. Furthermore, we will follow the standard notation, cf. [5] and [8].

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2 The Reissner-Mindlin model and the mixed formulation

Let us denote with $A = \Omega \times (-t/2, t/2)$ the region in $\mathbf{R}^3$ occupied by an undeformed elastic plate of thickness $t > 0$. The Reissner-Mindlin plate model (cf. [10]) describes the bending behaviour of the plate in terms of the transverse displacements $w(t)$ and of the rotation $\underline{\theta}(t)$ of the fibers normal to the midplane Ω .

In the case of a clamped plate, the stationary problem consists of finding the pair $(\underline{\theta}(t), w(t))$ that minimizes the functional

$$\Pi_t(\underline{\theta}(t), w(t)) = \frac{1}{2}\int_\Omega \mathcal{C}\mathcal{E}\underline{\theta}(t) : \mathcal{E}\underline{\theta}(t) + \frac{\lambda t^{-2}}{2}\int_\Omega |\underline{\theta}(t) - \underline{\nabla}\, w(t)|^2 - \int_\Omega f w(t)\, dx\, dy \tag{1}$$

over the space $V = \Theta \times W = (H^1_0(\Omega))^2 \times H^1_0(\Omega)$. In (1) $\mathcal{C}$ is a positive-definite fourth order symmetric tensor in which the Young's modulus E and the Poisson's ratio ν enter. Furthermore $\mathcal{E}\,\underline{\theta}(t)$ is the symmetric gradient of the field $\underline{\theta}(t)$ and $\lambda = Ek/2(1+\nu)$, with k shear correction factor (usually taken as 5/6). Note that the first term in (1) is related to the bending energy of the plate, the second to the shear energy, while the last corresponds to the external energy. Korn's inequality assures that $a(\cdot,\cdot) = \int_\Omega \mathcal{C}\mathcal{E}(\cdot) : \mathcal{E}(\cdot)$ is a coercive form over Θ so that there exists a unique solution $(\underline{\theta}(t), w(t))$ in V of the

Problem P_t: For $t > 0$ fixed, find $(\underline{\theta}(t), w(t))$ in V such that

$$a(\underline{\theta}(t), \underline{\eta}) + \lambda t^{-2}\,(\underline{\theta}(t) - \underline{\nabla}\, w(t), \underline{\eta} - \underline{\nabla}\, v) = \int_\Omega f v \qquad \forall(\underline{\eta}, v) \in V. \tag{2}$$

It is well-known that a straightforward finite element discretization based on formulation (2) generally fails the approximation because of the shear locking phenomenon (cf. [10]). Essentially, when the plate is numerically thin, the shear term in (2) imposes the Kirchhoff constraint which is too restrictive for standard low-order elements. Many methods have been presented (and studied) in order to overcome this undesirable lack of convergence. Several of them are based on a mixed formulation of problem (1). This means that one introduces the scaled shear stress (cf. e.g. [5])

$$\underline{\gamma} = \lambda t^{-2}(\underline{\theta} - \underline{\nabla}\, w)$$

as an independent unknown of the problem. Therefore one is led to consider the new functional

$$\tilde{\Pi}_t(\underline{\theta}, w, \underline{\gamma}) = \frac{1}{2}a(\underline{\theta}, \underline{\theta}) - \frac{\lambda^{-1}t^2}{2}||\underline{\gamma}||^2_{0,\Omega} + (\underline{\gamma}, \underline{\theta} - \underline{\nabla}\, w) - (f, w) \tag{3}$$

on $V \times (L^2(\Omega))^2$. Hence, the Euler-Lagrange equations of (3) gives the following mixed variational plate problem:

Problem $\tilde{P}_t$: For $t > 0$ fixed, find $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$ in $V \times (L^2(\Omega))^2$ such that

$$\begin{cases} a(\underline{\theta}(t), \underline{\eta}) + (\underline{\gamma}(t), \underline{\eta} - \underline{\nabla}\, v) = (f, v) & \forall (\underline{\eta}, v) \in V \\ (\underline{s}, \underline{\theta}(t) - \underline{\nabla}\, w(t)) - \lambda^{-1} t^2 (\underline{\gamma}(t), \underline{s}) = 0 & \forall \underline{s} \in (L^2(\Omega))^2 \end{cases} \tag{4}$$

Following the notation of [5], let us now introduce the differential operator

$$\text{rot} \, : \, \underline{\chi} \longrightarrow \text{rot}\, \underline{\chi} = -\frac{\partial \chi_1}{\partial y} + \frac{\partial \chi_2}{\partial x}.$$

In what follows, we also need the Hilbert space $\Gamma = H_0(\text{rot}; \Omega)$ defined by

$$H_0(\text{rot}; \Omega) = \Big\{ \underline{\chi} \, : \underline{\chi} \in (L^2(\Omega))^2 \, , \quad \text{rot}\, \underline{\chi} \in L^2(\Omega) \, , \quad \underline{\chi} \cdot \underline{t} = 0 \quad \text{on} \quad \partial\Omega \Big\}$$

$$||\underline{\chi}||^2_{H_0(\text{rot};\Omega)} \, := \, ||\underline{\chi}||^2_{0,\Omega} + ||\text{rot}\, \underline{\chi}||^2_{0,\Omega}$$

(here $\underline{t}$ is the unit tangent to $\partial\Omega$) and its dual space

$$\Gamma' = H^{-1}(\text{div}\, ; \Omega) = \Big\{ \underline{\gamma} \, : \underline{\gamma} \in (H^{-1}(\Omega))^2 \quad \text{div}\, \underline{\gamma} \in H^{-1}(\Omega) \Big\}.$$

The space Γ' will be equipped with the norm

$$||\underline{\gamma}||^2_{\Gamma'} \, := \, ||\underline{\gamma}||^2_{-1,\Omega} + ||\text{div}\, \underline{\gamma}||^2_{-1,\Omega},$$

which is easily seen to be equivalent to the natural dual norm induced by $H_0(\text{rot}\, ; \Omega)$.

We have that the following proposition holds (for a proof see [5]).

Proposition 2.1 *Given $t > 0$ and $f \in H^{-1}(\Omega)$, there exists a unique triple $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$ in $V \times (L^2(\Omega))^2$ satisfying equations (4). One has the estimate*

$$||\underline{\theta}(t)||_1 + ||w(t)||_1 + ||\underline{\gamma}(t)||_{\Gamma'} + t||\underline{\gamma}(t)||_0 \leq C||f||_{-1}. \tag{5}$$

Moreover, $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$ converges in $V \times \Gamma'$ to the solution $(\underline{\theta}_0, w_0, \underline{\gamma}_0)$ of the Kirchhoff problem:

Problem $\tilde{P}_0$: Find $(\underline{\theta}_0, w_0, \underline{\gamma}_0)$ in $V \times \Gamma'$ such that

$$\begin{cases} a(\underline{\theta}_0, \underline{\eta}) + < \underline{\gamma}_0, \underline{\eta} - \underline{\nabla}\, v >= (f, v) & \forall (\underline{\eta}, v) \in V \\ < \underline{s}, \underline{\theta}_0 - \underline{\nabla}\, w_0 >= 0 & \forall \underline{s} \in \Gamma', \end{cases} \tag{6}$$

where the brackets denote the duality between Γ and Γ'.

□

In the sequel, we will need the following regularity results (cf. e.g. [3] and [7]).

