A POSTERIORI ERROR ANALYSIS OF THE LINKED INTERPOLATION TECHNIQUE FOR PLATE BENDING PROBLEMS

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Abstract. We develop a posteriori error estimates for the so-called 'Linked Interpolation Technique' to approximate the solution of plate bending problems. We show that the proposed (residual-based) estimator is both reliable and efficient.

Key words. Reissner-Mindlin plates, finite element methods, a posteriori error analysis.

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1. Introduction. In this paper we present an *a posteriori* error analysis for the so-called 'Linked Interpolation Technique' (cf. [2], [3] and [22], for instance) to approximate the solution of the Reissner-Mindlin plate problem.

It is worth noticing that the main effort concerning the finite element discretization of the plate bending problems has been focused on proposing and analyzing *locking-free* schemes. As a consequence, most of the mathematical literature on the subject is addressed to establish *a priori* error estimates. We mention here, in a totally non-exhaustive way, the works [1], [5], [7], [13], [16], [19], [21], and the references therein. On the contrary, when considering the *a posteriori* error analysis for plates, only very few results are available (see [8], [9] and [15]).

In this work we consider the so-called 'Linked Interpolation Technique', focusing on two triangular elements: the first one is the low-order element proposed in [22] (see also [23]), while the second one is the quadratic scheme proposed in [3]. An *a priori* error analysis has been developed for both the methods in [17, 18] and [3], respectively. We also remark that our *a posteriori* error analysis may be straightforwardly extended to other schemes taking advantage of the 'Linked Interpolation Technique', such as the quadrilateral elements considered in [2] and [3], for example.

An outline of the paper is as follows. In Section 2 we briefly recall the Reissner-Mindlin problem, together with a mixed variational formulation and some useful regularity results. The 'Linked Interpolation Technique' is described in Section 3, where we develop an *a priori* analysis, for the sake of completeness (see also [17] or [18]). Section 4 is devoted to the *a posteriori* error estimates. In particular we introduce our estimator, and we prove its *reliability* (Section 4.1) and *efficiency* (Section 4.2). We point out that in the paper we consider the case of a clamped plate *only for simplicity.* Indeed, both the *a priori* and the *a posteriori* error analysis can be easily adapted to cover other relevant boundary conditions.

Throughout the paper we will use standard notations for Sobolev norms and seminorms. Moreover, we will denote with C a generic constant *independent* of the mesh parameter h and the plate thickness t, which may take different values in different occurrences.

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2. The Reissner-Mindlin problem. The Reissner-Mindlin equations for a clamped plate with polygonal mid-plane Ω require to find $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ such that

$$\begin{cases} -\operatorname{div} \mathbf{C}\,\varepsilon(\boldsymbol{\theta}) - \boldsymbol{\gamma} = 0 & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{\gamma} = g & \text{in } \Omega, \\ \boldsymbol{\gamma} = \mu t^{-2}(\boldsymbol{\nabla} w - \boldsymbol{\theta}) & \text{in } \Omega, \\ \boldsymbol{\theta} = 0, \ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here, $\mathbf{C}$ is the tensor of bending moduli, $\boldsymbol{\theta}$ represents the rotations, w the transversal displacement, $\boldsymbol{\gamma}$ the scaled shear stresses and g a given transversal load. Moreover, ε is the usual symmetric gradient operator, μ is the shear modulus, and t is the thickness. The classical variational formulation of problem (2.1) is

$$\begin{cases} \text{Find } (\boldsymbol{\theta}, w, \boldsymbol{\gamma}) \in \boldsymbol{\Theta} \times W \times (L^2(\Omega))^2 : \\ a(\boldsymbol{\theta}, \boldsymbol{\eta}) + (\boldsymbol{\nabla} v - \boldsymbol{\eta}, \boldsymbol{\gamma}) = (g, v) & (\boldsymbol{\eta}, v) \in \boldsymbol{\Theta} \times W, \\ (\boldsymbol{\nabla} w - \boldsymbol{\theta}, \boldsymbol{\tau}) - \mu^{-1} t^2 (\boldsymbol{\gamma}, \boldsymbol{\tau}) = 0 & \boldsymbol{\tau} \in (L^2(\Omega))^2, \end{cases} \tag{2.2}$$

where $\boldsymbol{\Theta} = (H_0^1(\Omega))^2$, $W = H_0^1(\Omega)$, $(\cdot,\cdot)$ is the inner-product in $L^2(\Omega)$ and

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta}) \ .$$

Following [10], we write the pair $(\boldsymbol{\theta}, w)$ as

$$(\boldsymbol{\theta}, w) = (\boldsymbol{\theta}_0 + \boldsymbol{\theta}_r, w_0 + w_r), \tag{2.3}$$

where the pair $(\boldsymbol{\theta}_0, , w_0)$ is the solution of the *limit problem*:

$$\begin{cases} \text{Find } (\boldsymbol{\theta}_0, w_0, \boldsymbol{\gamma}_0) \in \boldsymbol{\Theta} \times W \times \boldsymbol{\Gamma} : \\ a(\boldsymbol{\theta}_0, \boldsymbol{\eta}) + < \boldsymbol{\nabla} v - \boldsymbol{\eta}, \boldsymbol{\gamma}_0 >= (g, v) & (\boldsymbol{\eta}, v) \in \boldsymbol{\Theta} \times W, \\ < \boldsymbol{\nabla} w_0 - \boldsymbol{\theta}_0, \boldsymbol{\tau} >= 0 & \boldsymbol{\tau} \in \boldsymbol{\Gamma}, \end{cases} \tag{2.4}$$

and $(\boldsymbol{\theta}_r, w_r)$ can be thought as a remainder. Furthermore, $\boldsymbol{\Gamma} = H^{-1}(\operatorname{div}, \Omega)$ and $< \cdot, \cdot >$ is the duality pairing between $H_0(\operatorname{rot}, \Omega)$ and $H^{-1}(\operatorname{div}, \Omega)$. One has (cf. [10])

PROPOSITION 2.1. *Suppose that Ω is convex and $g \in L^2(\Omega)$. Then it holds*

$$||w_0||_3 + ||\boldsymbol{\theta}||_2 + ||\boldsymbol{\gamma}||_0 + t\,||\boldsymbol{\gamma}||_1 \le C(||g||_{-1} + t\,||g||_0), \tag{2.5}$$

$$||\boldsymbol{\theta}_r||_1 \le Ct\,||g||_{-1}, \tag{2.6}$$

$$||w_r||_2 \le Ct(||g||_{-1} + t\,||g||_0). \tag{2.7}$$

□

3. The Linked Interpolation Scheme and an a priori analysis. In this Section we present the general idea of the Linked Interpolation Technique (see [3] and [22], for instance), together with two examples of triangular elements. Furthermore, for the sake of completeness, we develop an a priori error analysis, focusing on the lowest-order element (see [17] and [18]).

3.1. The Linked Interpolation Scheme. Let $\{\mathcal{T}_h\}_{h>0}$ be a sequence of decompositions of Ω into triangular elements T, satisfying the usual compatibility conditions (see [12]). We also assume that the family $\{\mathcal{T}_h\}_{h>0}$ is *regular*, i.e. there exists a constant $\sigma > 0$ such that

$$h_T \le \sigma\rho_T \qquad \forall T \in \mathcal{T}_h, \tag{3.1}$$

where h_T is the diameter of the element T and ρ_T is the maximum diameter of the circles contained in T. We recall (see [12], for instance) that regularity implies the *minimum angle* condition: there exists a constant $\alpha > 0$ such that

$$\alpha_T \ge \alpha \qquad \forall T \in \mathcal{T}_h, \tag{3.2}$$

where α_T denotes the smallest inner angle of T. Moreover, given the decomposition $\mathcal{T}_h$ we will denote with $\mathcal{E}_h$ the set of the edges e of the triangles $T \in \mathcal{T}_h$. We now select the finite element spaces $\boldsymbol{\Theta}_h \subset \boldsymbol{\Theta}$, $W_h \subset W$, $\boldsymbol{\Gamma}_h \subset L^2(\Omega)^2$, together with a *suitable* linear operator (the so-called *linking operator*)

$$L : \boldsymbol{\Theta}_h \longrightarrow H^1_0(\Omega). \tag{3.3}$$

We then form the following finite dimensional subspace of $\boldsymbol{X} := \boldsymbol{\Theta} \times W$:

$$\boldsymbol{X}_h = \big\{(\boldsymbol{\eta}_h, v_h^*) = (\boldsymbol{\eta}_h, v_h + L\boldsymbol{\eta}_h) : \boldsymbol{\eta}_h \in \boldsymbol{\Theta}_h\,,\ v_h \in W_h\big\}, \tag{3.4}$$

and we finally consider the discrete problem

$$\begin{cases} \text{Find } (\boldsymbol{\theta}_h, w_h^*; \boldsymbol{\gamma}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h : \\ a(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + (\boldsymbol{\gamma}_h, \boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h) = (g, v_h^*) \qquad (\boldsymbol{\eta}_h, v_h^*) \in \boldsymbol{X}_h, \\ (\boldsymbol{\nabla} w_h^* - \boldsymbol{\theta}_h, \boldsymbol{\tau}_h) - \mu^{-1}t^2(\boldsymbol{\gamma}_h, \boldsymbol{\tau}_h) = 0 \qquad \boldsymbol{\tau}_h \in \boldsymbol{\Gamma}_h. \end{cases} \tag{3.5}$$

REMARK 3.1. *We point out that eliminating $\boldsymbol{\gamma}_h$ from system* (3.5)*, our scheme is equivalent to the following problem involving only the rotations and the vertical displacements:*

$$\begin{cases} \textit{Find } (\boldsymbol{\theta}_h, w_h^*) \in \boldsymbol{X}_h : \\ a(\boldsymbol{\theta}_h, \boldsymbol{\eta}_h) + \mu t^{-2}\big(P_h(\boldsymbol{\nabla} w_h^* - \boldsymbol{\theta}_h), P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h)\big) = (g, v_h) \quad \forall (\boldsymbol{\eta}_h, v_h^*) \in \boldsymbol{X}_h, \end{cases} \tag{3.6}$$

where P_h denotes the L^2-projection operator onto $\boldsymbol{\Gamma}_h$.

We are now ready to present the following two elements (for other methods based on the same strategy, see e.g. [2, 3]).

