A NEW CLASS OF MIXED FINITE ELEMENT METHODS FOR REISSNER–MINDLIN PLATES

C. LOVADINA †

Abstract. A new class of finite elements for Reissner–Mindlin plate problem is presented. The family is based on a modified mixed formulation recently introduced by Arnold and Brezzi. A result of stability and convergence uniformly in the thickness is provided.

Key words. Plate, shear locking, finite element.

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1. Introduction. The Reissner–Mindlin theory is widely used to describe the bending behaviour of an elastic plate loaded by a transverse force. Despite its simple formulation, it was suddenly noticed that discretization by means of finite elements was not at all straightforward, since in most cases numerical experiments exhibited a bad lack of convergence whenever the plate thickness was "too small". Nowadays this undesirable phenomenon, known as *shear locking* (cf. [11]), is well understood: as the thickness tends to zero, the Reissner–Mindlin model enforces the Kirchhoff constraint, which is usually too severe on the discrete level when low–order elements are employed. More precisely, Kirchhoff constraint arises from the shear stress term which has to be taken into account in Reissner–Mindlin formulation.

The most popular way to overcome shear locking phenomenon is to reduce the influence of the shear energy by considering a suitable mixed formulation. A number of locking free elements based on this philosophy have been proposed and studied (cf. [6], [8]). Unfortunately, most of the mixed formulations require a clever choice of spaces of discretization in order to success the approximation and they do not generalize to shells. Although many methods for both Koiter and Naghdi shells are used in engineering practice (cf [4], [15]), the problem of their convergence and stability has remained completely unsolved.

Arnold and Brezzi (cf. [2]) have recently proposed a modification to the standard mixed formulation of Reissner–Mindlin plate problem which seems to be capable to allow a wider choice of elements. More importantly, its generalization to the Naghdi shell problem is the only one for which a first theoretical analysis has been developped so far (cf. [3]). Having this in mind, we face the study of elements for Reissner–Mindlin plates based on the modified formulation as a first step of research, firmly believing that they can lead to competitive methods for shells.

In this paper we present a new class of finite elements based on the formulation given in [2]. For our family we prove stability and convegence uniformly in the thickness.

An outline of the paper is as follows.

In Section 2 we first recall the Reissner–Mindlin model. Then we briefly introduce both the standard mixed formulation and the modified one.

Section 3 is the core of the paper. By using a macroelement technique we prove a general stability result. The key idea is the possibility, under suitable hypotheses, to construct an interpolation operator *à la Fortin* (cf. [7]) not componentwise but mixing the components of rotations and vertical displacements.

†Istituto di Analisi Numerica del C.N.R., Via Abbiategrasso 209, 27100 Pavia, Italia.

In Section 4 we will apply the results of Section 3 to the case of the family $P_{k+2} - P_{k+2} - P_k$, showing that conditions of Theorem 3.1 are fulfilled for this choice. We conclude the paper by stating a uniform convergence result which follows immediately from our analysis and standard mixed finite element theory.

An appendix in which we briefly discuss the implementation features of our methods is also provided.

Our notations agree with those commonly used in the literature (cf. [7] and [10], for instance).

2. The Reissner–Mindlin model and the modified mixed formulation.

Let us denote with $A = \Omega \times (-t/2, t/2)$ the region in $\mathbf{R}^3$ occupied by an undeformed elastic plate of thickness $t > 0$. It's well known that the Reissner–Mindlin plate model describes the bending behaviour of the plate in terms of the transverse displacements and of the fiber rotations normal to the midplane Ω .