Proposition 2.2 *Suppose that Ω is convex and $f \in L^2(\Omega)$. Let $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$ be the solution of problem (4). Then the following estimate holds*

$$||\underline{\theta}(t)||_2 + ||w(t)||_2 + ||\underline{\gamma}(t)||_{H(\text{div})} + t||\underline{\gamma}(t)||_1 \leq C||f||_0, \tag{7}$$

where

$$||\underline{\gamma}(t)||^2_{H(\text{div})} = ||\underline{\gamma}(t)||^2_0 + ||\text{div}\underline{\gamma}(t)||^2_0.$$

□

We now write the pair $(\underline{\theta}(t), w(t))$ as

$$(\underline{\theta}(t), w(t)) = (\underline{\theta}_0 + \underline{\theta}_r(t), w_0 + w_r(t)), \tag{8}$$

where $(\underline{\theta}_0, , w_0)$ is defined by (6) and $(\underline{\theta}_r(t), w_r(t))$ can be thought as a remainder. In [7] it has been proved that one has the following result

Proposition 2.3 *Suppose that Ω is convex and $f \in L^2(\Omega)$. Then it holds*

$$||w_0||_3 + ||\underline{\theta}(t)||_2 + ||\underline{\gamma}(t)||_0 + t||\underline{\gamma}(t)||_1 \leq C(||f||_{-1} + t||f||_0), \tag{9}$$

$$||\underline{\theta}_r(t)||_1 \leq Ct||f||_{-1}, \tag{10}$$

$$||w_r(t)||_2 \leq Ct(||f||_{-1} + t||f||_0). \tag{11}$$

□

3 The discretized method and error analysis

From now on, for the sake of simplicity and without loss of generality, we will choose $\lambda = 1$. Hence the mixed variational problem reads as follows

Problem $\tilde{P}_t$: For $t > 0$ fixed, find $(\underline{\theta}, w, \underline{\gamma})$ in $V \times (L^2(\Omega))^2$ such that

$$\begin{cases} a(\underline{\theta}, \underline{\eta}) + (\underline{\gamma}, \underline{\eta} - \underline{\nabla}\, v) = (f, v) & \forall (\underline{\eta}, v) \in V \\ (\underline{\text{s}}, \underline{\theta} - \underline{\nabla}\, w) - t^2(\underline{\gamma}, \underline{\text{s}}) = 0 & \forall \underline{\text{s}} \in (L^2(\Omega))^2. \end{cases} \tag{12}$$

Let us now introduce a sequence $\{\mathcal{T}_h\}_{h>0}$ of partitionings of Ω into triangles. We will also suppose the regularity of $\{\mathcal{T}_h\}_{h>0}$ (cf. [8]), in the sense that there exists a constant $\sigma > 0$ such that

$$h_T \leq \sigma \rho_T \qquad \forall T \in \mathcal{T}_h, \tag{13}$$

where h_T is the diameter of triangle T and ρ_T is the maximum diameter of the circles contained in T.

A standard discretization of problem (12) consists in choosing finite dimensional spaces $\Theta_h \subset \Theta$, $W_h \subset W$ and $\Gamma_h \subset (L^2(\Omega))^2$ and in considering the discrete problem

find $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$ *in* $\Theta_h \times W_h \times \Gamma_h$ *such that*

$$\begin{cases} a(\underline{\theta}_h, \underline{\eta}) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla}\, v) = (f, v) & \forall (\underline{\eta}, v) \in \Theta_h \times W_h \\ (\underline{\text{s}}, \underline{\theta}_h - \underline{\nabla}\, w_h) - t^2(\underline{\gamma}_h, \underline{\text{s}}) = 0 & \forall \underline{\text{s}} \in \Gamma_h. \end{cases} \tag{14}$$

In [15] a differrent approach has been used in the discretization procedure. The idea is to “augment” the deflection space by means of the rotational degrees of freedom. More precisely, a suitable linear and bounded operator $L : \Theta_h \longrightarrow H^1_0(\Omega)$ is defined and the new finite element space

$$V_h = \left\{ (\underline{\eta}_h,\ v_h + L\underline{\eta}_h) : \quad \underline{\eta}_h \in \Theta_h,\ v_h \in W_h \right\} \tag{15}$$

is chosen to approximate the kinematic unknowns. Hence, the discretized problem becomes

$$\textit{find } (\underline{\theta}_h, w_h^*; \underline{\gamma}_h) \textit{ in } V_h \times \Gamma_h \textit{ such that}$$

$$\begin{cases} a(\underline{\theta}_h, \underline{\eta}) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla}\, v) = (f, v) & \forall (\underline{\eta}, v) \in V_h \\ (\underline{\mathrm{s}}, \underline{\theta}_h - \underline{\nabla}\, w_h^*) - t^2 (\underline{\gamma}_h, \underline{\mathrm{s}}) = 0 & \forall \underline{\mathrm{s}} \in \Gamma_h. \end{cases} \tag{16}$$

Noting that, due to (15), one has

$$w_h^* = w_h + L\underline{\theta}_h, \tag{17}$$

the problem (16) is obviously equivalent to the problem

$$\textit{find } (\underline{\theta}_h, w_h, \underline{\gamma}_h) \textit{ in } \Theta_h \times W_h \times \Gamma_h \textit{ such that}$$

$$\begin{cases} a(\underline{\theta}_h, \underline{\eta}) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla}\,(v + L\underline{\eta})) = (f, v + L\underline{\eta}) & \forall (\underline{\eta}, v) \in \Theta_h \times W_h \\ (\underline{\mathrm{s}}, \underline{\theta}_h - \underline{\nabla}\,(w_h + L\underline{\theta}_h)) - t^2 (\underline{\gamma}_h, \underline{\mathrm{s}}) = 0 & \forall \underline{\mathrm{s}} \in \Gamma_h. \end{cases} \tag{18}$$

Let us now define the approximation spaces and the operator L. Following [13], we set

$$\Theta_h = \left\{ \underline{\eta}_h \in \Theta \,:\, \underline{\eta}_{h|T} \in (P_1(T) \oplus B_3(T))^2 \quad \forall T \in \mathcal{T}_h \right\}$$

$$W_h = \left\{ v_h \in W \,:\, v_{h|T} \in P_1(T) \quad \forall T \in \mathcal{T}_h \right\}$$

$$\Gamma_h = \left\{ \underline{\mathrm{s}}_h \in (L^2(\Omega))^2 \,:\, \underline{\mathrm{s}}_{h|T} \in (P_0(T))^2 \quad \forall T \in \mathcal{T}_h \right\}, \tag{19}$$

where $P_r(T)$ is the space of polynomials defined on T of degree at most r and $B_3(T)$ is the space of cubic bubbles defined on T. In order to define the linear operator L, let us first introduce for each $T \in \mathcal{T}_h$ the functions

$$\varphi_i = \lambda_j \lambda_k\,, \tag{20}$$

where $\{\lambda_i\}_{1 \le i \le 3}$ are the barycentric coordinates of the triangle T and the indices (i, j, k) form a permutation of the set $(1, 2, 3)$. In a sense, the function φ_i is an edge bubble relatively to the edge e_i of T. Let us now set

$$EB(T) = \mathrm{Span}\, \{\varphi_i\}_{1 \le i \le 3}\,. \tag{21}$$

The operator L is locally defined (cf. [13]) as

$$L_{|T}\underline{\eta}_h = \sum_{i=1}^{3} \alpha_i \varphi_i \in EB(T), \tag{22}$$

by requiring that

$$(\underline{\eta}_h - \underline{\nabla}\, L\underline{\eta}_h) \cdot \underline{\tau}_i = \quad \text{constant along each } e_i, \tag{23}$$

where $\underline{\tau}_i$ is the tangential vector to the edge e_i. It is very easy to prove

Lemma 3.1 *For each $T \in \mathcal{T}_h$ the operator*

$$L_{|T} : (P_1(T) \oplus B_3(T))^2 \longrightarrow EB(T)$$

is well-defined, i.e. given $\underline{\eta}_h \in (P_1(T) \oplus B_3(T))^2$, there exists a unique $\psi \in EB(T)$ such that condition (23) is fulfilled.