3.1.1. The linear element. This element (see [22]) is described by the finite element spaces

$$\boldsymbol{\Theta}_h = \big\{\boldsymbol{\eta} \in \boldsymbol{\Theta} : \boldsymbol{\eta}_{|T} \in (P_1(T) \oplus B_3(T))^2\big\}, \tag{3.7}$$

$$W_h = \big\{v \in W : v_{|T} \in P_1(T)\big\}, \tag{3.8}$$

$$\boldsymbol{\Gamma}_h = \big\{\boldsymbol{\tau} \in L^2(\Omega)^2 : \boldsymbol{\tau}_{|T} \in P_0(T)^2\big\}, \tag{3.9}$$

where $P_k(T)$ is the space of polynomials of degree at most k defined on T and $B_3(T) = P_3(T) \cap H^1_0(T)$ is the space of cubic bubbles on T. The *linking operator* $L : \mathbf{\Theta}_h \longrightarrow H^1_0(\Omega)$ is defined as follows. For each $T \in \mathcal{T}_h$, we set

$$\varphi_i = \lambda_j \lambda_k \qquad \text{and} \qquad EB_2(T) = \text{Span}\,\{\varphi_i\}_{1\le i\le 3}\ , \tag{3.10}$$

where $\{\lambda_i\}_{1\le i\le 3}$ are the barycentric coordinates of the triangle T and the indices (i,j,k) form a permutation of the set $(1,2,3)$. Then, the operator L is locally defined as

$$L\boldsymbol{\eta}_{h|T} = \sum_{i=1}^{3} \alpha_i \varphi_i \in EB_2(T), \tag{3.11}$$

where the coefficients α_i are determined by requiring that

$$(\nabla L\boldsymbol{\eta}_h - \boldsymbol{\eta}_h)\cdot \mathbf{t} \quad \text{is constant on each } e\ . \tag{3.12}$$

Above, $\mathbf{t}$ denotes the tangential vector to the edge e. We recall that for the linking operator it holds (see [17] and [18])

$$||L\boldsymbol{\eta}_h||_{0,T} \le Ch_T ||\nabla L\boldsymbol{\eta}_h||_{0,T} \quad , \quad ||\nabla L\boldsymbol{\eta}_h||_{0,T} \le Ch_T |\boldsymbol{\eta}_h|_{1,T} \tag{3.13}$$

3.1.2. The quadratic element. This element (see [3]) is described by the finite element spaces

$$\mathbf{\Theta}_h = \big\{\boldsymbol{\eta} \in \mathbf{\Theta} : \boldsymbol{\eta}_{|T} \in P_2(T)^2 \oplus (P_1(T)^2 \oplus \nabla B_3(T))b_T\big\}, \tag{3.14}$$

$$W_h = \big\{v \in W : v_{|T} \in P_2(T) \oplus B_3(T)\big\}, \tag{3.15}$$

$$\mathbf{\Gamma}_h = \big\{\boldsymbol{\tau} \in L^2(\Omega)^2 : \boldsymbol{\tau}_{|T} \in P_1(T)^2 \oplus \nabla B_3(T)\big\}, \tag{3.16}$$

where $b_T = 27\lambda_1\lambda_2\lambda_3$. The *linking operator* $L : \mathbf{\Theta}_h \longrightarrow H^1_0(\Omega)$ is defined as follows. For each $T \in \mathcal{T}_h$, we set

$$\varphi_i = \lambda_j \lambda_k (\lambda_k - \lambda_j) \qquad \text{and} \qquad EB_3(T) = \text{Span}\,\{\varphi_i\}_{1\le i\le 3}\ , \tag{3.17}$$

where the indices (i,j,k) form a permutation of the set $(1,2,3)$. Then, the operator L is locally defined as

$$L\boldsymbol{\eta}_{h|T} = \sum_{i=1}^{3} \alpha_i \varphi_i \in EB_3(T), \tag{3.18}$$

where the coefficients α_i's are determined by requiring that

$$(\nabla L\boldsymbol{\eta}_h - \boldsymbol{\eta}_h)\cdot \mathbf{t} \quad \text{is linear on each } e\ . \tag{3.19}$$

For this linking operator it holds (see [3])

$$||L\boldsymbol{\eta}_h||_{0,T} \le Ch_T ||\nabla L\boldsymbol{\eta}_h||_{0,T} \quad , \quad ||\nabla L\boldsymbol{\eta}_h||_{0,T} \le Ch_T^2 |\boldsymbol{\eta}_h|_{2,T} \le Ch_T |\boldsymbol{\eta}_h|_{1,T} \tag{3.20}$$

3.2. A priori error estimates. Following the lines of [10, 17, 19, 21], we prove *a priori* error estimates with respect to the norms

$$|||(\boldsymbol{\eta}, v)|||_h^2 := ||\boldsymbol{\eta}||_1^2 + ||v||_1^2 + \sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||\boldsymbol{\nabla} v - \boldsymbol{\eta}||_{0,T}^2 \qquad \forall(\boldsymbol{\eta}, v) \in \boldsymbol{\Theta} \times W \quad (3.21)$$

and

$$||\boldsymbol{\tau}||_{-1} + t\,||\boldsymbol{\tau}||_0 \qquad \forall \boldsymbol{\tau} \in L^2(\Omega)^2 \ . \quad (3.22)$$

We will also use the following discrete norm

$$||\boldsymbol{\tau}||_h^2 := \sum_{T\in\mathcal{T}_h} h_T^2 ||\boldsymbol{\tau}||_{0,T}^2 + t^2 ||\boldsymbol{\tau}||_0^2 \qquad \forall \boldsymbol{\tau} \in L^2(\Omega)^2 \ . \quad (3.23)$$

Before proceeding, we need the following lemma, which establishes a suitable norm equivalence in the used finite element spaces.

LEMMA 3.1. *Consider the finite element spaces and the linking operator detailed in Section 3.1.1 (or in Section 3.1.2), and let* P_h *denote the* L^2*-projection operator on* $\boldsymbol{\Gamma}_h$*. Then for each* $(\boldsymbol{\eta}_h, v_h^*) \in \boldsymbol{X}_h$ *it holds*

$$\left(||\boldsymbol{\eta}_h||_1^2 + \sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h)||_{0,T}^2 \right)^{1/2} \le |||(\boldsymbol{\eta}_h, v_h^*)|||_h \quad (3.24)$$

and

$$|||(\boldsymbol{\eta}_h, v_h^*)|||_h \le C \left(||\boldsymbol{\eta}_h||_1^2 + \sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h)||_{0,T}^2 \right)^{1/2} . \quad (3.25)$$

Proof. Since (3.24) is trivial, we only consider (3.25). Therefore, take $\boldsymbol{\eta}_h \in \boldsymbol{\Theta}_h$, $v_h \in W_h$ and form $(\boldsymbol{\eta}_h, v_h^*) = (\boldsymbol{\eta}_h, v_h + L\boldsymbol{\eta}_h) \in \boldsymbol{X}_h$. We first notice that

$$\begin{aligned} ||\boldsymbol{\nabla} v_h^*||_0^2 &\le 2\left(||\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h||_0^2 + ||\boldsymbol{\eta}_h||_0^2\right) \\ &\le C \left(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h||_{0,T}^2 + ||\boldsymbol{\eta}_h||_1^2 \right) , \end{aligned} \quad (3.26)$$

so that, by Poincarè's inequality, we have

$$||v_h^*||_1^2 \le C \left(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h||_{0,T}^2 + ||\boldsymbol{\eta}_h||_1^2 \right) , \quad (3.27)$$

Next, we write $\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h$ as

$$\begin{aligned} \boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h &= \boldsymbol{\nabla} v_h + \boldsymbol{\nabla} L\boldsymbol{\eta}_h - \boldsymbol{\eta}_h = P_h \boldsymbol{\nabla} v_h + \boldsymbol{\nabla} L\boldsymbol{\eta}_h - \boldsymbol{\eta}_h \\ &= P_h \boldsymbol{\nabla} v_h^* - (P_h \boldsymbol{\nabla} L\boldsymbol{\eta}_h - \boldsymbol{\nabla} L\boldsymbol{\eta}_h) - \boldsymbol{\eta}_h \\ &= P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h) - (P_h \boldsymbol{\nabla} L\boldsymbol{\eta}_h - \boldsymbol{\nabla} L\boldsymbol{\eta}_h) + (P_h\boldsymbol{\eta}_h - \boldsymbol{\eta}_h) \ . \end{aligned} \quad (3.28)$$

Therefore, we have

$$\begin{aligned} ||\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h||_{0,T} &\le ||P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h)||_{0,T} \\ &\quad + ||P_h \boldsymbol{\nabla} L\boldsymbol{\eta}_h - \boldsymbol{\nabla} L\boldsymbol{\eta}_h||_{0,T} + ||P_h\boldsymbol{\eta}_h - \boldsymbol{\eta}_h||_{0,T} \ . \end{aligned} \tag{3.29}$$

Since (see also (3.13) and (3.20))

$$||P_h \boldsymbol{\nabla} L\boldsymbol{\eta}_h - \boldsymbol{\nabla} L\boldsymbol{\eta}_h||_{0,T} \le 2||\boldsymbol{\nabla} L\boldsymbol{\eta}_h||_{0,T} \le Ch_T|\boldsymbol{\eta}_h|_{1,T} \tag{3.30}$$

and

$$||P_h\boldsymbol{\eta}_h - \boldsymbol{\eta}_h||_{0,T} \le Ch_T|\boldsymbol{\eta}_h|_{1,T} \ , \tag{3.31}$$

from (3.29) we obtain

$$\begin{aligned} \frac{1}{h_T^2+t^2}||\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h||_{0,T}^2 &\le C\left(\frac{1}{h_T^2+t^2}||P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h)||_{0,T}^2 + \frac{h_T^2}{h_T^2+t^2}|\boldsymbol{\eta}_h|_{1,T}^2\right) \\ &\le C\left(\frac{1}{h_T^2+t^2}||P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h)||_{0,T}^2 + |\boldsymbol{\eta}_h|_{1,T}^2\right) \ . \end{aligned} \tag{3.32}$$

Therefore, we get

$$\sum_{T\in\mathcal{T}_h}\frac{1}{h_T^2+t^2}||\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h||_{0,T}^2 \le C\left(\sum_{T\in\mathcal{T}_h}\frac{1}{h_T^2+t^2}||P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h)||_{0,T}^2 + ||\boldsymbol{\eta}_h||_1^2\right) \ . \tag{3.33}$$

Using (3.27) and (3.31) we deduce estimate (3.25). ☐

It is now useful to set

$$\begin{aligned} \mathcal{A}(\boldsymbol{\theta}, w, \boldsymbol{\gamma}; \boldsymbol{\eta}, v, \boldsymbol{\tau}) :=& a(\boldsymbol{\theta}, \boldsymbol{\eta}) + (\boldsymbol{\nabla} v - \boldsymbol{\eta}, \boldsymbol{\gamma}) \\ &- (\boldsymbol{\nabla} w - \boldsymbol{\theta}, \boldsymbol{\tau}) + \mu^{-1}t^2(\boldsymbol{\gamma}, \boldsymbol{\tau}). \end{aligned} \tag{3.34}$$

Therefore, the continuous problem (2.2) reads

$$\begin{cases} \text{Find } (\boldsymbol{\theta}, w; \boldsymbol{\gamma}) \in \boldsymbol{X} \times L^2(\Omega)^2 \text{ s.t.} \\ \mathcal{A}(\boldsymbol{\theta}, w, \boldsymbol{\gamma}; \boldsymbol{\eta}, v, \boldsymbol{\tau}) = (g, v) \qquad \forall(\boldsymbol{\eta}, v; \boldsymbol{\tau}) \in \boldsymbol{X} \times L^2(\Omega)^2 \ , \end{cases} \tag{3.35}$$

while the discrete problem (3.5) is

$$\begin{cases} \text{Find } (\boldsymbol{\theta}_h, w_h^*; \boldsymbol{\gamma}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h \text{ s.t.} \\ \mathcal{A}(\boldsymbol{\theta}_h, w_h^*, \boldsymbol{\gamma}_h; \boldsymbol{\eta}_h, v_h^*, \boldsymbol{\tau}_h) = (g, v_h^*) \qquad \forall(\boldsymbol{\eta}_h, v_h^*; \boldsymbol{\tau}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h. \end{cases} \tag{3.36}$$

We have the following stability result, for which we only sketch the proof, since it takes advantage of the same techniques detailed in [10] and [17].