If we restrict our attention to the case of a clamped plate, the problem consists in finding the couple $(\underline{\theta}(t), w(t))$ that minimizes the following functional

$$\Pi_t(\underline{\theta}(t), w(t)) = \frac{1}{2} a(\underline{\theta}(t), \underline{\theta}(t)) + \frac{\lambda t^{-2}}{2} ||\underline{\theta}(t) - \underline{\nabla}\, w(t)||^2_{0,\Omega} - \int_\Omega f w(t)\, dx\, dy \qquad (2.1)$$

over the space $V = \Theta \times W = \left(H^1_0(\Omega)\right)^2 \times H^1_0(\Omega)$, where

(i) $\underline{\theta} = (\theta_1, \theta_2)$ and w are the fiber rotations and the transverse displacements, respectively;

(ii) $a(\cdot,\cdot) : \Theta \times \Theta \to \mathbf{R}$ is a bilinear form defined by

$$a(\underline{\theta}, \underline{\eta}) = \frac{E}{12(1-\nu^2)} \int_\Omega \left\{ \left(\frac{\partial \theta_1}{\partial x} + \nu \frac{\partial \theta_2}{\partial y} \right) \frac{\partial \eta_1}{\partial x} + \left(\nu \frac{\partial \theta_1}{\partial x} + \frac{\partial \theta_2}{\partial y} \right) \frac{\partial \eta_2}{\partial y} + \right.$$
$$\left. + \frac{1-\nu}{2} \left(\frac{\partial \theta_1}{\partial y} + \frac{\partial \theta_2}{\partial x} \right) \left(\frac{\partial \eta_1}{\partial y} + \frac{\partial \eta_2}{\partial x} \right) \right\} dx\, dy$$

where E is the Young's modulus and ν is the Poisson's ratio $(0 < \nu < 1/2)$;

(iii) $\lambda = Ek/2(1+\nu)$ with k shear correction factor (usually taken as 5/6);

(iv) ft^3 is the transverse load force density per unit area.

By Korn's inequality, $a(\cdot,\cdot)$ is a coercive form over Θ and this implies that there exists a unique solution $(\underline{\theta}(t), w(t))$ in V of the

Problem P_t:. *For $t > 0$ fixed, find $(\underline{\theta}(t), w(t))$ in V such that*

$$a(\underline{\theta}(t), \underline{\eta}) + \lambda t^{-2} \left(\underline{\theta}(t) - \underline{\nabla}\, w(t), \underline{\eta} - \underline{\nabla}\, v\right) = \int_\Omega f v\, dx\, dy \qquad \forall (\underline{\eta}, v) \in V \qquad (2.2)$$

The idea of discretizing by means of finite elements based on formulation (2.2) is typically a very naive one, because of the well known shear locking phenomenon (cf. [11]).

A common way to escape this undesirable lack of convergence is to discretize a mixed formulation derived by introducing the scaled shear stress

$$\underline{\xi} = \lambda t^{-2} (\underline{\theta} - \underline{\nabla}\, w)$$

as independent unknown. This leads to consider the saddle point problem for the functional

$$\tilde{\Pi}_t(\underline{\theta}, w, \underline{\xi}) = \frac{1}{2}a(\underline{\theta}, \underline{\theta}) - \frac{\lambda^{-1}t^2}{2}||\underline{\xi}||^2_{0,\Omega} + (\underline{\xi}, \underline{\theta} - \underline{\nabla}\, w) - (f, w) \tag{2.3}$$

on $V \times (L^2(\Omega))^2$. Hence the variational plate problem now reads as follows: PROBLEM $\tilde{P}_t$:. *For $t > 0$ fixed, find $(\underline{\theta}(t), w(t), \underline{\xi}(t))$ in $V \times (L^2(\Omega))^2$ such that*

$$\begin{cases} a(\underline{\theta}(t), \underline{\eta}) + (\underline{\xi}, \underline{\eta} - \underline{\nabla}\, v) = (f, v) & \forall(\underline{\eta}, v) \in V \\ (\underline{\tau}, \underline{\theta}(t) - \underline{\nabla}\, w(t)) - \lambda^{-1}t^2(\underline{\xi}, \underline{\tau}) = 0 & \forall \underline{\tau} \in (L^2(\Omega))^2 \end{cases} \tag{2.4}$$