Proof: Fix an edge e_i in T. Using the representation (22) and the fact that $\varphi_k = 0$ along e_i whenever $k \neq i$ (cf. (20)), condition (23) becomes

$$(\underline{\eta}_h - \underline{\nabla}\, L\underline{\eta}_h) \cdot \underline{\tau}_i = (\underline{\eta}_h - \alpha_i \underline{\nabla}\, \varphi_i) \cdot \underline{\tau}_i = c_i, \tag{24}$$

for a constant c_i. Proving the lemma thus means that the constants α_i and c_i in (24) are uniquely determined by $\underline{\eta}_h$. We first note that, as φ_i vanishes at the endpoints of the edge e_i, it holds

$$\int_{e_i} \underline{\nabla}\, \varphi_i \cdot \underline{\tau}_i = \int_{e_i} \frac{\partial \varphi_i}{\partial \underline{\tau}_i} = 0, \tag{25}$$

i.e. the linear function $\underline{\nabla}\, \varphi_i \cdot \underline{\tau}_i$ has zero mean value on e_i. Using this and integrating (24) along the edge e_i gives

$$\int_{e_i} \underline{\eta}_h \cdot \underline{\tau}_i = c_i |e_i|, \tag{26}$$

and hence

$$c_i = \frac{1}{|e_i|} \int_{e_i} \underline{\eta}_h \cdot \underline{\tau}_i := \overline{\underline{\eta}_h \cdot \underline{\tau}_i}. \tag{27}$$

Furthermore, from (24) and (27) one obtains

$$\alpha_i \underline{\nabla}\, \varphi_i \cdot \underline{\tau}_i = \underline{\eta}_h \cdot \underline{\tau}_i - \overline{\underline{\eta}_h \cdot \underline{\tau}_i}. \tag{28}$$

As $\underline{\nabla}\, \varphi_i \cdot \underline{\tau}_i$ is a linear function on e_i with zero mean value, equation (28) in the unknown α_i is uniquely solvable. The proof is thus complete.

□

Since α_i is determined by the tangent trace of $\underline{\eta}_h \in \Theta_h$, it follows that the $L_{|T}$'s continuously match together to define a global operator

$$L : \Theta_h \longrightarrow L(\Theta_h) \subset H^1_0(\Omega).$$

Notice that, with this choice

$$H_0^1(\Omega) \cap \mathcal{L}_1^1(\Omega, \mathcal{T}_h) \subset W_h^* = (W_h + L(\Theta_h)) \subset H_0^1(\Omega) \cap \mathcal{L}_2^1(\Omega, \mathcal{T}_h),$$

where $\mathcal{L}_k^1(\Omega, \mathcal{T}_h)$ is the space of continuous functions which are locally polynomials of degree k (cf. [5]).

We are ready to prove

Lemma 3.2 *For the linear operator L the following continuity estimate holds*

$$||L\underline{\eta}_h||_1 \le Ch|\underline{\eta}_h|_1. \tag{29}$$

Proof: We proceed locally, on each triangle T. Let us notice that, from (22) and a simple scaling argument, one has

$$|L_{|T}\underline{\eta}_h|_{1,T} \le C\sum_{i=1}^{3}|\alpha_i|. \tag{30}$$

Moreover, from (28)

$$\begin{aligned} |\alpha_i|||\underline{\nabla}\,\varphi_i \cdot \underline{\tau}_i||_{L^\infty(e_i)} &\le ||\underline{\eta}_h \cdot \underline{\tau}_i - \overline{\underline{\eta}_h \cdot \underline{\tau}_i}||_{L^\infty(e_i)} \\ &\le h_T|\underline{\eta}_h|_{W^{1,\infty}(T)} \le C|\underline{\eta}_h|_{1,T}, \end{aligned} \tag{31}$$

from which one obtains

$$|\alpha_i| \le Ch_T|\underline{\eta}_h|_{1,T}. \tag{32}$$

It follows from (30) and (32) that it holds

$$|L_{|T}\underline{\eta}_h|_{1,T} \le Ch_T|\underline{\eta}_h|_{1,T}. \tag{33}$$

Estimate (29) easily follows from Poincaré's inequality.

□

Let us now come to the stability analysis of the method. We will define a mesh-dependent norm on the shear space Γ_h by setting

$$||\underline{\xi}||_h^2 := \sum_{T\in\mathcal{T}_h} h_T^2||\underline{\xi}||_{0,T}^2 + t^2||\underline{\xi}||_0^2. \tag{34}$$

Furthermore, given $(\underline{\theta}, w) \in \Theta_h \times W_h$, we define

$$||\underline{\theta}, w||_h^2 := ||\underline{\theta}||_1^2 + ||\underline{\nabla}\,w||_0^2 + \sum_{T\in\mathcal{T}_h}(h_T^2 + t^2)^{-1}||\underline{\theta} - \underline{\nabla}\,(w + L\underline{\theta})||_{0,T}^2.) \tag{35}$$

Hence, given $(\underline{\theta}, w, \underline{\xi}) \in \Theta_h \times W_h \times \Gamma_h$ we can set

$$|||\underline{\theta}, w, \underline{\xi}|||^2 := ||\underline{\theta}, w||_h^2 + ||\underline{\xi}||_h^2. \tag{36}$$

Moreover, let us set

$$\mathcal{A}(\underline{\theta}, w, \gamma; \underline{\eta}, v, \underline{\mathrm{s}}) := a(\underline{\theta}, \underline{\eta}) + (\underline{\gamma}, \underline{\eta} - \underline{\nabla}\, v) - (\underline{\mathrm{s}}, \underline{\theta} - \underline{\nabla}\, w) + t^2(\underline{\gamma}, \underline{\mathrm{s}}), \tag{37}$$

and

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_h, v_h, \underline{\mathrm{s}}_h) := a(\underline{\theta}_h, \underline{\eta}_h) + (\underline{\gamma}_h, \underline{\eta}_h - \underline{\nabla}\,(v_h + L\underline{\eta}_h)) - (\underline{\mathrm{s}}_h, \underline{\theta}_h - \underline{\nabla}\,(w_h + L\underline{\theta}_h)) + t^2(\underline{\gamma}_h, \underline{\mathrm{s}}_h). \tag{38}$$

Hence the continuous problem (12) can be written as

$$\textit{find } (\underline{\theta}, w, \underline{\gamma}) \textit{ in } \Theta \times W \times (L^2(\Omega))^2 \textit{ such that}$$

$$\mathcal{A}(\underline{\theta}, w, \gamma; \underline{\eta}, v, \underline{\mathrm{s}}) = (f, v) \qquad \forall(\underline{\eta}, v, \underline{\mathrm{s}}) \in \Theta \times W \times (L^2(\Omega))^2, \tag{39}$$

while the discretized one reads as follows

$$\textit{find } (\underline{\theta}_h, w_h, \underline{\gamma}_h) \textit{ in } \Theta_h \times W_h \times \Gamma_h \textit{ such that}$$

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}, v, \underline{\mathrm{s}}) = (f, v + L\underline{\eta}) \qquad \forall(\underline{\eta}, v, \underline{\mathrm{s}}) \in \Theta_h \times W_h \times \Gamma_h. \tag{40}$$

We are ready to prove our stability result. The technique used is very similar to that given in [7].