Proposition 3.2. *Given* $(\boldsymbol{\beta}_h, z_h^*; \boldsymbol{\rho}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$ *there exists* $(\boldsymbol{\eta}_h, v_h^*; \boldsymbol{\tau}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$ *such that*

$$\mathcal{A}(\boldsymbol{\beta}_h, z_h^*, \boldsymbol{\rho}_h; \boldsymbol{\eta}_h, v_h^*, \boldsymbol{\tau}_h) \ge C\left(|||(\boldsymbol{\beta}_h, z_h^*)|||_h^2 + ||\boldsymbol{\rho}_h||_h^2\right) \tag{3.37}$$

$$|||(\boldsymbol{\eta}_h, v_h^*)|||_h + ||\boldsymbol{\tau}_h||_h \le C\,(|||(\boldsymbol{\beta}_h, z_h^*)|||_h + ||\boldsymbol{\rho}_h||_h) \tag{3.38}$$

Proof. Let us $(\boldsymbol{\beta}_h, z_h^*; \boldsymbol{\rho}_h)$ be given in $\boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$. Using exactly the same arguments of [10] and [17] we get that there exists $(\boldsymbol{\eta}_h, v_h^*; \boldsymbol{\tau}_h)$ in $\boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$ such that

$$\mathcal{A}(\boldsymbol{\beta}_h, z_h^*, \boldsymbol{\rho}_h; \boldsymbol{\eta}_h, v_h^*, \boldsymbol{\tau}_h) \ge C \left(||\boldsymbol{\beta}_h||_1^2 + \sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||P_h(\boldsymbol{\nabla} z_h^* - \boldsymbol{\beta}_h)||_{0,T}^2 + ||\boldsymbol{\rho}_h||_h^2 \right) \tag{3.39}$$

and

$$\begin{aligned} ||\boldsymbol{\eta}_h||_1 + \Big(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||P_h(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h)||_{0,T}^2 \Big)^{1/2} + ||\boldsymbol{\tau}_h||_h \\ \le C \left(||\boldsymbol{\beta}_h||_1 + \Big(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||P_h(\boldsymbol{\nabla} z_h^* - \boldsymbol{\beta}_h)||_{0,T}^2 \Big)^{1/2} + ||\boldsymbol{\rho}_h||_h \right) \,. \end{aligned} \tag{3.40}$$

We now use Lemma 3.1 to infer that given $(\boldsymbol{\beta}_h, z_h^*; \boldsymbol{\rho}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$, there exists $(\boldsymbol{\eta}_h, v_h^*; \boldsymbol{\tau}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$ such that

$$\mathcal{A}(\boldsymbol{\beta}_h, z_h^*, \boldsymbol{\rho}_h; \boldsymbol{\eta}_h, v_h^*, \boldsymbol{\tau}_h) \ge C \left(|||(\boldsymbol{\beta}_h, z_h^*)|||_h^2 + ||\boldsymbol{\rho}_h||_h^2 \right) \tag{3.41}$$

and

$$|||(\boldsymbol{\eta}_h, v_h^*)|||_h + ||\boldsymbol{\tau}_h||_h \le C\,(|||(\boldsymbol{\beta}_h, z_h^*)|||_h + ||\boldsymbol{\rho}_h||_h) \,. \tag{3.42}$$

□

We are now ready to prove our error estimate (see also [18] and [17]). We focus on the lowest-order element detailed in Section 3.1.1, but a similar technique (together with the ideas developed in [19]) may be applied to appropriately treat the higher-order case of Section 3.1.2.

Proposition 3.3. *Suppose that Ω is a convex polygon and $g \in L^2(\Omega)$ and consider the element detailed in Section 3.1.1. Let $(\boldsymbol{\theta}, w; \boldsymbol{\gamma}) \in \boldsymbol{X} \times L^2(\Omega)^2$ and $(\boldsymbol{\theta}_h, w_h^*; \boldsymbol{\gamma}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$ be the solutions of problem* (3.35) *and* (3.36), *respectively. Then the following a priori estimates holds*

$$|||(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h^*)|||_h + ||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_{-1} + ||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_h \le C\,h(||g||_{-1} + t\,||g||_0) \,. \tag{3.43}$$

Proof. Since our method is consistent (cf. (3.35) and (3.36)) and stable (see Proposition 3.2), error estimates with respect to the norms in question can be established in the standard way. Hence, let

$$(\boldsymbol{\theta}_I, w_I^*; \boldsymbol{\gamma}_I) = (\boldsymbol{\theta}_I, w_I + L\boldsymbol{\theta}_I; \boldsymbol{\gamma}_I) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h \tag{3.44}$$

be a suitable interpolant (to be specified later) of the continuous solution $(\boldsymbol{\theta}, w^*; \boldsymbol{\gamma})$. Corresponding to $(\boldsymbol{\theta}_h - \boldsymbol{\theta}_I, w_h^* - w_I^*; \boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$ there exists (see Proposition 3.2) $(\boldsymbol{\eta}_h, v_h^*; \boldsymbol{\tau}_h) \in \boldsymbol{X}_h \times \boldsymbol{\Gamma}_h$ such that

$$\begin{aligned} \mathcal{A}(\boldsymbol{\theta}_h - \boldsymbol{\theta}_I, w_h^* - w_I^*, \boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I; \boldsymbol{\eta}_h, v_h^*, \boldsymbol{\tau}_h) \ge C\big(|||(\boldsymbol{\theta}_h - \boldsymbol{\theta}_I, w_h^* - w_I^*)|||_h^2 \\ + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I||_h^2\big) \,, \end{aligned} \tag{3.45}$$

and

$$|||(\boldsymbol{\eta}_h, v_h^*)|||_h + ||\boldsymbol{\tau}_h||_h \leq C\,(|||(\boldsymbol{\theta}_h - \boldsymbol{\theta}_I, w_h^* - w_I^*)|||_h + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I||_h)\ . \tag{3.46}$$

By consistency it holds

$$\begin{aligned}\mathcal{A}(\boldsymbol{\theta}_h - \boldsymbol{\theta}_I, w_h^* - w_I^*, \boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I; \boldsymbol{\eta}_h, v_h^*, \boldsymbol{\tau}_h) &= \mathcal{A}(\boldsymbol{\theta} - \boldsymbol{\theta}_I, w - w_I^*, \boldsymbol{\gamma} - \boldsymbol{\gamma}_I; \boldsymbol{\eta}_h, v_h^*, \boldsymbol{\tau}_h)\\ &= a(\boldsymbol{\theta} - \boldsymbol{\theta}_I, \boldsymbol{\eta}_h) + (\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_I)\\ &- \big(\boldsymbol{\nabla}(w - w_I^*) - (\boldsymbol{\theta} - \boldsymbol{\theta}_I), \boldsymbol{\tau}_h\big) + \mu^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_I, \boldsymbol{\tau}_h)\\ &= (I) + (II) + (III) + (IV)\ .\end{aligned} \tag{3.47}$$

To bound the four terms above, we first choose the interpolants $\boldsymbol{\theta}_I$, w_I^* and $\boldsymbol{\gamma}_I$ as follows. According to the splitting (2.3), $\boldsymbol{\theta}_I$ is given by

$$\boldsymbol{\theta}_I := \mathcal{I}\boldsymbol{\theta} = \mathcal{I}\boldsymbol{\theta}_0 + \mathcal{I}\boldsymbol{\theta}_r\ , \tag{3.48}$$

where $\mathcal{I}$ is the Lagrange interpolating operator. To define w_I^*, we need to specify w_I (cf. (3.44)). Again, the splitting (2.3) suggests to set

$$w_I := \mathcal{I}w = \mathcal{I}w_0 + \mathcal{I}w_r\ . \tag{3.49}$$

Therefore, w_I^* turns out to be $w_I^* = w_I + L\boldsymbol{\theta}_I = \mathcal{I}w + L(\mathcal{I}\boldsymbol{\theta})$. Finally, $\boldsymbol{\gamma}_I$ is simply the L^2-projection of $\boldsymbol{\gamma}$ onto $\boldsymbol{\Gamma}_h$.

Estimate for (I). Using the H^1-continuity of the bilinear form $a(\cdot,\cdot)$, standard approximation results and estimate (2.5) we have

$$(I) = a(\boldsymbol{\theta} - \boldsymbol{\theta}_I, \boldsymbol{\eta}_h) \leq Ch||\boldsymbol{\theta}||_2||\boldsymbol{\eta}_h||_1 \leq Ch(||g||_{-1} + t\,||g||_0)||\boldsymbol{\eta}_h||_1\ . \tag{3.50}$$

Estimate for (II). We notice that

$$\begin{aligned}(II) =&(\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_I)\\ &\leq \left(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2}||\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h||_{0,T}^2\right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} (h_T^2 + t^2)||\boldsymbol{\gamma} - \boldsymbol{\gamma}_I||_{0,T}^2\right)^{1/2},\end{aligned} \tag{3.51}$$

by which, using again (2.5) and standard approximation estimates, we get

$$(II) \leq Ch(||g||_{-1} + t\,||g||_0)\left(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2}||\boldsymbol{\nabla} v_h^* - \boldsymbol{\eta}_h||_{0,T}^2\right)^{1/2}. \tag{3.52}$$

Estimate for (III).

$$\begin{aligned}(III) &= -\big(\boldsymbol{\nabla}(w - w_I^*) - (\boldsymbol{\theta} - \boldsymbol{\theta}_I), \boldsymbol{\tau}_h\big)\\ &\leq \left(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2}||\boldsymbol{\nabla}(w - w_I^*) - (\boldsymbol{\theta} - \boldsymbol{\theta}_I)||_{0,T}^2\right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} (h_T^2 + t^2)||\boldsymbol{\tau}_h||_{0,T}^2\right)^{1/2}.\end{aligned} \tag{3.53}$$

We now notice that we have (see (2.3), (3.44) and (3.48)–(3.49))

$$\begin{aligned}\boldsymbol{\nabla}(w - w_I^*) - (\boldsymbol{\theta} - \boldsymbol{\theta}_I) = &\Big\{\boldsymbol{\nabla}\big(w_0 - \mathcal{I}w_0 - L(\mathcal{I}\boldsymbol{\theta}_0)\big) - (\boldsymbol{\theta}_0 - \mathcal{I}\boldsymbol{\theta}_0)\Big\} \\ &+ \Big\{\boldsymbol{\nabla}\big(w_r - \mathcal{I}w_r - L(\mathcal{I}\boldsymbol{\theta}_r)\big) - (\boldsymbol{\theta}_r - \mathcal{I}\boldsymbol{\theta}_r)\Big\} \ .\end{aligned} \tag{3.54}$$

In [17] it has been proved that

$$\big|\boldsymbol{\nabla}\big(w_0 - \mathcal{I}w_0 - L(\mathcal{I}\boldsymbol{\theta}_0)\big)\big|_{0,T} \le Ch_T^2 |w_0|_{3,T} \ , \tag{3.55}$$

while standard approximation results give

$$|\boldsymbol{\theta}_0 - \mathcal{I}\boldsymbol{\theta}_0|_{0,T} \le Ch_T^2 |\boldsymbol{\theta}_0|_{2,T} \tag{3.56}$$

$$|\boldsymbol{\theta}_r - \mathcal{I}\boldsymbol{\theta}_r|_{0,T} \le Ch_T^2 |\boldsymbol{\theta}_r|_{2,T} \ . \tag{3.57}$$