In agreement with the notations of [7], let us now introduce the differential operators

$$\underline{\mathrm{rot}}\ :\ \varphi \longrightarrow \underline{\mathrm{rot}}\, \varphi = \left\{ \frac{\partial \varphi}{\partial y}\, ,\ -\frac{\partial \varphi}{\partial x} \right\}$$

$$\mathrm{rot}\ :\ \underline{\chi} \longrightarrow \mathrm{rot}\, \underline{\chi} = -\frac{\partial \chi_1}{\partial y} + \frac{\partial \chi_2}{\partial x}.$$

In what follows, we also need the Hilbert space $\Gamma = H_0(\mathrm{rot}; \Omega)$ defined by

$$H_0(\mathrm{rot}; \Omega) = \left\{ \underline{\chi}\ :\ \underline{\chi} \in (L^2(\Omega))^2\, , \quad \mathrm{rot}\, \underline{\chi} \in L^2(\Omega)\, , \quad \underline{\chi}\ \cdot\ \underline{t} = 0 \quad \text{on} \quad \partial\Omega \right\}$$

$$||\underline{\chi}||^2_{H_0(\mathrm{rot};\Omega)}\ :=\ ||\underline{\chi}||^2_{0,\Omega} + ||\mathrm{rot}\, \underline{\chi}||^2_{0,\Omega}$$

(here $\underline{t}$ is the unit tangent to $\partial\Omega$) and its dual space

$$\Gamma' = H^{-1}(\mathrm{div}\, ; \Omega) = \left\{ \underline{\gamma}\ :\ \underline{\gamma} \in (H^{-1}(\Omega))^2 \quad \mathrm{div}\, \underline{\gamma} \in H^{-1}(\Omega) \right\}.$$

The space Γ' will be equipped with the norm

$$||\underline{\gamma}||^2_{\Gamma'}\ :=\ ||\underline{\gamma}||^2_{-1,\Omega} + ||\mathrm{div}\, \underline{\gamma}||^2_{-1,\Omega}$$

which is equivalent to the natural dual norm induced by $H_0(\mathrm{rot}\, ; \Omega)$.

We are now ready to claim that the following proposition holds (for a proof see [7], for instance).

PROPOSITION 2.1. *Given $t > 0$, there exists a unique triple $(\underline{\theta}(t), w(t), \underline{\xi}(t))$ in $V \times (L^2(\Omega))^2$ satisfying equations (2.4). Furthermore*

$$\underline{\theta}(t) \rightharpoonup \underline{\theta}_0 \quad \text{in} \quad \Theta$$

$$w(t) \rightharpoonup w_0 \quad \text{in} \quad \mathrm{W}$$

$$\underline{\xi}(t) \rightharpoonup \underline{\xi}_0 \quad \text{in} \quad \Gamma'$$

and $(\underline{\theta}_0, w_0, \underline{\xi}_0)$ in $V \times \Gamma'$ solves

$$\begin{cases} a(\underline{\theta}_0, \underline{\eta}) - < \underline{\xi}_0, \underline{\eta} - \underline{\nabla}\, v >_{\Gamma' \times \Gamma} = (f, v) & \forall(\underline{\eta}, v) \in V \\ < \underline{\tau}, \underline{\theta}_0 - \underline{\nabla}\, w_0 >_{\Gamma' \times \Gamma} = 0 & \forall \underline{\tau} \in \Gamma' \qquad \square \end{cases} \tag{2.5}$$

Therefore, from the general theory of mixed methods, a choice $V_h \times \Gamma_h \subset V \times \Gamma'$ of finite element spaces will be stable if it satisfies (cf. [7]):

(S.1) $\forall (\underline{\eta}_h, v_h) \in V_h^* = \left\{ (\underline{\eta}_h, v_h) \in V_h : (\underline{\eta}_h - \underline{\nabla}\, v_h, \underline{\tau}_h) = 0 \quad \forall \underline{\tau}_h \in \Gamma_h \right\}$:

$$a(\underline{\eta}_h, \underline{\eta}_h) \geq \alpha_0 \left(||\underline{\eta}_h||^2_{1,\Omega} + ||v_h||^2_{1,\Omega} \right)$$

(S.2) $\forall\, \underline{\tau}_h \in \Gamma_h$:

$$\sup_{(\underline{\eta}_h, v_h) \in V_h} \frac{(\underline{\eta}_h - \underline{\nabla}\, v_h, \underline{\tau}_h)}{||(\underline{\eta}_h, v_h)||_{1,\Omega}} \geq \beta_0 ||\underline{\tau}_h||_{\Gamma'}$$

with α_0 and β_0 independent of h.