Proposition 3.1 *Given* $(\underline{\theta}_h, w_h, \underline{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h$*, there exists* $(\underline{\eta}_h, v_h, \underline{\mathrm{s}}_h) \in \Theta_h \times W_h \times \Gamma_h$ *such that*

$$|||\underline{\eta}_h, v_h, \underline{\mathrm{s}}_h||| \le C|||\underline{\theta}_h, w_h, \underline{\gamma}_h||| \tag{41}$$

and

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_h, v_h, \underline{\mathrm{s}}_h) \ge C|||\underline{\theta}_h, w_h, \underline{\gamma}_h|||^2. \tag{42}$$

Proof: Let us $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$ be given in $\Theta_h \times W_h \times \Gamma_h$. The proof will be performed in four steps.
i) Let us first choose $(\underline{\eta}_1, v_1, \underline{\mathrm{s}}_1) \in \Theta_h \times W_h \times \Gamma_h$ such that $\underline{\eta}_1 = \underline{\theta}_h$, $v_1 = w_h$ and $\underline{\mathrm{s}}_1 = \underline{\gamma}_h$. It is obvious that

$$|||\underline{\eta}_1, v_1, \underline{\mathrm{s}}_1||| = |||\underline{\theta}_h, w_h, \underline{\gamma}_h|||. \tag{43}$$

Furthermore it holds

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_1, v_1, \underline{\mathrm{s}}_1) = a(\underline{\theta}_h, \underline{\theta}_h) + t^2||\underline{\gamma}_h||_0^2. \tag{44}$$

By Korn's inequality it follows that

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_1, v_1, \underline{\mathrm{s}}_1) \ge C_1 \left(||\underline{\theta}_h||_1^2 + t^2||\underline{\gamma}_h||_0^2 \right). \tag{45}$$

ii) Choose now $(\underline{\eta}_2, v_2, \underline{\mathrm{s}}_2) \in \Theta_h \times W_h \times \Gamma_h$ such that $v_2 = 0$, $\underline{\mathrm{s}}_2 = 0$ and $\underline{\eta}_{2|T} = h_T^2 b_T \underline{\gamma}_h$, where b_T denotes the cubic bubble within the element T. On one hand it holds

$$|||\underline{\eta}_2, v_2, \underline{s}_2|||^2 = ||\underline{\eta}_2||_1^2 \leq C\left(\sum_T h_T^4 |b_T \underline{\gamma}_h|_{1,T}^2\right). \tag{46}$$

But an easy scaling argument (cf. [8]) shows that $|b_T\underline{\gamma}_h|_{1,T} \leq c h_T^{-1} |\underline{\gamma}_h|_{0,T}$ so that from (46) it follows

$$|||\underline{\eta}_2, v_2, \underline{s}_2|||^2 \leq C\left(\sum_T h_T^2 ||\underline{\gamma}_h||_{0,T}^2\right) \leq C|||\underline{\theta}_h, w_h, \underline{\gamma}_h|||^2. \tag{47}$$

On the other hand we get

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_2, v_2, \underline{s}_2) = a(\underline{\theta}_h, \underline{\eta}_2) + (\underline{\gamma}_h, \underline{\eta}_2), \tag{48}$$

since $L\underline{\eta}_2 = 0$. Now

$$(\underline{\gamma}_h, \underline{\eta}_2) = \sum_T h_T^2 (\underline{\gamma}_h, b_T \underline{\gamma}_h)_T \geq C\left(\sum_T h_T^2 ||\underline{\gamma}_h||_{0,T}^2\right), \tag{49}$$

due to a scaling argument. To control the first term in the right-hand side of (48), we note that for $\delta > 0$

$$a(\underline{\theta}_h, \underline{\eta}_2) \geq -\frac{M}{2\delta}||\underline{\theta}_h||_1^2 - \frac{\delta M}{2}||\underline{\eta}_2||_1^2 \geq -\frac{M}{2\delta}||\underline{\theta}_h||_1^2 - \frac{\delta C}{2}\sum_T h_T^2 ||\underline{\gamma}_h||_{0,T}^2, \tag{50}$$

where M is the continuity constant of the bilinear form $a(\cdot, \cdot)$. Taking δ sufficiently small, from (48), (49) and (50) we get

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_2, v_2, \underline{s}_2) \geq C_2 \sum_T h_T^2 ||\underline{\gamma}_h||_{0,T}^2 - C_3 ||\underline{\theta}_h||_1^2. \tag{51}$$

iii) Choose $(\underline{\eta}_3, v_3, \underline{s}_3) \in \Theta_h \times W_h \times \Gamma_h$ such that $\underline{\eta}_3 = 0$, $v_3 = 0$ and $\underline{s}_3 = \underline{\nabla}\, w_h$. It is an admissible choice since $w_{h|T} \in P_1(T)$. We have

$$|||\underline{\eta}_3, v_3, \underline{s}_3|||^2 = ||\underline{\nabla}\, w_h||_h^2 \leq C||\underline{\nabla}\, w_h||_0^2 \leq C|||\underline{\theta}_h, w_h, \underline{\gamma}_h|||^2. \tag{52}$$

It holds, with $\delta > 0$,

$$\begin{aligned}
\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_3, v_3, \underline{s}_3) &= -(\underline{\nabla}\, w_h, \underline{\theta}_h - \underline{\nabla}\,(w_h + L\underline{\theta}_h)) + t^2(\underline{\gamma}_h, \underline{\nabla}\, w_h) \\
&= ||\underline{\nabla}\, w_h||_0^2 - (\underline{\nabla}\, w_h, \underline{\theta}_h - \underline{\nabla}\, L\underline{\theta}_h) + t^2(\underline{\gamma}_h, \underline{\nabla}\, w_h) \\
&\geq ||\underline{\nabla}\, w_h||_0^2 - \frac{\delta}{2}||\underline{\nabla}\, w_h||_0^2 - \frac{1}{2\delta}||\underline{\theta}_h - \underline{\nabla}\, L\underline{\theta}_h||_0^2 + t^2(\underline{\gamma}_h, \underline{\nabla}\, w_h) \\
&\geq ||\underline{\nabla}\, w_h||_0^2 - \frac{\delta}{2}||\underline{\nabla}\, w_h||_0^2 - \frac{C}{2\delta}||\underline{\theta}_h||_1^2 + t^2(\underline{\gamma}_h, \underline{\nabla}\, w_h).
\end{aligned} \tag{53}$$

Moreover one has, with $\varepsilon > 0$,

$$t^2(\underline{\gamma}_h, \underline{\nabla}\, w_h) \geq t^2 \left(-\frac{1}{2\varepsilon} ||\underline{\gamma}_h||_0^2 - \frac{\varepsilon}{2} ||\underline{\nabla}\, w_h||_0^2 \right). \quad (54)$$

By (53) and (54), taking δ and ε sufficiently small, one finally gets

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_3, v_3, \underline{s}_3) \geq C_4 ||\underline{\nabla}\, w_h||_0^2 - C_5 ||\underline{\theta}_h||_1^2 - C_6 t^2 ||\underline{\gamma}_h||_0^2. \quad (55)$$

iv) Choose $(\underline{\eta}_4, v_4, \underline{s}_4) \in \Theta_h \times W_h \times \Gamma_h$ such that $\underline{\eta}_4 = 0$, $v_4 = 0$ and $\underline{s}_{4|T} = (h_T^2 + t^2)^{-1} P_0(\underline{\theta}_h - \underline{\nabla}\,(w_h + L\underline{\theta}_h))_{|T}$, P_0 being the L^2-projection operator over the piecewise constant functions.

Now it only suffices to take a suitable linear combination of $\{(\underline{\eta}_i, v_i, \underline{s}_i)\}_{i=1}^3$ so that by (43), (45), (47), (51), (52) and (55) it follows that (41) and (42) hold. The proof is then complete.