Furthermore, using also (3.13) it holds

$$\begin{aligned}\big|\boldsymbol{\nabla}\big(w_r - \mathcal{I}w_r - L(\mathcal{I}\boldsymbol{\theta}_r)\big)\big|_{0,T} &\le |\boldsymbol{\nabla}(w_r - \mathcal{I}w_r)|_{0,T} + |\boldsymbol{\nabla}L(\mathcal{I}\boldsymbol{\theta}_r)|_{0,T} \\ &\le |\boldsymbol{\nabla}(w_r - \mathcal{I}w_r)|_{0,T} + |\boldsymbol{\nabla}L(\mathcal{I}\boldsymbol{\theta}_r - \boldsymbol{\theta}_r)|_{0,T} + |\boldsymbol{\nabla}L(\boldsymbol{\theta}_r)|_{0,T} \\ &\le C\big(h_T|w_r|_{2,T} + h_T|\mathcal{I}\boldsymbol{\theta}_r - \boldsymbol{\theta}_r|_{1,T} + h_T|\boldsymbol{\theta}_r|_{1,T}\big) \\ &\le C\big(h_T|w_r|_{2,T} + h_T^2|\boldsymbol{\theta}_r|_{2,T} + h_T|\boldsymbol{\theta}_r|_{1,T}\big)\end{aligned} \tag{3.58}$$

From (3.54)–(3.58) we obtain

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||\boldsymbol{\nabla}(w - w_I^*) - (\boldsymbol{\theta} - \boldsymbol{\theta}_I)||_{0,T}^2 \\ &\qquad \le C \sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} \big(h_T^4|w_0|_{3,T}^2 + h_T^4|\boldsymbol{\theta}|_{2,T}^2 + h_T^2|w_r|_{2,T}^2 + h_T^2|\boldsymbol{\theta}_r|_{1,T}^2\big) \\ &\qquad \le Ch^2\big(|w_0|_3^2 + |\boldsymbol{\theta}|_2^2\big) + \sum_{T\in\mathcal{T}_h} \frac{h_T^2}{h_T^2 + t^2}\big(|w_r|_{2,T}^2 + |\boldsymbol{\theta}_r|_{1,T}^2\big) \\ &\qquad \le Ch^2\big(|w_0|_3^2 + |\boldsymbol{\theta}|_2^2\big) + \sum_{T\in\mathcal{T}_h} h_T^2 \left(\frac{|w_r|_{2,T}^2}{t^2} + \frac{|\boldsymbol{\theta}_r|_{1,T}^2}{t^2}\right) \\ &\qquad \le Ch^2 \left(|w_0|_3^2 + |\boldsymbol{\theta}|_2^2 + \frac{|w_r|_2^2}{t^2} + \frac{|\boldsymbol{\theta}_r|_1^2}{t^2}\right) \ .\end{aligned} \tag{3.59}$$

Using (2.5)–(2.7), from (3.59) it follows that

$$\begin{aligned}&\left(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} ||\boldsymbol{\nabla}(w - w_I^*) - (\boldsymbol{\theta} - \boldsymbol{\theta}_I)||_{0,T}^2\right)^{1/2} \\ &\qquad \le Ch\left(||w_0||_3 + ||\boldsymbol{\theta}||_2 + \frac{||w_r||_2}{t} + \frac{||\boldsymbol{\theta}_r||_1}{t}\right) \\ &\qquad \le Ch(||g||_{-1} + t\,||g||_0) \ .\end{aligned} \tag{3.60}$$

Therefore, we obtain (see (3.53))

$$(III) \leq Ch(||g||_{-1} + t\,||g||_0) \left(\sum_{T \in \mathcal{T}_h} (h_T^2 + t^2) ||\boldsymbol{\tau}_h||_{0,T}^2 \right)^{1/2} . \tag{3.61}$$

Estimate for (IV). We simply notice that

$$(IV) = \mu^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_I, \boldsymbol{\tau}_h) \leq Ct\,||\boldsymbol{\gamma} - \boldsymbol{\gamma}_I||_0 t\,||\boldsymbol{\tau}_h||_0 \leq Ch(||g||_{-1} + t\,||g||_0) t\,||\boldsymbol{\tau}_h||_0 . \tag{3.62}$$

Collecting (3.50), (3.52), (3.61) and (3.62), from (3.47) we get

$$\begin{aligned} &\mathcal{A}(\boldsymbol{\theta}_h - \boldsymbol{\theta}_I, w_h^* - w_I^*, \boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I; \boldsymbol{\eta}_h, v_h^*, \boldsymbol{\tau}_h) \\ &\qquad \leq Ch(||g||_{-1} + t\,||g||_0) \left(\|\!|(\boldsymbol{\eta}_h, v_h^*)|\!\|_h + ||\boldsymbol{\tau}_h||_h \right) . \end{aligned} \tag{3.63}$$

From (3.45), (3.46), (3.63) and the triangle inequality, we infer

$$\|\!|(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w_h^*)|\!\|_h + ||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_h \leq C\,h(||g||_{-1} + t\,||g||_0) . \tag{3.64}$$

To obtain the error in the H^{-1}-norm for the shears, we use the 'Pitkäranta-Verfürth trick' (cf. [20], [24] and also [21]). Hence, we recall that

$$||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_{-1} = \sup_{\boldsymbol{\eta} \in \boldsymbol{\Theta}} \frac{(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\eta})}{||\boldsymbol{\eta}||_1} . \tag{3.65}$$

For a generic $\boldsymbol{\eta} \in \boldsymbol{\Theta}$ we consider its Clemént's interpolant $\boldsymbol{\eta}^c \in \boldsymbol{\Theta}_h$ (see [12], for instance), and we write

$$(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\eta}) = (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\eta} - \boldsymbol{\eta}^c) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\eta}^c) . \tag{3.66}$$

On the one hand, we have

$$\begin{aligned} (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\eta} - \boldsymbol{\eta}^c) &\leq \left(\sum_{T \in \mathcal{T}_h} h_T^2 ||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_{0,T}^2 \right)^{1/2} \left(\sum_{T \in \mathcal{T}_h} h_T^{-2} ||\boldsymbol{\eta} - \boldsymbol{\eta}^c||_{0,T}^2 \right)^{1/2} \\ &\leq C||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_h ||\boldsymbol{\eta}||_1 . \end{aligned} \tag{3.67}$$

On the other hand, recalling (2.2) and (3.5), we get

$$(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\eta}^c) = a(\boldsymbol{\theta} - \boldsymbol{\theta}_h, \boldsymbol{\eta}^c) \leq C||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_1 ||\boldsymbol{\eta}^c||_1 \leq C||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_1 ||\boldsymbol{\eta}||_1 . \tag{3.68}$$

From (3.65)–(3.68), we obtain

$$||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_{-1} \leq C\left(||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_h + ||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_1\right) . \tag{3.69}$$

Estimate (3.43) now follows from (3.64) and (3.69). □

Using the technique in [10], one may also get the following improved estimates.

PROPOSITION 3.4. *Suppose that* Ω *is a convex polygon and* $g \in L^2(\Omega)$*. Then the following a priori estimates holds*

$$||\boldsymbol{\theta} - \boldsymbol{\theta}_h||_0 \leq Ch^2(||g||_{-1} + t\,||g||_0) \tag{3.70}$$

$$||w - w_h^*||_1 \leq Ch(h + t)(||g||_{-1} + t\,||g||_0) . \tag{3.71}$$

□

4. A posteriori error estimates. The aim of this section is to introduce suitable error estimator for the elements based on the 'Linked Interpolation Technique', and to prove its *reliability* and *efficiency.* To begin, for each $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$ we introduce the following quantities

$$\begin{aligned}\widetilde{\eta}_T^2 := h_T^2 || \operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h ||_{0,T}^2 + h_T^2(h_T^2 + t^2) || \operatorname{div} \boldsymbol{\gamma}_h + g_h ||_{0,T}^2 \\ + \frac{\mu^2}{h_T^2 + t^2} || \mu^{-1} t^2 \, \boldsymbol{\gamma}_h - (\boldsymbol{\nabla} w_h^* - \boldsymbol{\theta}_h) ||_{0,T}^2 \ ,\end{aligned} \tag{4.1}$$

$$\eta_e^2 := h_e || \, [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \mathbf{n}]\!] \, ||_{0,e}^2 + h_e (h_e^2 + t^2) || \, [\![\boldsymbol{\gamma}_h \cdot \mathbf{n}]\!] \, ||_{0,e}^2 \ , \tag{4.2}$$

where g_h is some approximation of the load g. Moreover, h_e is the length of the side e and $[\![\cdot]\!]$ denotes the jump operator. We then define a *local* indicator η_T as

$$\eta_T := \left(\widetilde{\eta}_T^2 + \sum_{e \subset \partial T} \eta_e^2 \right)^{1/2} \ , \tag{4.3}$$

and a *global* indicator η as

$$\eta := \left(\sum_{T \in \mathcal{T}_h} \widetilde{\eta}_T^2 + \sum_{e \in \mathcal{E}_h} \eta_e^2 \right)^{1/2} \ . \tag{4.4}$$

REMARK 4.1. *When considering the element described in Section 3.1.1, the expression in* (4.1) *becomes simpler, since we locally have* $\operatorname{div} \boldsymbol{\gamma}_h = 0$ (*see* (3.9)).

We now introduce some useful notation: given a generic $e \in \mathcal{E}_h$, we denote with ω_e the union of the triangles in $\mathcal{T}_h$ having e as a side. Furthermore, for $T \in \mathcal{T}_h$ we set ω_T as the union of the ω_e's, with $e \subset \partial T$. We proceed with the following result.

LEMMA 4.1. *Given $e \in \mathcal{E}_h$, let $P_k(e)$ be the space of polynomials of degree at most k defined on e. There exists a linear operator*

$$\Pi_e \ : \ P_k(e) \longrightarrow H_0^2(\omega_e) \tag{4.5}$$

such that for all $p_k \in P_k(e)$ it holds

$$C_1 ||p_k||_{0,e}^2 \leq \int_e p_k \big(\Pi_e p_k \big) \leq ||p_k||_{0,e}^2 \tag{4.6}$$

$$||\Pi_e p_k||_{0,\omega_e} \leq C_2 h_e^{1/2} ||p_k||_{0,e} \tag{4.7}$$

$$|\boldsymbol{\nabla}(\Pi_e p_k)|_{0,\omega_e} \leq C_3 h_e^{-1/2} ||p_k||_{0,e} \tag{4.8}$$

$$|\boldsymbol{\nabla}(\Pi_e p_k)|_{1,\omega_e} \leq C_4 h_e^{-3/2} ||p_k||_{0,e} \ . \tag{4.9}$$

Above, the constants C_i depend only on k and on the minimum angle of the triangles in the meshes $\mathcal{T}_h$.

Proof. We consider only the case of an *interior* edge e: if e is a *boundary* edge (i.e. $e \subset \partial\Omega$), the required modifications are obvious. Due to the minimum angle condition, there exists a *fixed* 'reference' rhomb $\widehat{D}$, as depicted in Fig. 4.1, where e.g.