Remark. Even though we only require the inclusion $\Gamma_h \subset \Gamma'$, it's obvious that every natural choice for Γ_h will always satisfy $\Gamma_h \subset (L^2(\Omega))^2$.

Note that finding finite element spaces that fulfill both condition (S.1) and (S.2) is not so trivial since $a(\cdot,\cdot)$ considered as a bilinear form on V is coercive only on $V^* = \left\{ (\underline{\eta}, v) \in V : \underline{\eta} = \underline{\nabla}\, v \right\}$ and not on the whole V. Hence, if we try to enrich V_h in order to get (S.2), then (S.1) may fail; on the other hand if we let V_h be small with respect to Γ_h , condition (S.1) will become easier to verify but (S.2) may be false.

Recently, Arnold and Brezzi (cf. [2]) have proposed a modification to the mixed formulation (2.4) by taking

$$\underline{\gamma} = \lambda (t^{-2} - \alpha)(\underline{\theta} - \underline{\nabla}\, w)$$

as Lagrange multiplier and setting

$$\tilde{a}(\underline{\theta}, w; \underline{\eta}, v) = a(\underline{\theta}, \underline{\eta}) + \lambda\alpha(\underline{\theta} - \underline{\nabla}\, w, \underline{\eta} - \underline{\nabla}\, v)$$

where α is a parameter such that $0 < \alpha < t^{-2}$ (cf. Appendix). Therefore the weak formulation for the plate problem turns out to be

$$\begin{cases} \tilde{a}(\underline{\theta}, w; \underline{\eta}, v) + (\underline{\gamma}, \underline{\eta} - \underline{\nabla}\, v) = (f, v) & \forall (\underline{\eta}, v) \in V \\ (\underline{\mathrm{s}}, \underline{\theta} - \underline{\nabla}\, w) - \dfrac{t^2}{\lambda(1 - \alpha t^2)} (\underline{\gamma}, \underline{\mathrm{s}}) = 0 & \forall \underline{\mathrm{s}} \in (L^2(\Omega))^2 \end{cases} \tag{2.6}$$

By Poincaré's inequality and Korn's inequality it's easy to show that $\tilde{a}(\cdot,\cdot)$ is indeed bounded and coercive on the whole V. This means that in choosing finite element spaces for the discretization of (2.6) we do not have to take care of condition (S.1) anymore: any choice satisfying (S.2) will be stable and the following proposition is true (cf. [7]).

Proposition 2.2. *Let us choose $\Theta_h \subset \Theta$, $W_h \subset W$, $\Gamma_h \subset \Gamma'$ and consider the discretized problem of (2.6): Find $(\underline{\theta}_h, w_h, \underline{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h = V_h \times \Gamma_h$, solution of equations*

$$\begin{cases} \tilde{a}(\underline{\theta}_h, w_h; \underline{\eta}_h, v_h) + (\underline{\gamma}_h, \underline{\eta}_h - \underline{\nabla}\, v_h) = (f, v_h) & \forall (\underline{\eta}_h, v_h) \in V_h \\ (\underline{\mathrm{s}}_h, \underline{\theta}_h - \underline{\nabla}\, w_h) - \dfrac{t^2}{\lambda(1 - \alpha t^2)} (\underline{\gamma}_h, \underline{\mathrm{s}}_h) = 0 & \forall \underline{\mathrm{s}}_h \in \Gamma_h \end{cases} \tag{2.7}$$