□

Before facing the problem of establishing an error bound for the method, let us now prove a lemma concerning the linear operator L. We have

Lemma 3.3 *Let us denote with $(\cdot)_I$ the Lagrange interpolating operator onto the space of linear functions. Then the linear and continuous operator*

$$\Pi_T : H^3(T) \longrightarrow P_2(T) \subset H^1(T) \quad (56)$$

defined by

$$v \longrightarrow v_I + L((\underline{\nabla}\, v)_I) \quad (57)$$

is P_2-invariant.

Proof: Let v be in $P_2(T)$. Then since $\underline{\nabla}\, v \in (P_1(T))^2$ it follows that

$$(\underline{\nabla}\, v)_I = \underline{\nabla}\, v$$

so that

$$\Pi_T v = v_I + L\underline{\nabla}\, v\,.$$

Hence the lemma is proved if one establishes

$$v - v_I = L\underline{\nabla}\, v\,. \quad (58)$$

Notice that, due to (20) and (21), $P_2(T)$ can be written as

$$P_2(T) = P_1(T) \oplus EB(T)\,. \quad (59)$$

Thus, take first $v \in P_1(T)$. It holds $v = v_I$ and

$$\underline{\nabla}\, v = (\underline{\nabla}\, v)_I = \text{constant}\,. \quad (60)$$

Hence, it holds

$$|\underline{\nabla}\, v|_{1,T} = |(\underline{\nabla}\, v)_I|_{1,T} = 0.$$

Recalling (33), one has

$$|L\underline{\nabla}\, v|_{1,T} = |L(\underline{\nabla}\, v)_I|_{1,T} \leq Ch_T|(\underline{\nabla}\, v)_I|_{1,T} = 0, \tag{61}$$

i.e.

$$L\underline{\nabla}\, v = 0\,. \tag{62}$$

Therefore (58) holds for linear functions.

Next, take $v \in EB(T)$. By the linearity of L and by (21), it follows that it is sufficient to check (58) for the functions φ_i's only ($1 \leq i \leq 3$). Hence, given φ_i, consider the equation

$$(\underline{\nabla}\, \varphi_i - \underline{\nabla}\, L\underline{\nabla}\, \varphi_i) \cdot \underline{\tau}_i = c_i \tag{63}$$

in the unknown $L\underline{\nabla}\, \varphi_i$. As $\underline{\nabla}\, \varphi_i \in (P_1(T))^2$ and the operator L is well-defined (cf. Lemma 3.1), equation (63) is uniquely solvable. It is easily seen (cf. also (28)), after having recalled that $\overline{\underline{\nabla}\, \varphi_i \cdot \underline{\tau}_i} = 0$), that the solution is given by

$$L\underline{\nabla}\, \varphi_i = \varphi_i\,. \tag{64}$$

Noting that $(\varphi_i)_I = 0$ (since φ_i vanishes at the vertices of the triangle T), it follows from (64) that (58) is fulfilled for the functions of a basis of $EB(T)$ also. The lemma is thus proved, by simply recalling (59).

□

The following Corollary easily follows from the standard approximation theory in Sobolev spaces (cf. [8], for instance).

Corollary 3.1 *For the linear operator Π_T defined as in Lemma 3.3 the following estimate holds*

$$||v - \Pi_T\, v||_{1,T} \leq Ch_T^2|v|_{3,T} \qquad \forall v \in H^3(T). \tag{65}$$

□

We are ready to prove our error bound

Proposition 3.2 *Suppose that Ω is a convex polygon and $f \in L^2(\Omega)$. Let $(\underline{\theta}, w, \gamma)$ be the solution of problem (12) and let $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$ be the solution of the discretized problem (18) once the choices (19)-(23) have been done. Then the following error estimate holds*

$$||\underline{\theta} - \underline{\theta}_h||_1 + ||w - w_h||_1 + \left(\sum_{T\in\mathcal{T}_h} h_T^2||\gamma - \underline{\gamma}_h||_{0,T}^2\right)^{1/2} + t||\gamma - \underline{\gamma}_h||_0 \leq Ch||f||_0. \tag{66}$$

Proof: As in equation (8), let us split $\underline{\theta}$ as $\underline{\theta} = \underline{\theta}_0 + \underline{\theta}_r$ and w as $w = w_0 + w_r$. Choose $\underline{\theta}_L$ (resp. w_L) as the usual Lagrange interpolant of $\underline{\theta}_0$ (resp. w_0). Moreover, choose $\underline{\theta}_C$ (resp. w_C) as the Clemént interpolant of $\underline{\theta}_r$ (resp. w_r). Then set

$$\underline{\theta}_I = \underline{\theta}_L + \underline{\theta}_C\,, \tag{67}$$

$$w_I = w_L + w_C\,. \tag{68}$$

By the standard approximation theory (cf. [8]), it holds (for $s = 0,\, 1$)

$$||\underline{\theta} - \underline{\theta}_I||_{s,T} \le ||\underline{\theta}_0 - \underline{\theta}_L||_{s,T} + ||\underline{\theta}_r - \underline{\theta}_C||_{s,T} \le Ch_T^{2-s}(|\underline{\theta}_0|_{2,T} + |\underline{\theta}_r|_{2,T}) \tag{69}$$

and

$$||w - w_I||_{s,T} \le ||w_0 - w_L||_{s,T} + ||w_r - w_C||_{s,T} \le Ch_T^{2-s}(|w_0|_{2,T} + |w_r|_{2,T}). \tag{70}$$

Finally, let us choose $\underline{\gamma}^* \in \Gamma_h$ such that

$$||\underline{\gamma} - \underline{\gamma}^*||_{\Gamma'} \le Ch||\underline{\gamma}||_{H(\mathrm{div})}, \tag{71}$$

$$||\underline{\gamma} - \underline{\gamma}^*||_0 \le Ch||\underline{\gamma}||_1. \tag{72}$$

For a proof of the existence of such an interpolant cf. e.g. [7].

By Proposition 3.1, given $(\underline{\theta}_h - \underline{\theta}_I, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*) \in \Theta_h \times W_h \times \Gamma_h$, there exists $(\underline{\eta}_h, v_h, \underline{\mathrm{s}}_h) \in \Theta_h \times W_h \times \Gamma_h$ such that

$$|||\underline{\eta}_h, v_h, \underline{\mathrm{s}}_h||| \le C|||\underline{\theta}_h - \underline{\theta}_I, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*||| \tag{73}$$

and

$$\mathcal{A}_h(\underline{\theta}_h - \underline{\theta}_I, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*; \underline{\eta}_h, v_h, \underline{\mathrm{s}}_h) \ge C|||\underline{\theta}_h - \underline{\theta}_I, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*|||^2. \tag{74}$$