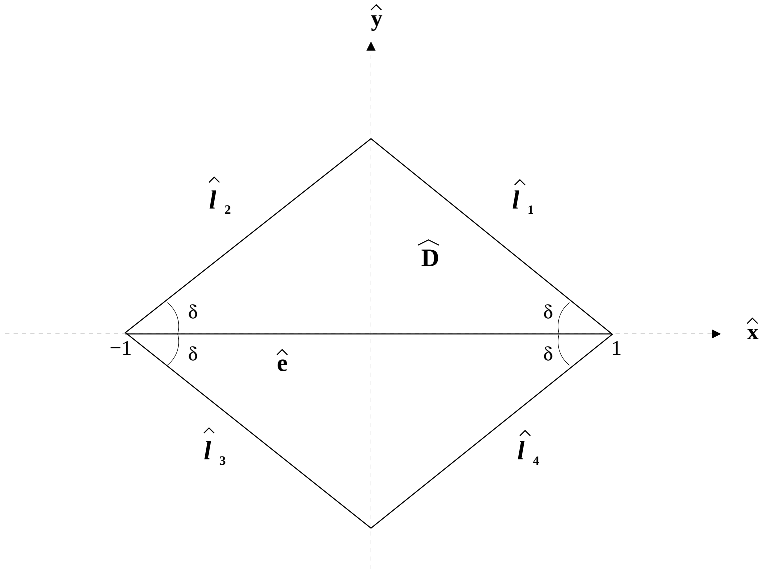

FIG. 4.1. *The 'reference' rhomb* $\widehat{D}$

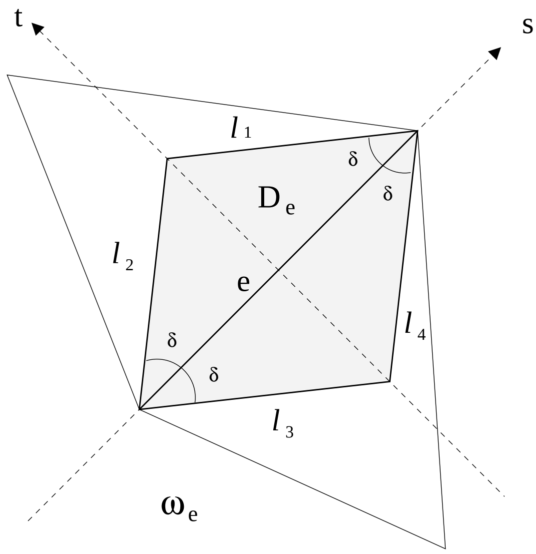

FIG. 4.2. *Relevant objects associated with the edge e*

$\delta = \alpha/2$ (see (3.2)), and with the following property: for each $e \in \mathcal{E}_h$ it is possible to determine a rhomb $D_e \subseteq \omega_e$ similar to $\widehat{D}$ (see Fig. 4.2). According to Fig. 4.2, on ω_e we now introduce local Cartesian coordinates (s,t), as well as the functions

$$d_i(s,t) = \text{"distance of } (s,t) \text{ from the edge } l_i\text{"},\ i = 1, ..., 4 \text{ (see Fig. 4.2).} \qquad (4.10)$$

Next, we define $\psi_e(s,t) : \omega_e \longrightarrow \mathbf{R}$ as

$$\psi_e(s,t) := \alpha_e \, \chi_{D_e}(s,t) \prod_{i=1}^{4} d_i(s,t)^2 \ , \qquad (4.11)$$

where $\chi_{D_e}(s,t)$ is the characteristic function of the set D_e, while α_e is a normalization constant in order to have $||\psi_e||_\infty = 1$. We also notice that in the coordinates (s,t) a generic polynomial $p_k \in P_k(e)$ can be simply written as $p_k(s)$. We are ready to define $\Pi_e : P_k(e) \longrightarrow H^2_0(\omega_e)$ by setting

$$\big(\Pi_e p_k\big)(s,t) := \psi_e(s,t)p_k(s) \qquad (s,t) \in \omega_e \ . \tag{4.12}$$

Estimates (4.6)–(4.9) easily follows from standard scaling arguments, using the *fixed* reference rhomb $\widehat{D}$. □

4.1. Upper bounds. We now prove that the indicator just introduced can be used as a *reliable* error estimator. We will prove our upper bounds for the *linear* element of Section 3.1.1 by means of a saturation assumption involving its *quadratic* version. Therefore, in order to avoid confusion, we will denote all the quantities relative to the quadratic element described in Section 3.1.2 by a "tilde". For example, the approximation spaces and linking operator in (3.14)–(3.19) will be renamed as $\widetilde{\mathbf{\Theta}}_h$, $\widetilde{W}_h$, $\widetilde{\mathbf{\Gamma}}_h$ and $\widetilde{L}$, respectively. Accordingly, we define

$$\widetilde{\boldsymbol{X}}_h = \big\{(\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}^*_h) = (\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h + \widetilde{L}\widetilde{\boldsymbol{\eta}}_h) : \widetilde{\boldsymbol{\eta}}_h \in \widetilde{\mathbf{\Theta}}_h \ , \ \widetilde{v}_h \in \widetilde{W}_h\big\}. \tag{4.13}$$

We need to make the following

Saturation assumption: Let $(\boldsymbol{\theta}_h, w^*_h, \boldsymbol{\gamma}_h) \in \boldsymbol{X}_h \times \mathbf{\Gamma}_h$ (resp. $(\widetilde{\boldsymbol{\theta}}_h, \widetilde{w}^*_h, \widetilde{\boldsymbol{\gamma}}_h) \in \widetilde{\boldsymbol{X}}_h \times \widetilde{\mathbf{\Gamma}}_h$) be the discrete solution using the linear (resp. quadratic) element. We assume that there exists $0 < \rho < 1$ such that

$$\begin{aligned} &|||(\boldsymbol{\theta} - \widetilde{\boldsymbol{\theta}}_h, w - \widetilde{w}^*_h)|||_h + ||\boldsymbol{\gamma} - \widetilde{\boldsymbol{\gamma}}_h||_{-1} + t\,||\boldsymbol{\gamma} - \widetilde{\boldsymbol{\gamma}}_h||_0 \\ &\qquad \le \rho\Big(|||(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w^*_h)|||_h + ||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_{-1} + t\,||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_0\Big). \end{aligned} \tag{4.14}$$

□

By using the saturation assumption (4.14), it is easily seen that one gets the reliability estimate

$$\begin{aligned} &|||(\boldsymbol{\theta} - \boldsymbol{\theta}_h, w - w^*_h)|||_h + ||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_{-1} + t\,||\boldsymbol{\gamma} - \boldsymbol{\gamma}_h||_0 \\ &\qquad \le C\left(\sum_{T\in\mathcal{T}_h}\Big(\eta^2_T + h^2_T(h^2_T + t^2)||g - g_h||^2_{0,T}\Big)\right)^{1/2} , \end{aligned} \tag{4.15}$$

provided one is able to bound

$$|||(\widetilde{\boldsymbol{\theta}}_h - \boldsymbol{\theta}_h, \widetilde{w}^*_h - w^*_h)|||_h + ||\widetilde{\boldsymbol{\gamma}}_h - \boldsymbol{\gamma}_h||_{-1} + t\,||\widetilde{\boldsymbol{\gamma}}_h - \boldsymbol{\gamma}_h||_0 \ . \tag{4.16}$$

To this aim, we need the next result, which states that $\boldsymbol{X}_h \subseteq \widetilde{\boldsymbol{X}}_h$, and that functions in $\widetilde{\boldsymbol{X}}_h$ can be approximated by functions in $\boldsymbol{X}_h$.

Lemma 4.2. *It holds $\boldsymbol{X}_h \subseteq \widetilde{\boldsymbol{X}}_h$; moreover, given $(\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}^*_h) \in \widetilde{\boldsymbol{X}}_h$, there exists $(\boldsymbol{\eta}_h, v^*_h) \in \boldsymbol{X}_h$ such that*

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h} h^{-2}_T\left(||\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h||^2_{0,T} + \frac{1}{h^2_T + t^2}||\widetilde{v}^*_h - v^*_h||^2_{0,T}\right) \\ &\qquad + \sum_{e\in\mathcal{E}_h} h^{-1}_e\left(||\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h||^2_{0,e} + \frac{1}{h^2_e + t^2}||\widetilde{v}^*_h - v^*_h||^2_{0,e}\right) \le C|||(\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}^*_h)|||^2_h \ . \end{aligned} \tag{4.17}$$

Proof. First, we need to show that a generic $(\boldsymbol{\eta}_h, v_h^*) = (\boldsymbol{\eta}_h, v_h + L\boldsymbol{\eta}_h) \in \boldsymbol{X}_h$ can be written as $(\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h^*) = (\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h + \widetilde{L}\widetilde{\boldsymbol{\eta}}_h) \in \widetilde{\boldsymbol{X}}_h$, for suitable $\widetilde{\boldsymbol{\eta}}_h \in \widetilde{\boldsymbol{\Theta}}_h$ and $\widetilde{v}_h \in \widetilde{W}_h$. This forces $\widetilde{\boldsymbol{\eta}}_h = \boldsymbol{\eta}_h$, which is an admissible choice, since obviously $\boldsymbol{\Theta}_h \subseteq \widetilde{\boldsymbol{\Theta}}_h$. Noting that $v_h + L\boldsymbol{\eta}_h \in \widetilde{W}_h$, we set $\widetilde{v}_h = v_h + L\boldsymbol{\eta}_h$. We now observe (see (3.18)–(3.19)) that $\widetilde{L}\widetilde{\boldsymbol{\eta}}_h = \widetilde{L}\boldsymbol{\eta}_h = 0$. Indeed, given $\boldsymbol{\eta}_h \in \boldsymbol{\Theta}_h$, the equation

$$(\nabla \widetilde{L}\boldsymbol{\eta}_h - \boldsymbol{\eta}_h) \cdot \mathbf{t} \quad \text{is linear on each } e \tag{4.18}$$

has unique solution $\widetilde{L}\boldsymbol{\eta}_h = 0$, since $\boldsymbol{\eta}_h$ is already linear on each edge e. Therefore we have

$$(\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h^*) = (\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h + \widetilde{L}\widetilde{\boldsymbol{\eta}}_h) = (\boldsymbol{\eta}_h, v_h + L\boldsymbol{\eta}_h + \widetilde{L}\boldsymbol{\eta}_h) = (\boldsymbol{\eta}_h, v_h + L\boldsymbol{\eta}_h) = (\boldsymbol{\eta}_h, v_h^*) \ ,$$

which proves $\boldsymbol{X}_h \subseteq \widetilde{\boldsymbol{X}}_h$.