Problem (2.7) has a unique solution; moreover, if condition (S.2) is satisfied and $(\underline{\theta}, w, \underline{\gamma})$ is the solution of (2.6), then there exists a constant C independent of t and h such that

$$||\underline{\theta} - \underline{\theta}_h||_{1,\Omega} + ||w - w_h||_{1,\Omega} + |||\underline{\gamma} - \underline{\gamma}_h|||_t \leq$$
$$\leq \mathrm{C}\big(\inf_{\underline{\eta}_h \in \Theta_h} ||\underline{\eta} - \underline{\eta}_h||_{1,\Omega} + \inf_{v_h \in W_h} ||w - v_h||_{1,\Omega} + \inf_{\underline{s}_h \in \Gamma_h} |||\underline{\gamma} - \underline{s}_h|||_t \big) \qquad (2.8)$$

where

$$|||\underline{\gamma}|||_t := ||\underline{\gamma}||_{-1,\Omega} + ||\mathrm{div}\, \underline{\gamma}||_{-1,\Omega} + t||\underline{\gamma}||_{0,\Omega}$$

3. A stability result. Let us start by recalling some standard definitions and notations useful to the subsequent analysis. First of all, we say that a family $\{\mathcal{T}_h\}_{h>0}$ of triangulations of Ω is *regular* if there exists a constant $\sigma > 0$ such that

$$h_T \leq \sigma \rho_T \qquad \forall\, T \in \bigcup_{h>0} \mathcal{T}_h$$

where h_T is the diameter of triangle T and ρ_T is the maximum diameter of the circles contained in T. Furthermore, a *macroelement* is the union of a fixed number of neighboring triangles along a well-defined pattern. A macroelement $M = \cup_{i=1}^m T_i$ is said to be equivalent to a reference macroelement $\widehat{M} = \cup_{i=1}^m \widehat{T}_i$ if there is a mapping $F_M : \widehat{M} \longrightarrow M$ for which the following conditions are fulfilled (cf. [13]):

(1) F_M is a continuous bijection.

(2) $T_i = F_M(\widehat{T}_i) \qquad \forall\, i \quad 1 \leq i \leq m$

(3) $F_{M|\widehat{T}_i} = F_{T_i} \circ F^{-1}_{\widehat{T}_i}$

where F_{T_i} and $F_{\widehat{T}_i}$ are the affine functions from the standard reference triangle onto T_i and $\widehat{T}_i$, respectively.

From a given triangulation $\mathcal{T}_h$ of Ω it's always possible to derive (obviously not in a unique fashion) a "macroelement mesh" $\mathcal{M}_h$ in such a way that each $T \in \mathcal{T}_h$ is covered by some macroelement M in $\mathcal{M}_h$ and each macroelement M is equivalent to a certain reference macroelement $\widehat{M}$.

Let us now introduce finite element spaces $V_h \times \Gamma_h \subset V \times \Gamma'$ for the discretization of (2.6). For every macroelement M in $\mathcal{M}_h$ we will denote with $V_{0,M}$ the space

$$V_{0,M} = \big\{ (\underline{\eta}_h, v_h) \in V_h : \quad (\underline{\eta}_h, v_h) = (\underline{0}, 0) \quad \mathrm{in}\, \Omega \setminus M \big\}.$$

We further define the space Γ_M as

$$\Gamma_M = \big\{ \underline{s}_h \in \Gamma_h : \quad \underline{s}_h = 0 \quad \mathrm{in}\, \Omega \setminus M \big\}.$$

Finally, let us set

$$N_M = \big\{ \underline{s}_M \in \Gamma_M : \quad (\underline{s}_M, \underline{\eta}_M - \underline{\nabla}\, v_M) = 0 \qquad \forall (\underline{\eta}_M, v_M) \in V_{0,M} \big\}.$$

In what follows, we will always assume that the family $\{\mathcal{T}_h\}_{h>0}$ *arising from finite element discretization will be regular; moreover, there will be only a fixed finite number of reference macroelements* $\{\widehat{M}_1, ..., \widehat{M}_r\}$ *to which each macroelement* $M \in \cup_{h>0} \mathcal{M}_h$ *is equivalent.*

We are now in the position to state and prove our main result of this section.