By recalling (37), (38) and noting that $(\underline{\theta}, w, \underline{\gamma})$ (resp. $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$) is the solution of the continuous (resp. discretized) problem, one has, choosing $(\underline{\eta}, v, \underline{\mathrm{s}}) = (\underline{\eta}_h, v_h + L\underline{\eta}_h, \underline{\mathrm{s}}_h)$ in equation (39),

$$\begin{aligned}
\mathcal{A}_h(\underline{\theta}_h - \underline{\theta}_I, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*; \underline{\eta}_h, v_h, \underline{\mathrm{s}}_h) &= a(\underline{\theta}_h - \underline{\theta}_I, \underline{\eta}_h) + (\underline{\gamma}_h - \underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\
&- (\underline{\mathrm{s}}_h, \underline{\theta}_h - \underline{\theta}_I - \underline{\nabla}(w_h - w_I + L(\underline{\theta}_h - \underline{\theta}_I))) + t^2(\underline{\gamma}_h - \underline{\gamma}^*, \underline{\mathrm{s}}_h) \\
&= (f, v_h + L\underline{\eta}_h) - a(\underline{\theta}_I, \underline{\eta}_h) - (\underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\
&+ (\underline{\mathrm{s}}_h, \underline{\theta}_I - \underline{\nabla}(w_I + L\underline{\theta}_I)) - t^2(\underline{\gamma}^*, \underline{\mathrm{s}}_h) \\
&= a(\underline{\theta}, \underline{\eta}_h) + (\underline{\gamma}, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) - (\underline{\mathrm{s}}_h, \underline{\theta} - \underline{\nabla}\, w) + t^2(\underline{\gamma}, \underline{\mathrm{s}}_h) \\
&- a(\underline{\theta}_I, \underline{\eta}_h) - (\underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\
&+ (\underline{\mathrm{s}}_h, \underline{\theta}_I - \underline{\nabla}(w_I + L\underline{\theta}_I)) - t^2(\underline{\gamma}^*, \underline{\mathrm{s}}_h)
\end{aligned}$$

$$
\begin{aligned}
&= a(\underline{\theta} - \underline{\theta}_I, \underline{\eta}_h) + (\underline{\gamma} - \underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\
&- (\underline{\mathrm{s}}_h, \underline{\theta} - \underline{\theta}_I - \underline{\nabla}(w - w_I - L\underline{\theta}_I)) + t^2(\underline{\gamma} - \underline{\gamma}^*, \underline{\mathrm{s}}_h).
\end{aligned} \tag{75}
$$

Hence, we get from (74)

$$
\begin{aligned}
C|||\underline{\theta}_h - \underline{\theta}_I, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*|||^2 &\leq a(\underline{\theta} - \underline{\theta}_I, \underline{\eta}_h) + (\underline{\gamma} - \underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\
&- (\underline{\mathrm{s}}_h, \underline{\theta} - \underline{\theta}_I - \underline{\nabla}(w - w_I - L\underline{\theta}_I)) + t^2(\underline{\gamma} - \underline{\gamma}^*, \underline{\mathrm{s}}_h).
\end{aligned} \tag{76}
$$

We are now ready to estimate the four terms in the right-hand side of (76).

i) By the continuity of the bilinear form $a(\cdot, \cdot)$ and (69) one gets

$$
a(\underline{\theta} - \underline{\theta}_I, \underline{\eta}_h) \leq M||\underline{\theta} - \underline{\theta}_I||_1||\underline{\eta}_h||_1 \leq Ch(|\underline{\theta}_0|_2 + |\underline{\theta}_r|_2)||\underline{\eta}_h||_1. \tag{77}
$$

By estimate (9), since $\underline{\theta}_0 = \underline{\nabla}\, w_0$, one easily obtains

$$
a(\underline{\theta} - \underline{\theta}_I, \underline{\eta}_h) \leq Ch(||f||_{-1} + t||f||_0)||\underline{\eta}_h||_1. \tag{78}
$$

ii) For the second term it follows from (71) and (7)

$$
\begin{aligned}
(\underline{\gamma} - \underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) &\leq C||\underline{\gamma} - \underline{\gamma}^*||_{\Gamma'}||\underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)||_{\Gamma} \\
&\leq Ch||\underline{\gamma}||_{H(\mathrm{div})}(||\underline{\eta}_h||_1 + ||\underline{\nabla}\, v_h||_0) \\
&\leq Ch||f||_0(||\underline{\eta}_h||_1 + ||\underline{\nabla}\, v_h||_0).
\end{aligned} \tag{79}
$$

iii) To treat

$$
-(\underline{\mathrm{s}}_h, \underline{\theta} - \underline{\theta}_I - \underline{\nabla}(w - w_I - L\underline{\theta}_I)),
$$

let us split it into two terms:

$$
\begin{aligned}
-(\underline{\mathrm{s}}_h, \underline{\theta} - \underline{\theta}_I - \underline{\nabla}(w - w_I - L\underline{\theta}_I)) = &- (\underline{\mathrm{s}}_h, \underline{\theta}_0 - \underline{\theta}_L - \underline{\nabla}(w_0 - w_L - L\underline{\theta}_L)) \\
&- (\underline{\mathrm{s}}_h, \underline{\theta}_r - \underline{\theta}_C - \underline{\nabla}(w_r - w_C - L\underline{\theta}_C)) \\
&= T_1 + T_2
\end{aligned} \tag{80}
$$

First consider

$$
T_1 = -(\underline{\mathrm{s}}_h, \underline{\theta}_0 - \underline{\theta}_L - \underline{\nabla}(w_0 - w_L - L\underline{\theta}_L)).
$$

Since $\underline{\theta}_0 = \underline{\nabla}\, w_0$ it follows that

$$\underline{\theta}_L = (\underline{\nabla}\, w_0)_L, \tag{81}$$

where $(\underline{\nabla}\, w_0)_L$ denotes the Lagrangian interpolant of the vector valued function $\underline{\nabla}\, w_0$. Hence

$$T_1 = -(\underline{\mathrm{s}}_h, \underline{\theta}_0 - \underline{\theta}_L - \underline{\nabla}\,(w_0 - w_L - L(\underline{\nabla}\, w_0)_L)). \tag{82}$$

But, by the standard approximation theory, one locally has

$$\begin{aligned} -(\underline{\mathrm{s}}_h, \underline{\theta}_0 - \underline{\theta}_L)_T &\leq Ch_T||\underline{\mathrm{s}}_h||_{0,T} h_T^{-1} ||\underline{\theta}_0 - \underline{\theta}_L||_{0,T} \\ &\leq Ch_T||\underline{\mathrm{s}}_h||_{0,T} h_T |\underline{\theta}_0|_{2,T}. \end{aligned} \tag{83}$$

From (83) it follows that

$$-(\underline{\mathrm{s}}_h, \underline{\theta}_0 - \underline{\theta}_L) \leq Ch \left(\sum_T h_T^2 ||\underline{\mathrm{s}}_h||_{0,T}^2 \right)^{1/2} |\underline{\theta}_0|_2. \tag{84}$$

Moreover, by Lemma 3.3, it holds

$$\begin{aligned} -(\underline{\mathrm{s}}_h, -\underline{\nabla}\,(w_0 - w_L - L(\underline{\nabla}\, w_0)_L))_T &\leq Ch_T||\underline{\mathrm{s}}_h||_{0,T} h_T^{-1} |w_0 - \Pi_T w_0|_{1,T} \\ &\leq Ch_T||\underline{\mathrm{s}}_h||_{0,T} h_T |w_0|_{3,T}, \end{aligned} \tag{85}$$

so that

$$-(\underline{\mathrm{s}}_h, -\underline{\nabla}\,(w_0 - w_L - L(\underline{\nabla}\, w_0)_L)) \leq Ch \left(\sum_T h_T^2 ||\underline{\mathrm{s}}_h||_{0,T}^2 \right)^{1/2} |w_0|_3. \tag{86}$$

Hence, by (84), (86) and estimate (9), we infer that

$$T_1 \leq Ch(|w_0|_3 + |\underline{\theta}_0|_2)||\underline{\mathrm{s}}_h||_h \leq Ch(||f||_{-1} + t||f||_0)||\underline{\mathrm{s}}_h||_h. \tag{87}$$

Let us next treat $T_2 = -(\underline{\mathrm{s}}_h, \underline{\theta}_r - \underline{\theta}_C - \underline{\nabla}\,(w_r - w_C - L\underline{\theta}_C))$. Since

$$t^2 \underline{\gamma} = \underline{\theta} - \underline{\nabla}\, w\,,$$

and

$$\underline{\theta}_0 - \underline{\nabla}\, w_0 = 0\,,$$

it follows that

$$t^2 \underline{\gamma} = \underline{\theta}_r - \underline{\nabla}\, w_r. \tag{88}$$

Now, define $t^2 \underline{\gamma}_C$ as follows.

$$t^2\underline{\gamma}_C = \underline{\theta}_C - (\underline{\nabla}\, w_r)_C, \tag{89}$$

where $\underline{\theta}_C$ is the Clemént interpolant of $\underline{\theta}$ and $(\underline{\nabla}\, w_r)_C$ is the Clemént interpolant of $\underline{\nabla}\, w_r$.