To prove estimate (4.17), let $(\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h^*) = (\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h + \widetilde{\boldsymbol{\eta}}_h) \in \widetilde{\boldsymbol{X}}_h$ be given. We define (recalling that $\mathcal{I}$ denotes the Lagrange interpolating operator):

$$\boldsymbol{\eta}_h = \mathcal{I}\widetilde{\boldsymbol{\eta}}_h \in \boldsymbol{\Theta}_h \qquad , \qquad v_h = \mathcal{I}\widetilde{v}_h \in W_h \ . \tag{4.19}$$

Accordingly, we set

$$(\boldsymbol{\eta}_h, v_h^*) = \big(\mathcal{I}\widetilde{\boldsymbol{\eta}}_h, \mathcal{I}\widetilde{v}_h + L(\mathcal{I}\widetilde{\boldsymbol{\eta}}_h)\big) \in \boldsymbol{X}_h \ . \tag{4.20}$$

By standard approximation results and scaling arguments, we have

$$\sum_{T\in\mathcal{T}_h} h_T^{-2}||\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h||_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}||\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h||_{0,e}^2 \le C||\widetilde{\boldsymbol{\eta}}_h||_1^2 \ . \tag{4.21}$$

To continue, let us note that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \frac{h_T^{-2}}{h_T^2+t^2}||\widetilde{v}_h^* - v_h^*||_{0,T}^2 \le 2\sum_{T\in\mathcal{T}_h} \frac{h_T^{-2}}{h_T^2+t^2}||\widetilde{v}_h - v_h||_{0,T}^2 \\ + 2\sum_{T\in\mathcal{T}_h} \frac{h_T^{-2}}{h_T^2+t^2}||\widetilde{L}\widetilde{\boldsymbol{\eta}}_h - L\boldsymbol{\eta}_h||_{0,T}^2 \ .\end{aligned} \tag{4.22}$$

From standard approximation theory we have

$$\begin{aligned}||\widetilde{v}_h - v_h||_{0,T}^2 &\le Ch_T^4|\widetilde{v}_h|_{2,T}^2 = Ch_T^4|\nabla\widetilde{v}_h|_{1,T}^2 \\ &\le Ch_T^4\left(|\nabla\widetilde{v}_h^* - \widetilde{\boldsymbol{\eta}}_h|_{1,T}^2 + |\widetilde{\boldsymbol{\eta}}_h - \nabla\widetilde{L}\widetilde{\boldsymbol{\eta}}_h|_{1,T}^2\right) \ .\end{aligned} \tag{4.23}$$

Using an inverse inequality and (3.20) we get

$$||\widetilde{v}_h - v_h||_{0,T}^2 \le Ch_T^2||\nabla\widetilde{v}_h^* - \widetilde{\boldsymbol{\eta}}_h||_{0,T}^2 + Ch_T^4|\widetilde{\boldsymbol{\eta}}_h|_{1,T}^2 \ . \tag{4.24}$$

Therefore, we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\frac{h_T^{-2}}{h_T^2+t^2}||\widetilde{v}_h-v_h||_{0,T}^2 &\le C\sum_{T\in\mathcal{T}_h}\frac{1}{h_T^2+t^2}||\boldsymbol{\nabla}\widetilde{v}_h^*-\widetilde{\boldsymbol{\eta}}_h||_{0,T}^2\\ &\quad+C\sum_{T\in\mathcal{T}_h}\frac{h_T^2}{h_T^2+t^2}|\widetilde{\boldsymbol{\eta}}_h|_{1,T}^2\\ &\le C\left(\sum_{T\in\mathcal{T}_h}\frac{1}{h_T^2+t^2}||\boldsymbol{\nabla}\widetilde{v}_h^*-\widetilde{\boldsymbol{\eta}}_h||_{0,T}^2+||\widetilde{\boldsymbol{\eta}}_h||_1^2\right)\ .\end{aligned}\tag{4.25}$$

Furthermore, from (3.13), (3.20) and (4.19), we have

$$\begin{aligned}||\widetilde{L}\widetilde{\boldsymbol{\eta}}_h-L\boldsymbol{\eta}_h||_{0,T}^2 &\le 2\left(||\widetilde{L}\widetilde{\boldsymbol{\eta}}_h||_{0,T}^2+||L\boldsymbol{\eta}_h||_{0,T}^2\right)\\ &\le Ch_T^4\left(|\widetilde{\boldsymbol{\eta}}_h|_{1,T}^2+|\boldsymbol{\eta}_h|_{1,T}^2\right)\le Ch_T^4|\widetilde{\boldsymbol{\eta}}_h|_{1,T}^2\ .\end{aligned}\tag{4.26}$$

As a consequence, we get

$$\sum_{T\in\mathcal{T}_h}\frac{h_T^{-2}}{h_T^2+t^2}||\widetilde{L}\widetilde{\boldsymbol{\eta}}_h-L\boldsymbol{\eta}_h||_{0,T}^2\le C\sum_{T\in\mathcal{T}_h}\frac{h_T^2}{h_T^2+t^2}|\widetilde{\boldsymbol{\eta}}_h|_{1,T}^2\le C||\widetilde{\boldsymbol{\eta}}_h||_{1,T}^2\ .\tag{4.27}$$

Using (4.25) and (4.27), from (4.22) we have

$$\sum_{T\in\mathcal{T}_h}\frac{h_T^{-2}}{h_T^2+t^2}||\widetilde{v}_h^*-v_h^*||_{0,T}^2\le C|||(\widetilde{\boldsymbol{\eta}}_h,\widetilde{v}_h^*)|||_h^2\ .\tag{4.28}$$

The shape regularity of $\mathcal{T}_h$, scaling arguments, and estimate (4.28) show that

$$\sum_{e\in\mathcal{E}_h}\frac{h_e^{-1}}{h_e^2+t^2}||\widetilde{v}_h^*-v_h^*||_{0,e}^2\le C\sum_{T\in\mathcal{T}_h}\frac{h_T^{-2}}{h_T^2+t^2}||\widetilde{v}_h^*-v_h^*||_{0,T}^2\le C|||(\widetilde{\boldsymbol{\eta}}_h,\widetilde{v}_h^*)|||_h^2\ .\tag{4.29}$$

Collecting (4.21), (4.28) and (4.29), we infer estimate (4.17). □

We are now ready to prove the following proposition.

PROPOSITION 4.3. *We have*

$$\begin{aligned}&|||(\widetilde{\boldsymbol{\theta}}_h-\boldsymbol{\theta}_h,\widetilde{w}_h^*-w_h^*)|||_h+||\widetilde{\boldsymbol{\gamma}}_h-\boldsymbol{\gamma}_h||_{-1}+t\,||\widetilde{\boldsymbol{\gamma}}_h-\boldsymbol{\gamma}_h||_0\\ &\qquad\le C\left(\sum_{T\in\mathcal{T}_h}\left(\eta_T^2+h_T^2(h_T^2+t^2)||g-g_h||_{0,T}^2\right)\right)^{1/2}\ .\end{aligned}\tag{4.30}$$

Proof. Consider $(\widetilde{\boldsymbol{\theta}}_h-\boldsymbol{\theta}_h,\widetilde{w}_h^*-w_h^*;\widetilde{\boldsymbol{\gamma}}_h-\boldsymbol{\gamma}_h)\in\widetilde{\boldsymbol{X}}_h\times\widetilde{\boldsymbol{\Gamma}}_h$. Discrete stability for the *quadratic* element (see Proposition 3.2) implies that there exists $(\widetilde{\boldsymbol{\eta}}_h,\widetilde{v}_h^*;\widetilde{\boldsymbol{\tau}}_h)$ in $\widetilde{\boldsymbol{X}}_h\times\widetilde{\boldsymbol{\Gamma}}_h$ such that

$$
|||(\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h^*)|||_h + ||\widetilde{\boldsymbol{\tau}}_h||_h \leq 1 \tag{4.31}
$$

and

$$
\begin{aligned}
C\Big(|||(\widetilde{\boldsymbol{\theta}}_h - \boldsymbol{\theta}_h, \widetilde{w}_h^* - w_h^*)|||_h + ||\widetilde{\boldsymbol{\gamma}}_h - \boldsymbol{\gamma}_h||_h \Big) & \\
\leq \Big\{ a(\widetilde{\boldsymbol{\theta}}_h - \boldsymbol{\theta}_h, \widetilde{\boldsymbol{\eta}}_h) + (\widetilde{\boldsymbol{\gamma}}_h - \boldsymbol{\gamma}_h, \boldsymbol{\nabla} \widetilde{v}_h^* - \widetilde{\boldsymbol{\eta}}_h) \Big\} & \\
+ \Big\{ - \big(\boldsymbol{\nabla}(\widetilde{w}_h^* - w_h^*) - (\widetilde{\boldsymbol{\theta}}_h - \boldsymbol{\theta}_h), \widetilde{\boldsymbol{\tau}}_h \big) + \mu^{-1} t^2 (\widetilde{\boldsymbol{\gamma}}_h - \boldsymbol{\gamma}_h, \widetilde{\boldsymbol{\tau}}_h) \Big\} & \\
= (I) + (II) \ . &
\end{aligned} \tag{4.32}
$$

On one hand, since $(\widetilde{\boldsymbol{\theta}}_h, \widetilde{w}_h^*; \widetilde{\boldsymbol{\gamma}}_h)$ (resp. $(\boldsymbol{\theta}_h, w_h^*; \boldsymbol{\gamma}_h)$) solves the higher-order (resp. low-order) discrete problem, we have

$$
\begin{aligned}
(I) &= a(\widetilde{\boldsymbol{\theta}}_h - \boldsymbol{\theta}_h, \widetilde{\boldsymbol{\eta}}_h) + (\widetilde{\boldsymbol{\gamma}}_h - \boldsymbol{\gamma}_h, \boldsymbol{\nabla} \widetilde{v}_h^* - \widetilde{\boldsymbol{\eta}}_h) \\
&= (g, \widetilde{v}_h^*) - a(\boldsymbol{\theta}_h, \widetilde{\boldsymbol{\eta}}_h) - (\boldsymbol{\gamma}_h, \boldsymbol{\nabla} \widetilde{v}_h^* - \widetilde{\boldsymbol{\eta}}_h) \\
&= (g, \widetilde{v}_h^* - v_h^*) - a(\boldsymbol{\theta}_h, \widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h) - \big(\boldsymbol{\gamma}_h, \boldsymbol{\nabla}(\widetilde{v}_h^* - v_h^*) - (\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h) \big) \ ,
\end{aligned} \tag{4.33}
$$

where we choose $(\boldsymbol{\eta}_h, v_h^*) \in \boldsymbol{X}_h$ satisfying estimate (4.17). An elementwise integration by parts gives

$$
\begin{aligned}
(I) = \sum_{T \in \mathcal{T}_h} \Big\{ \int_T \big(\operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h \big) \cdot (\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h) - \int_{\partial T} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \mathbf{n} \cdot (\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h) \Big\} & \\
+ \sum_{T \in \mathcal{T}_h} \Big\{ \int_T \big(\operatorname{div} \boldsymbol{\gamma}_h + g \big) (\widetilde{v}_h^* - v_h^*) - \int_{\partial T} \boldsymbol{\gamma}_h \cdot \mathbf{n}\, (\widetilde{v}_h^* - v_h^*) \Big\} &
\end{aligned} \tag{4.34}
$$

by which

$$
\begin{aligned}
(I) = \sum_{T \in \mathcal{T}_h} \int_T \big(\operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h \big) \cdot (\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h) - \sum_{e \in \mathcal{E}_h} \int_e [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \mathbf{n}]\!] \cdot (\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h) & \\
+ \sum_{T \in \mathcal{T}_h} \int_T \big(\operatorname{div} \boldsymbol{\gamma}_h + g \big) (\widetilde{v}_h^* - v_h^*) - \sum_{e \in \mathcal{E}_h} \int_e [\![\boldsymbol{\gamma}_h \cdot \mathbf{n}]\!] (\widetilde{v}_h^* - v_h^*) \ . &
\end{aligned} \tag{4.35}
$$

Hence, it holds

$$
\begin{aligned}
(I) \leq C \Bigg(&\Big(\sum_{T\in\mathcal{T}_h} h_T^2 || \operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h ||_{0,T}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} h_T^{-2} ||\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h ||_{0,T}^2 \Big)^{1/2} \\
&+ \Big(\sum_{e\in\mathcal{E}_h} h_e || \, [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h)\mathbf{n}]\!] \, ||_{0,e}^2 \Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_h} h_e^{-1} ||\widetilde{\boldsymbol{\eta}}_h - \boldsymbol{\eta}_h ||_{0,e}^2 \Big)^{1/2} \\
&+ \Big(\sum_{T\in\mathcal{T}_h} h_T^2 (h_T^2 + t^2) || \operatorname{div} \boldsymbol{\gamma}_h + g ||_{0,T}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 (h_T^2 + t^2)} ||\widetilde{v}_h^* - v_h^* ||_{0,T}^2 \Big)^{1/2} \\
&+ \Big(\sum_{e\in\mathcal{E}_h} h_e (h_e^2 + t^2) || \, [\![\boldsymbol{\gamma}_h \cdot \mathbf{n}]\!] \, ||_{0,e}^2 \Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_h} \frac{1}{h_e (h_e^2 + t^2)} ||\widetilde{v}_h^* - v_h^* ||_{0,e}^2 \Big)^{1/2} \Bigg) \, .
\end{aligned}
\tag{4.36}
$$