THEOREM 3.1. *Given* $\Theta_h \subset \Theta, \quad W_h \subset W, \quad \Gamma_h \subset \Gamma'$ *subspaces for discretization of problem (2.6), suppose that the following conditions are met:*

(C1) *There exists a linear operator $\pi_h : W \longrightarrow W_h$ such that*

(C1.1) $||\pi_h v||_{1,\Omega} \leq c||v||_{1,\Omega}$, c independent of h

(C1.2) $\int_\Omega \underline{\nabla}(\pi_h v - v)\cdot \underline{\mathrm{s}}_h = 0 \qquad \forall \underline{\mathrm{s}}_h \in \Gamma_h$

(C2) *From $\mathcal{T}_h$ it is possible to derive $\mathcal{M}_h$ in such a way that*

(C2.1) $\Gamma_h = \bigoplus_{M\in\mathcal{M}_h} \Gamma_M$ (direct sum in L^2)

(C2.2) each N_M is the trivial space.

Then condition (S.2) of the previous section is fulfilled.

Proof. Let $(\underline{\eta}, v) \in V$ be given. Fix an arbitrary macroelement $M \in \mathcal{M}_h$; let us denote with i_M the index $1 \leq i_M \leq r$ such that M is equivalent to $\widehat{M}_{i_M}$ and with $h_M = \max_i h_{T_i}$ if $M = \cup_i^m T_i$. Consider the problem to find $(\underline{\eta}_M^B, v_M^B) \in V_{0,M}$ solution of

$$\int_M (\underline{\eta}_M^B - \underline{\nabla}\, v_M^B)\cdot \underline{\mathrm{s}}_M = \int_M (\underline{\eta} - \Pi_1 \underline{\eta})\cdot \underline{\mathrm{s}}_M \qquad \forall \underline{\mathrm{s}}_M \in \Gamma_M\, , \tag{3.1}$$

where $\Pi_1\underline{\eta}$ is the Clément interpolated of $\underline{\eta}$.

By (C2.2) it follows that system (3.1) is solvable. Let us take the solution of minimum norm. An easy scaling argument and the features of the Clément interpolating operator (cf. [7]) show that there exists $c(\widehat{M}_{i_M}) > 0$ such that

$$|\,\underline{\eta}_M^B\,|_{1,M}^2 + h_M^{-2}|\,v_M^B\,|_{1,M}^2 \leq c(\widehat{M}_{i_M})||\,\underline{\eta}\,||_{1,M}^2 \tag{3.2}$$

Since h_M is obviously bounded by mes(Ω), inequality (3.2) implies that

$$\exists\, c_1(\widehat{M}_{i_M}) > 0 \;: \quad |\,\underline{\eta}_M^B\,|_{1,M}^2 + |\,v_M^B\,|_{1,M}^2 \leq c_1(\widehat{M}_{i_M})\,||\,\underline{\eta}\,||_{1,M}^2. \tag{3.3}$$

Now let us set

$$\begin{cases} \underline{\eta}^I = \Pi_1\underline{\eta} + \sum_M \underline{\eta}_M^B \\ v^I = \pi_h v + \sum_M v_M^B \end{cases} \tag{3.4}$$