We have

$$\begin{aligned} T_2 &= -(\underline{\mathrm{s}}_h, t^2\underline{\gamma} - t^2\underline{\gamma}_C + \underline{\nabla}\, w_C - (\underline{\nabla}\, w_r)_C + \underline{\nabla}\, L\underline{\theta}_C) \\ &= -(\underline{\mathrm{s}}_h, t^2\underline{\gamma} - t^2\underline{\gamma}_C) - (\underline{\mathrm{s}}_h, \underline{\nabla}\, w_C - (\underline{\nabla}\, w_r)_C) - (\underline{\mathrm{s}}_h, \underline{\nabla}\, L\underline{\theta}_C). \end{aligned} \tag{90}$$

Let us estimate the three terms above. We first have

$$-(\underline{\mathrm{s}}_h, t^2\underline{\gamma} - t^2\underline{\gamma}_C) \le Cht||\underline{\gamma}||_1 t||\underline{\mathrm{s}}_h||_0\,, \tag{91}$$

and it follows by estimate (9)

$$-(\underline{\mathrm{s}}_h, t^2\underline{\gamma} - t^2\underline{\gamma}_C) \le Ch(||f||_{-1} + t||f||_0)t||\underline{\mathrm{s}}_h||_0\,, \tag{92}$$

Furthermore, it holds

$$\begin{aligned} -(\underline{\mathrm{s}}_h, \underline{\nabla}\, w_C - (\underline{\nabla}\, w_r)_C) &= -(\underline{\mathrm{s}}_h, \underline{\nabla}\, w_C - \underline{\nabla}\, w_r + \underline{\nabla}\, w_r - (\underline{\nabla}\, w_r)_C) \\ &\le ||\underline{\mathrm{s}}_h||_0||\underline{\nabla}\, w_C - \underline{\nabla}\, w_r||_0 + ||\underline{\mathrm{s}}_h||_0||\underline{\nabla}\, w_r - (\underline{\nabla}\, w_r)_C||_0 \\ &\le Ch||\underline{\mathrm{s}}_h||_0(|w_r|_2 + |\underline{\nabla}\, w_r|_1) \\ &\le Ch||\underline{\mathrm{s}}_h||_0|w_r|_2. \end{aligned} \tag{93}$$

By recalling estimate (11), we therefore get

$$-(\underline{\mathrm{s}}_h, \underline{\nabla}\, w_C - (\underline{\nabla}\, w_r)_C) \le Ch(||f||_{-1} + t||f||_0)t||\underline{\mathrm{s}}_h||_0. \tag{94}$$

Finally, we have

$$-(\underline{\mathrm{s}}_h, \underline{\nabla}\, L\underline{\theta}_C) \le ||\underline{\mathrm{s}}_h||_0|L\underline{\theta}_C|_1 \le Ch||\underline{\mathrm{s}}_h||_0|\underline{\theta}_C|_1 \le Ch||\underline{\mathrm{s}}_h||_0|\underline{\theta}_r|_1. \tag{95}$$

Recalling (10), we thus obtain

$$-(\underline{\mathrm{s}}_h, \underline{\nabla}\, L\underline{\theta}_C) \le Ch||f||_{-1}t||\underline{\mathrm{s}}_h||_0. \tag{96}$$

Collecting estimates (92), (94) and (96), by (90) we obtain

$$T_2 \le Ch(||f||_{-1} + ||f||_0)t||\underline{\mathrm{s}}_h||_0 \le Ch(||f||_{-1} + t||f||_0)||\underline{\mathrm{s}}_h||_h. \tag{97}$$

Recalling that

$$-(\underline{\mathrm{s}}_h, \underline{\theta} - \underline{\theta}_I - \underline{\nabla}\,(w - w_I - L\underline{\theta}_I)) = T_1 + T_2,$$

we infer from estimates (87) and (97)

$$-(\underline{\mathrm{s}}_h, \underline{\theta} - \underline{\theta}_I - \underline{\nabla}\,(w - w_I - L\underline{\theta}_I)) \leq Ch(||f||_{-1} + t||f||_0)||\underline{\mathrm{s}}_h||_h. \tag{98}$$

iv) For the last term of (76) we have

$$t^2(\underline{\gamma} - \underline{\gamma}^*, \underline{\mathrm{s}}_h) \leq Cht||\underline{\gamma}||_1\, t||\underline{\mathrm{s}}_h||_0, \tag{99}$$

so that by estimate (9) we get

$$t^2(\underline{\gamma} - \underline{\gamma}^*, \underline{\mathrm{s}}_h) \leq Ch(||f||_{-1} + t||f||_0)||\underline{\mathrm{s}}_h||_h, \tag{100}$$

Collecting (78), (79), (98) and (100), from (76) it follows

$$|||\underline{\theta}_h - \underline{\theta}_I, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*|||^2 \leq Ch\,(||f||_{-1} + (1+t)||f||_0)\,|||\underline{\eta}_h, v_h, \underline{\mathrm{s}}_h||| \leq Ch|||\underline{\eta}_h, v_h, \underline{\mathrm{s}}_h||| \tag{101}$$

Now, using (73), one has

$$|||\underline{\theta}_h - \underline{\theta}_I, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*||| \leq Ch||f||_0 \tag{102}$$

and error estimate (66) follows from the triangle inequality.

□

Finally, we give an error estimate for the approximated deflection defined by equation (17).

Corollary 3.2 *In the hypotheses of Proposition 3.2, the following error bound holds*

$$||w - w_h^*||_1 \leq Ch||f||_0, \tag{103}$$

where (cf. (17))

$$w_h^* = w_h + L\underline{\theta}_h. \tag{104}$$

Proof: We have

$$||w - w_h^*||_1 = ||w - w_h - L\underline{\theta}_h||_1 \leq ||w - w_h||_1 + ||L\underline{\theta}_h||_1. \tag{105}$$

By (66) it follows that

$$||w - w_h||_1 \leq Ch||f||_0 \tag{106}$$

and

$$||\underline{\theta}_h||_1 \leq C||f||_0. \tag{107}$$

Hence by (105) and Lemma 3.2, we get

$$||w - w_h^*||_1 \leq Ch||f||_0 + Ch||\underline{\theta}_h||_1||f||_0 \leq Ch||f||_0. \tag{108}$$

□

4 Comparison between the method of Taylor-Auricchio and the method of Brezzi-Fortin-Stenberg

In [6] Brezzi, Fortin and Stenberg proposed a linear mixed plate element based on a Galerkin least-squares formulation of the problem. They chose

$$\tilde{\Theta}_h = \{\underline{\eta}_h \in \Theta \,:\, \underline{\eta}_{h|T} \in (P_1(T))^2 \quad \forall T \in \mathcal{T}_h\}$$

$$W_h = \{v_h \in W \,:\, v_{h|T} \in P_1(T) \quad \forall T \in \mathcal{T}_h\}$$

$$\tilde{\Gamma}_h = \{\underline{s}_h \in H_0(\mathrm{rot},\Omega) \,:\, \underline{s}_{h|T} \in TR_0(T) \quad \forall T \in \mathcal{T}_h\}, \tag{109}$$

and they considered the discretized problem

find $(\underline{\tilde{\theta}}_h, \tilde{w}_h, \underline{\tilde{\gamma}}_h)$ *in* $\tilde{\Theta}_h \times W_h \times \tilde{\Gamma}_h$ *such that*

$$\begin{cases} a(\underline{\tilde{\theta}}_h, \underline{\eta}) + (\underline{\tilde{\gamma}}_h, R\underline{\eta} - \underline{\nabla}\, v) = (f, v) & \forall(\underline{\eta}, v) \in \tilde{\Theta}_h \times W_h \\ (\underline{s}, R\underline{\tilde{\theta}}_h - \underline{\nabla}\, \tilde{w}_h) - (t^2 + \delta h^2)(\underline{\tilde{\gamma}}_h, \underline{s}) = 0 & \forall \underline{s} \in \tilde{\Gamma}_h. \end{cases} \tag{110}$$