Using Lemma 4.2, we get

$$
\begin{aligned}
(I) \leq C \Bigg(&\Big(\sum_{T\in\mathcal{T}_h} h_T^2 || \operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h ||_{0,T}^2 \Big)^{1/2} + \Big(\sum_{e\in\mathcal{E}_h} h_e || \, [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h)\mathbf{n}]\!] \, ||_{0,e}^2 \Big)^{1/2} \\
&+ \Big(\sum_{T\in\mathcal{T}_h} h_T^2 (h_T^2 + t^2) || \operatorname{div} \boldsymbol{\gamma}_h + g ||_{0,T}^2 \Big)^{1/2} + \Big(\sum_{e\in\mathcal{E}_h} h_e (h_e^2 + t^2) || \, [\![\boldsymbol{\gamma}_h \cdot \mathbf{n}]\!] \, ||_{0,e}^2 \Big)^{1/2} \Bigg) \\
&\times |\!|\!| (\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h^*) |\!|\!|_h .
\end{aligned}
\tag{4.37}
$$

Therefore, one has

$$
\begin{aligned}
(I) \leq C \Bigg(&\Big(\sum_{T\in\mathcal{T}_h} h_T^2 || \operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h ||_{0,T}^2 \Big)^{1/2} + \Big(\sum_{e\in\mathcal{E}_h} h_e || \, [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h)\mathbf{n}]\!] \, ||_{0,e}^2 \Big)^{1/2} \\
&+ \Big(\sum_{T\in\mathcal{T}_h} h_T^2 (h_T^2 + t^2) || \operatorname{div} \boldsymbol{\gamma}_h + g_h ||_{0,T}^2 \Big)^{1/2} + \Big(\sum_{T\in\mathcal{T}_h} h_T^2 (h_T^2 + t^2) || g - g_h ||_{0,T}^2 \Big)^{1/2} \\
&+ \Big(\sum_{e\in\mathcal{E}_h} h_e (h_e^2 + t^2) || \, [\![\boldsymbol{\gamma}_h \cdot \mathbf{n}]\!] \, ||_{0,e}^2 \Big)^{1/2} \Bigg) \, |\!|\!| (\widetilde{\boldsymbol{\eta}}_h, \widetilde{v}_h^*) |\!|\!|_h .
\end{aligned}
\tag{4.38}
$$

On the other hand, since $(\widetilde{\boldsymbol{\theta}}_h, \widetilde{w}_h^*; \widetilde{\boldsymbol{\gamma}}_h)$ solves the higher-order discrete problem, we have

$$
\begin{aligned}
(II) &= -\big(\boldsymbol{\nabla}(\widetilde{w}_h^* - w_h^*) - (\widetilde{\boldsymbol{\theta}}_h - \boldsymbol{\theta}_h), \widetilde{\boldsymbol{\tau}}_h \big) + \mu^{-1} t^2 (\widetilde{\boldsymbol{\gamma}}_h - \boldsymbol{\gamma}_h, \widetilde{\boldsymbol{\tau}}_h) \\
&= - \big(\mu^{-1} t^2 \, \boldsymbol{\gamma}_h - (\boldsymbol{\nabla} w_h^* - \boldsymbol{\theta}_h), \widetilde{\boldsymbol{\tau}}_h \big) \\
&\leq \left(\sum_{T\in\mathcal{T}_h} \frac{1}{h_T^2 + t^2} || \mu^{-1} t^2 \, \boldsymbol{\gamma}_h - (\boldsymbol{\nabla} w_h^* - \boldsymbol{\theta}_h) ||_{0,T}^2 \right)^{1/2} \left(\sum_{T\in\mathcal{T}_h} (h_T^2 + t^2) ||\widetilde{\boldsymbol{\tau}}_h||_{0,T}^2 \right)^{1/2} .
\end{aligned}
\tag{4.39}
$$

As a consequence, from (4.32), (4.38), (4.39), using (4.31) and recalling definitions (4.1)–(4.3), we have

$$|\!|\!|(\widetilde{\boldsymbol{\theta}}_h-\boldsymbol{\theta}_h,\widetilde{w}_h^*-w_h^*)|\!|\!|_h+||\widetilde{\boldsymbol{\gamma}}_h-\boldsymbol{\gamma}_h||_h \le C\left(\sum_{T\in\mathcal{T}_h}\left(\eta_T^2+h_T^2(h_T^2+t^2)||g-g_h||_{0,T}^2\right)\right)^{1/2} . \tag{4.40}$$

The same arguments as in (3.65)–(3.69), applied to $\widetilde{\boldsymbol{\gamma}}_h-\boldsymbol{\gamma}_h$, give

$$||\widetilde{\boldsymbol{\gamma}}_h-\boldsymbol{\gamma}_h||_{-1} \le C\left(||\widetilde{\boldsymbol{\gamma}}_h-\boldsymbol{\gamma}_h||_h+||\widetilde{\boldsymbol{\theta}}_h-\boldsymbol{\theta}_h||_1\right) . \tag{4.41}$$

Combining (4.40) and (4.41) we infer estimate (4.30). The proof is complete. □

4.2. Lower bounds. We now prove the *efficiency* of our error estimator by establishing the following proposition.

PROPOSITION 4.4. *Let $(\boldsymbol{\theta},w;\boldsymbol{\gamma})$ (resp. $(\boldsymbol{\theta}_h,w_h^*;\boldsymbol{\gamma}_h)$) be the solution of the continuous (resp. discrete) problem. Given $T\in\mathcal{T}_h$, it holds*

$$\begin{aligned}\eta_T \le C\Bigg(&\frac{1}{(h_T^2+t^2)^{1/2}}\big|\big|\boldsymbol{\nabla}(w_h^*-w)-(\boldsymbol{\theta}_h-\boldsymbol{\theta})\big|\big|_{0,T}+||\boldsymbol{\theta}_h-\boldsymbol{\theta}||_{1,\omega_T}\\ &+||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{-1,\omega_T}+t\,||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{0,\omega_T}+\big(\sum_{T\subset\omega_T}h_T^2(h_T^2+t^2)||g-g_h||_{0,T}^2\big)^{1/2}\Bigg) ,\end{aligned} \tag{4.42}$$

where η_T is defined by (4.1)–(4.3).

Proof. Fix $T\in\mathcal{T}_h$ and a generic edge $e\subset\partial T$. We proceed in three steps.
First step. Since

$$\mu^{-1}t^2\boldsymbol{\gamma}=\boldsymbol{\nabla}w-\boldsymbol{\theta} , \tag{4.43}$$

we get

$$\begin{aligned}&\frac{1}{(h_T^2+t^2)^{1/2}}||\mu^{-1}t^2\,\boldsymbol{\gamma}_h-(\boldsymbol{\nabla}w_h^*-\boldsymbol{\theta}_h)||_{0,T}\\ &=\frac{1}{(h_T^2+t^2)^{1/2}}\big|\big|\mu^{-1}t^2\,(\boldsymbol{\gamma}_h-\boldsymbol{\gamma})-\big(\boldsymbol{\nabla}(w_h^*-w)-(\boldsymbol{\theta}_h-\boldsymbol{\theta})\big)\big|\big|_{0,T}\\ &\le C\left(t\,||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{0,T}+\frac{1}{(h_T^2+t^2)^{1/2}}\big|\big|\boldsymbol{\nabla}(w_h^*-w)-(\boldsymbol{\theta}_h-\boldsymbol{\theta})\big|\big|_{0,T}\right)\end{aligned} \tag{4.44}$$

Second step. We choose

$$\boldsymbol{\eta}_T=h_T^2(\operatorname{div}\mathbf{C}\,\varepsilon(\boldsymbol{\theta}_h)+\boldsymbol{\gamma}_h)\,b_T , \tag{4.45}$$

where b_T is the standard cubic bubble on T. We observe that

$$|\boldsymbol{\eta}_T|_{1,T}\le Ch_T||\operatorname{div}\mathbf{C}\,\varepsilon(\boldsymbol{\theta}_h)+\boldsymbol{\gamma}_h||_{0,T} . \tag{4.46}$$

Taking advantage of the equilibrium equation

$$-\operatorname{div}\mathbf{C}\,\varepsilon(\boldsymbol{\theta})-\boldsymbol{\gamma}=\mathbf{0} , \tag{4.47}$$

we get

$$
\begin{aligned}
&h_T^2 || \operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h ||_{0,T}^2 \\
&\quad \leq C\big(\operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h, \boldsymbol{\eta}_T \big) = C\big(\operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h - \boldsymbol{\theta}) + (\boldsymbol{\gamma}_h - \boldsymbol{\gamma}), \boldsymbol{\eta}_T \big) \\
&\quad = C\left(-a(\boldsymbol{\theta}_h - \boldsymbol{\theta}, \boldsymbol{\eta}_T) + (\boldsymbol{\gamma}_h - \boldsymbol{\gamma}, \boldsymbol{\eta}_T) \right) \\
&\quad \leq C\left(||\boldsymbol{\theta}_h - \boldsymbol{\theta}||_{1,T} + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}||_{-1,T} \right) |\boldsymbol{\eta}_T|_{1,T} \ .
\end{aligned}
\tag{4.48}
$$

Using (4.46), from (4.48) we thus obtain

$$
h_T || \operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h ||_{0,T} \leq C\left(||\boldsymbol{\theta}_h - \boldsymbol{\theta}||_{1,T} + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}||_{-1,T} \right) \ . \tag{4.49}
$$

Next, we choose

$$
\boldsymbol{\eta}_e = h_e P([\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \mathbf{n}]\!])\, b_e \ , \tag{4.50}
$$

where P is the prolongation operator introduced in [25] and b_e is the usual 'edge' bubble on e. We observe that it holds

$$
\Big(\sum_{T \subset \omega_e} h_T^{-2} ||\boldsymbol{\eta}_e||_{0,T}^2 \Big)^{1/2} \leq C |\boldsymbol{\eta}_e|_{1,\omega_e} \leq C h_e^{1/2} || \, [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \mathbf{n}]\!] \, ||_{0,e} \ . \tag{4.51}
$$

Integrating by parts and using again the equilibrium equation (4.47), we have

$$
\begin{aligned}
&h_e || \, [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \mathbf{n}]\!] \, ||_{0,e}^2 \\
&\quad \leq C \int_e [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \mathbf{n}]\!] \cdot \boldsymbol{\eta}_e = C \left(\int_{\omega_e} \operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \cdot \boldsymbol{\eta}_e + \int_{\omega_e} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) : \varepsilon(\boldsymbol{\eta}_e) \right) \\
&\quad = C \Big(\big(\operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h, \boldsymbol{\eta}_e \big) + a(\boldsymbol{\theta}_h - \boldsymbol{\theta}, \boldsymbol{\eta}_e) - (\boldsymbol{\gamma}_h - \boldsymbol{\gamma}, \boldsymbol{\eta}_e) \Big) \\
&\quad \leq C \Bigg(\Big(\sum_{T \subset \omega_e} h_T^2 || \operatorname{div} \mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) + \boldsymbol{\gamma}_h ||_{0,T}^2 \Big)^{1/2} \Big(\sum_{T \subset \omega_e} h_T^{-2} ||\boldsymbol{\eta}_e||_{0,T}^2 \Big)^{1/2} \\
&\quad + \Big(||\boldsymbol{\theta}_h - \boldsymbol{\theta}||_{1,\omega_e} + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}||_{-1,\omega_e} \Big) |\boldsymbol{\eta}_e|_{1,\omega_e} \Bigg) \ .
\end{aligned}
\tag{4.52}
$$