Thanks to (C2.1), every $\underline{\mathrm{s}}_h \in \Gamma_h$ can be written in a unique fashion as $\underline{\mathrm{s}}_h = \sum_M \underline{\mathrm{s}}_{h,M}$ where $\underline{\mathrm{s}}_{h,M} \in \Gamma_M$; hence we have

$$\int_\Omega (\underline{\eta}^I - \underline{\nabla}\, v^I) \cdot \underline{\mathrm{s}}_h = \text{ (by (3.4))}$$

$$= \sum_M \int_M \Big[\Pi_1 \underline{\eta} + \underline{\eta}^B_M - \underline{\nabla}\,(v^B_M + \pi_h v) \Big] \cdot \underline{\mathrm{s}}_{h,M} =$$

$$= \sum_M \Big[\int_M (\Pi_1 \underline{\eta} + \underline{\eta}^B_M - \underline{\nabla}\, v^B_M) \cdot \underline{\mathrm{s}}_{h,M} - \int_M \underline{\nabla}\, \pi_h v \cdot \underline{\mathrm{s}}_{h,M} \Big] =$$

$$= \text{(by (C1.2) and (3.1))} = \sum_M \Big(\int_M \underline{\eta} \cdot \underline{\mathrm{s}}_{h,M} - \int_M \underline{\nabla}\, v \cdot \underline{\mathrm{s}}_{h,M} \Big) =$$

$$= \text{(again (C2.1))} = \int_\Omega (\underline{\eta} - \underline{\nabla}\, v) \cdot \underline{\mathrm{s}}_h$$

Hence for every $(\underline{\eta}, v) \in V$ we have found $\Pi_h(\underline{\eta}, v) = (\underline{\eta}^I, v^I) \in V_h$ such that

$$\int_\Omega (\underline{\eta}^I - \underline{\nabla}\, v^I) \cdot \underline{\mathrm{s}}_h = \int_\Omega (\underline{\eta} - \underline{\nabla}\, v) \cdot \underline{\mathrm{s}}_h \qquad \forall \underline{\mathrm{s}}_h \in \Gamma_h \tag{3.5}$$

Let us now estimate $|\underline{\eta}^I|^2_{1,\Omega} + |v^I|^2_{1,\Omega}$:

$$|\underline{\eta}^I|^2_{1,\Omega} + |v^I|^2_{1,\Omega} = \Big| \Pi_1 \underline{\eta} + \sum_M \underline{\eta}^B_M \Big|^2_{1,\Omega} + \Big| \pi_h v + \sum_M v^B_M \Big|^2_{1,\Omega} \leq$$

$$\leq 2 \Big(|\Pi_1 \underline{\eta}|^2_{1,\Omega} + |\pi_h v|^2_{1,\Omega} + \sum_M \big(|\, \underline{\eta}^B_M \,|^2_{1,M} + |\, v^B_M \,|^2_{1,M} \big) \Big) \leq$$

$$\leq \text{(by 3.3 and (C1.1))} \leq 2 \Big(c \left(||v||^2_{1,\Omega} + ||\underline{\eta}||^2_{1,\Omega} \right) + \sum_M c_1(\widehat{M}_{i_M}) ||\underline{\eta}||^2_{1,M} \Big).$$

Note that, above, we have also used the continuity of the operator Π_1. Since there's only a finite number of reference macroelements, we obtain

$$|\underline{\eta}^I|^2_{1,\Omega} + |v^I|^2_{1,\Omega} \leq C_1 \big(||v||^2_{1,\Omega} + ||\underline{\eta}||^2_{1,\Omega} \big) \tag{3.6}$$

with $C_1 = 2 \max \{c,\ c_1(\widehat{M}_{i_M}), ...,\ c_1(\widehat{M}_{i_M})\}$. Therefore, by Poincaré's inequality, we finally have

$$||\Pi_h(\underline{\eta}, v)||_{1,\Omega} \leq C ||(\underline{\eta}, v)||_{1,\Omega} \tag{3.7}$$

with C independent of h.

It's well-known (cf. [7], for instance) that (3.5) together with (3.7) implies condition (S.2) of the previous section and the proof is complete. □

Remark. Note that in Theorem 3.1 we have done nothing but employing Fortin's criterion (cf. [7]). So far every interpolation operator built in order to meet this criterion split into $\Pi_h = (\pi_1, \pi_2)$ where $\pi_1 : \Theta \longrightarrow \Theta_h$ and $\pi_2 : W \longrightarrow W_h$; hence the operator acted componentwise. Here we have presented a construction of Π_h for which this splitting is not necessary (cf. equation (3.1)). There are several cases where the difficulty consists in building π_1 : in such a situation our result may be of great help (cf. Section 4).