In (109) $TR_0(T)$ is the rotated Raviart-Thomas space of the lowest degree (cf. [6]), defined by

$$TR_0(T) = (P_1(T))^2 \oplus (y, -x)P_0(T). \tag{111}$$

It can be shown that $\tilde{\Gamma}_h$ consists of piecewise linear functions with a continuous and constant tangential component along each edge.

Moreover, in (110) δ is a positive constant and R is a linear operator

$$R : H_0^1(\Omega)^2 \cap H_0(\mathrm{rot},\Omega) \longrightarrow \tilde{\Theta}_h$$

defined by

$$\int_e (R\underline{\eta} - \underline{\eta}) \cdot \underline{\tau}_e = 0 \qquad \forall e \text{ edge in } \mathcal{T}_h\,. \tag{112}$$

For problem (110) they were able to prove the error bound (cf. [6])

$$||\underline{\theta} - \underline{\tilde{\theta}}_h||_1 + ||w - \tilde{w}_h||_1 + (t+h)||\underline{\gamma} - \underline{\tilde{\gamma}}_h||_0 \le Ch(||\underline{\theta}||_2 + ||w||_2 + ||\underline{\gamma}||_{H(\mathrm{div})} + t||\underline{\gamma}||_1). \tag{113}$$

Let us come back to the method under consideration in the present paper. For the sake of reader's convenience, let us recall that we are considering the discretized problem

find $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$ *in* $\Theta_h \times W_h \times \Gamma_h$ *such that*

$$\begin{cases} a(\underline{\theta}_h, \underline{\eta}) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla}\,(v + L\underline{\eta})) = (f, v + L\underline{\eta}) & \forall(\underline{\eta}, v) \in \Theta_h \times W_h \\ (\underline{s}, \underline{\theta}_h - \underline{\nabla}\,(w_h + L\underline{\theta}_h)) - t^2(\underline{\gamma}_h, \underline{s}) = 0 & \forall \underline{s} \in \Gamma_h. \end{cases} \tag{114}$$

where

$$\Theta_h = \{\underline{\eta}_h \in \Theta \, : \, \underline{\eta}_{h|T} \in (P_1(T) \oplus B_3(T))^2 \quad \forall T \in \mathcal{T}_h\}$$

$$W_h = \{v_h \in W \, : \, v_{h|T} \in P_1(T) \quad \forall T \in \mathcal{T}_h\}$$

$$\Gamma_h = \{\underline{\mathrm{s}}_h \in (L^2(\Omega))^2 \, : \, \underline{\mathrm{s}}_{h|T} \in (P_0(T))^2 \quad \forall T \in \mathcal{T}_h\}. \tag{115}$$

Because of the structure of Θ_h it is possible to eliminate by static condensation the rotation bubble degrees of freedom (cf. [1] and [12]). In this way, it could be shown that from problem (114) one gets an equivalent problem having, essentially, the following structure

find $(\underline{\theta}'_h, w'_h, \underline{\gamma}'_h)$ *in* $\tilde{\Theta}_h \times W_h \times \Gamma_h$ *such that*

$$\begin{cases} a(\underline{\theta}'_h, \underline{\eta}) + (\underline{\gamma}'_h, \underline{\eta} - \underline{\nabla}\,((v + L\underline{\eta})) = (f, v + L\underline{\eta}) & \forall (\underline{\eta}, v) \in \tilde{\Theta}_h \times W_h \\ (\underline{\mathrm{s}}, \underline{\theta}'_h - \underline{\nabla}\,(w'_h + L\underline{\theta}'_h)) - \sum_T (t^2 + c_T h_T^2)(\underline{\gamma}'_h, \underline{\mathrm{s}})_T = 0 & \forall \underline{\mathrm{s}} \in \Gamma_h, \end{cases} \tag{116}$$

where c_T is a positive constant independent of h.

Let us now establish the following

Lemma 4.1 *For each* $\underline{\eta} \in \tilde{\Theta}_h$ *it holds*

$$\underline{\eta} - \underline{\nabla}\, L\underline{\eta} = R\underline{\eta}. \tag{117}$$

Proof: Since $L\underline{\eta} \in H_0^1(\Omega)$ it follows that

$$\underline{\eta} - \underline{\nabla}\, L\underline{\eta} \in H_0(\mathrm{rot}, \Omega).$$

Now notice that $\underline{\eta} - \underline{\nabla}\, L\underline{\eta}$ is a piecewise linear function for which it holds (cf. (23))

$$(\underline{\eta} - \underline{\nabla}\, L\underline{\eta}) \cdot \underline{\tau}_e = \text{constant} \qquad \forall e \text{ edge in } \mathcal{T}_h. \tag{118}$$

Hence $(\underline{\eta} - \underline{\nabla}\, L\underline{\eta}) \in \tilde{\Gamma}_h$. Moreover, integrating (118) along the edge e and recalling (25) and (112), one obtains

$$\int_e (\underline{\eta} - \underline{\nabla}\, L\underline{\eta}) \cdot \underline{\tau}_e = \int_e \underline{\eta} \cdot \underline{\tau}_e = \int_e R\underline{\eta} \cdot \underline{\tau}_e \qquad \forall e \text{ edge in } \mathcal{T}_h, \tag{119}$$

from which (117) follows.

□

By Lemma 4.1 one easily realizes that problem (116) can be written as *find* $(\underline{\theta}'_h, w'_h, \underline{\gamma}'_h)$ *in* $\tilde{\Theta}_h \times W_h \times \Gamma_h$ *such that*

$$\begin{cases} a(\underline{\theta}'_h, \underline{\eta}) + (\underline{\gamma}'_h, R\underline{\eta} - \underline{\nabla}\, v) = (f, v + L\underline{\eta}) & \forall(\underline{\eta}, v) \in \tilde{\Theta}_h \times W_h \\ (\underline{s}, R\underline{\theta}'_h - \underline{\nabla}\, w'_h) - \sum_T (t^2 + c_T h_T^2)(\underline{\gamma}'_h, \underline{s})_T = 0 & \forall \underline{s} \in \Gamma_h, \end{cases} \tag{120}$$

A comparison between formulation (110) and formulation (120) shows that the two methods are indeed very similar. The main difference stands in that the Brezzi-Fortin-Stenberg method uses a rotated Raviart-Thomas approximation for the shear stress field, while in Taylor-Auricchio method a further projection over the piecewise constant functions is employed. Consequentely, even though the order of convergence is the same for both methods (cf. (66) and (113)), one expects a better performance of the scheme presented in [6] than the one developed in [13]. Moreover, the Taylor-Auricchio method does not seem to be cheaper in the implementation procedure.

Note added in proof. A few months after sending the paper for publication we learnt that some results of ours have been proved, using different techniques, by Lyly in the paper:

M. Lyly. On the connection between some linear triangular Reissner-Mindlin plate bending elements, to appear in *Numer. Math.*

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