Therefore, using (4.51) and (4.49), from (4.52) we get

$$
h_e^{1/2} || \, [\![\mathbf{C}\, \varepsilon(\boldsymbol{\theta}_h) \mathbf{n}]\!] \, ||_{0,e} \leq C\left(||\boldsymbol{\theta}_h - \boldsymbol{\theta}||_{1,\omega_e} + ||\boldsymbol{\gamma}_h - \boldsymbol{\gamma}||_{-1,\omega_e} \right) \ . \tag{4.53}
$$

Third step. We first define

$$
\varphi_T = (\operatorname{div} \boldsymbol{\gamma}_h + g_h)\, b_T^2 \ . \tag{4.54}
$$

We observe that $\varphi_T \in H_0^2(T)$ and one has

$$
\begin{aligned}
|\varphi_T|_{1,T} &\leq C h_T^{-1} || \operatorname{div} \boldsymbol{\gamma}_h + g_h ||_{0,T} \\
|\boldsymbol{\nabla} \varphi_T|_{1,T} &\leq C h_T^{-2} || \operatorname{div} \boldsymbol{\gamma}_h + g_h ||_{0,T} \ .
\end{aligned}
\tag{4.55}
$$

We then set

$$
v_T = h_T^2 (h_T^2 + t^2)\, \varphi_T \ . \tag{4.56}
$$

Using the equilibrium equation

$$-\operatorname{div}\boldsymbol{\gamma} = g\ , \tag{4.57}$$

we get

$$\begin{aligned} h_T^2(h_T^2+t^2)\|\operatorname{div}\boldsymbol{\gamma}_h + g_h\|_{0,T}^2 &\le C\big(\operatorname{div}\boldsymbol{\gamma}_h + g_h, v_T\big) \\ &= C\Big(\big(\operatorname{div}(\boldsymbol{\gamma}_h-\boldsymbol{\gamma}), v_T\big) + (g_h - g, v_T)\Big)\ . \end{aligned} \tag{4.58}$$

We now separately treat the two terms at the right-hand side of (4.58). Integrating by parts, recalling (4.54) and (4.56), and using (4.55), we have

$$\begin{aligned} &\big(\operatorname{div}(\boldsymbol{\gamma}_h-\boldsymbol{\gamma}), v_T\big) = -(\boldsymbol{\gamma}_h-\boldsymbol{\gamma}, \boldsymbol{\nabla} v_T) \\ &= -h_T^4(\boldsymbol{\gamma}_h-\boldsymbol{\gamma}, \boldsymbol{\nabla}\varphi_T) - t^2h_T^2(\boldsymbol{\gamma}_h-\boldsymbol{\gamma}, \boldsymbol{\nabla}\varphi_T) \\ &\le \|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}\|_{-1,T}h_T^4|\boldsymbol{\nabla}\varphi_T|_{1,T} + t\,\|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}\|_{0,T}h_T^2 t\,\|\boldsymbol{\nabla}\varphi_T\|_{0,T} \\ &\le C\Big(\|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}\|_{-1,T}h_T^2\|\operatorname{div}\boldsymbol{\gamma}_h+g_h\|_{0,T} + t\,\|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}\|_{0,T}h_T t\,\|\operatorname{div}\boldsymbol{\gamma}_h+g_h\|_{0,T}\Big) \\ &\le C\Big(\|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}\|_{-1,T} + t\,\|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}\|_{0,T}\Big)\, h_T(h_T^2+t^2)^{1/2}\|\operatorname{div}\boldsymbol{\gamma}_h+g_h\|_{0,T}\ . \end{aligned} \tag{4.59}$$

Furthermore, it holds

$$\begin{aligned} (g_h-g, v_T) &\le h_T(h_T^2+t^2)^{1/2}\|g_h-g\|_{0,T}\, h_T(h_T^2+t^2)^{1/2}\|\varphi_T\|_{0,T} \\ &\le Ch_T(h_T^2+t^2)^{1/2}\|g_h-g\|_{0,T}\, h_T(h_T^2+t^2)^{1/2}\|\operatorname{div}\boldsymbol{\gamma}_h+g_h\|_{0,T}\ . \end{aligned} \tag{4.60}$$

Therefore, using (4.59) and (4.60), from (4.58) we infer

$$\begin{aligned} h_T(h_T^2+t^2)^{1/2}\|\operatorname{div}\boldsymbol{\gamma}_h+g_h\|_{0,T} &\le C\Big(\|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}\|_{-1,T} \\ &+ t\,\|\boldsymbol{\gamma}_h-\boldsymbol{\gamma}\|_{0,T} + h_T(h_T^2+t^2)^{1/2}\|g_h-g\|_{0,T}\Big)\ . \end{aligned} \tag{4.61}$$

Next, we define

$$\varphi_e = \Pi_e([\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!])\ , \tag{4.62}$$

where Π_e is the linear operator of Lemma 4.1. Therefore, we have

$$\|\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,\|_{0,e}^2 \le C\int_e [\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,\varphi_e \tag{4.63}$$

$$\|\varphi_e\|_{0,\omega_e} \le Ch_e^{1/2}\|\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,\|_{0,e} \tag{4.64}$$

$$\|\boldsymbol{\nabla}\varphi_e\|_{0,\omega_e} \le Ch_e^{-1/2}\|\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,\|_{0,e} \tag{4.65}$$

$$|\boldsymbol{\nabla}\varphi_e|_{1,\omega_e} \le Ch_e^{-3/2}\|\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,\|_{0,e}\ . \tag{4.66}$$

We then set

$$v_e = h_e(h_e^2+t^2)\,\varphi_e\ . \tag{4.67}$$

Integrating by parts using (4.63) and the equilibrium equation (4.57), we get

$$
\begin{aligned}
h_e(h_e^2+t^2)||\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,||_{0,e}^2 &\le C\int_e [\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,v_e \\
&\le C\left(\int_{\omega_e} v_e\operatorname{div}\boldsymbol{\gamma}_h+\int_{\omega_e}\boldsymbol{\gamma}_h\cdot\boldsymbol{\nabla} v_e\right) \\
&= C\Big((\operatorname{div}\boldsymbol{\gamma}_h+g,v_e)+(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\nabla} v_e)\Big) \\
&= C\Big((\operatorname{div}\boldsymbol{\gamma}_h+g_h,v_e)+(g-g_h,v_e)+(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\nabla} v_e)\Big)\ .
\end{aligned}
\tag{4.68}
$$

We now estimate the three terms above. Recalling (4.67) and using (4.64), we obtain

$$
\begin{aligned}
(\operatorname{div}\boldsymbol{\gamma}_h+g_h,v_e) &= h_e(h_e^2+t^2)\big(\operatorname{div}\boldsymbol{\gamma}_h+g_h,\varphi_e\big) \\
&= \sum_{T\subset\omega_e}\int_T\Big(h_e(h_e^2+t^2)^{1/2}(\operatorname{div}\boldsymbol{\gamma}_h+g_h)\Big)\Big((h_e^2+t^2)^{1/2}\varphi_e\Big) \\
&\le \Big(\sum_{T\subset\omega_e} h_e^2(h_e^2+t^2)||\operatorname{div}\boldsymbol{\gamma}_h+g_h||_{0,T}^2\Big)^{1/2}\Big(\sum_{T\subset\omega_e}(h_e^2+t^2)||\varphi_e||_{0,T}^2\Big)^{1/2} \\
&\le \Big(\sum_{T\subset\omega_e} h_T^2(h_T^2+t^2)||\operatorname{div}\boldsymbol{\gamma}_h+g_h||_{0,T}^2\Big)^{1/2}\Big(\sum_{T\subset\omega_e}(h_e^2+t^2)||\varphi_e||_{0,T}^2\Big)^{1/2} \\
&\le C\Big(\sum_{T\subset\omega_e} h_T^2(h_T^2+t^2)||\operatorname{div}\boldsymbol{\gamma}_h+g_h||_{0,T}^2\Big)^{1/2} h_e^{1/2}(h_e^2+t^2)^{1/2}||\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,||_{0,e}\ .
\end{aligned}
\tag{4.69}
$$

The same argument shows that it holds

$$
(g-g_h,v_e)\le C\Big(\sum_{T\subset\omega_e} h_T^2(h_T^2+t^2)||g-g_h||_{0,T}^2\Big)^{1/2} h_e^{1/2}(h_e^2+t^2)^{1/2}||\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,||_{0,e}\ .
\tag{4.70}
$$

We now notice that

$$
(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\nabla} v_e)=h_e^3(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\nabla}\varphi_e)+h_et^2(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\nabla}\varphi_e)\ . \tag{4.71}
$$

On one hand, using (4.66), we have

$$
\begin{aligned}
h_e^3(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\nabla}\varphi_e) &\le ||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{-1,\omega_e}h_e^3|\boldsymbol{\nabla}\varphi_e|_{1,\omega_e} \\
&\le C||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{-1,\omega_e}h_e^{3/2}||\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,||_{0,e}\ .
\end{aligned}
\tag{4.72}
$$

On the other hand, from (4.65) we get

$$
\begin{aligned}
h_et^2(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\nabla}\varphi_e) &\le t\,||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{0,\omega_e}h_et\,||\boldsymbol{\nabla}\varphi_e||_{0,\omega_e} \\
&\le Ct\,||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{0,\omega_e}h_e^{1/2}t\,||\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,||_{0,e}\ .
\end{aligned}
\tag{4.73}
$$

Therefore, using (4.72) and (4.73) from (4.71) we obtain

$$
(\boldsymbol{\gamma}_h-\boldsymbol{\gamma},\boldsymbol{\nabla} v_e)\le C\big(||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{-1,\omega_e}+t\,||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{0,\omega_e}\big)h_e^{1/2}(h_e^2+t^2)^{1/2}||\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,||_{0,e}\ .
\tag{4.74}
$$

Collecting (4.69), (4.70) and (4.74), we infer from (4.68) that

$$h_e^{1/2}(h_e^2+t^2)^{1/2}||\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,||_{0,e}\leq C\left(\Big(\sum_{T\subset\omega_e}h_T^2(h_T^2+t^2)||\operatorname{div}\boldsymbol{\gamma}_h+g_h||_{0,T}^2\Big)^{1/2}\right.$$
$$\left.+||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{-1,\omega_e}+t\,||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{0,\omega_e}+\Big(\sum_{T\subset\omega_e}h_T^2(h_T^2+t^2)||g-g_h||_{0,T}^2\Big)^{1/2}\right)\,. \tag{4.75}$$

Hence, from (4.61) we get

$$h_e^{1/2}(h_e^2+t^2)^{1/2}||\,[\![\boldsymbol{\gamma}_h\cdot\mathbf{n}]\!]\,||_{0,e}\leq C\Big(||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{-1,\omega_e}$$
$$+\,t\,||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}||_{0,\omega_e}+\Big(\sum_{T\subset\omega_e}h_T^2(h_T^2+t^2)||g-g_h||_{0,T}^2\Big)^{1/2}\Big)\,. \tag{4.76}$$

Estimate (4.42) now follows from (4.44), (4.49), (4.53), (4.61) and (4.76). □

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