4. Analysis of the class $(P_{k+2} - P_{k+2} - P_k)$**.** In this section we will apply our results to show convergence and stability of a new class of mixed finite element methods based on formulation (2.6).

For every $k \geq 0$ let us introduce the following spaces of discretization:

$$
\begin{aligned}
&\Theta_h = \{ \underline{\eta}_h \in \Theta \, : \, \underline{\eta}_{h|T} \in (P_{k+2}(T))^2 \quad \forall T \in \mathcal{T}_h \} \\
&W_h = \{ v_h \in W \, : \, v_{h|T} \in P_{k+2}(T) \quad \forall T \in \mathcal{T}_h \} \\
&\Gamma_h = \{ \underline{s}_h \in (L^2(\Omega))^2 \, : \, \underline{s}_{h|T} \in (P_k(T))^2 \quad \forall T \in \mathcal{T}_h \}
\end{aligned}
\tag{4.1}
$$

The goal of the next proposition is to verify, under an extremely weak hypothesis on $\mathcal{T}_h$, that conditions of Theorem 3.1 are fulfilled.

PROPOSITION 4.1. *Let us choose the approximation spaces as in (4.1) and let us suppose that every $M \in \cup_{h>0} \mathcal{M}_h$ is the union of at least three triangles. Then conditions of Theorem (3.1) are met.*

Proof. It's easily seen that we can build $\pi_h : W \longrightarrow W_h$ such that (C1.1) and (C1.2) are true. Indeed, set first $\pi_h^1 : W \longrightarrow W_h$ as the interpolating operator of Clément; we have (cf. [7])

$$
\sum_{T \in \mathcal{T}_h} h_T^{2s-2} | \, v - \pi_h^1 v \, |_{s,T}^2 \leq c \, || \, v \, ||_{1,\Omega}^2 \qquad s = 0 \, , \, 1 \tag{4.2}
$$

Secondly, for every $v \in W$ take $\pi_h^2 v \in W_h$ satisfying:

$$
\begin{cases}
\pi_h^2 v(a) = 0 & \forall \, a \text{ vertex of } T \in \mathcal{T}_h \\
\displaystyle\int_e (\pi_h^2 v - v) p_k = 0 & \forall p_k \in P_k(e) \; \forall \, e \text{ edge of } T \in \mathcal{T}_h \\
\displaystyle\int_T (\pi_h^2 v - v) q_{k-1} = 0 & \forall q_{k-1} \in P_{k-1}(T) \text{ for } k > 0
\end{cases}
\tag{4.3}
$$

Standard arguments show that

$$
|| \, \pi_h^2 v \, ||_{1,T} \leq c \big(\, h_T^{-1} | \, v \, |_{0,T} + | \, v \, |_{1,T} \, \big) \tag{4.4}
$$

If we now set

$$
\pi_h v = \pi_h^1 v + \pi_h^2 \big(\, v - \pi_h^1 v \big) \tag{4.5}
$$

it's easy to see from (4.2), (4.3) and (4.4) that $\pi_h : W \longrightarrow W_h$ is a linear bounded operator for which (C1.1) and (C1.2) are fulfilled.

As far as condition (C2.1) is concerned, note that $\underline{s}_h \in \Gamma_h$ is fully discontinuous along adjacent triangles, thus (C2.1) is trivially satisfied.

Therefore it remains to verify (C2.2).

Fix a macroelement $M = \cup_{i=1}^m T_i$ and take $\underline{s}_M \in N_M$. By choosing $(\underline{0}, v_M) \in V_{0,M}$, integrating by parts yields

$$0 = \int_M \underline{\nabla}\, v_M \cdot \underline{s}_M = \sum_{i=1}^m \Big\{ -\int_{T_i} v_M \mathrm{div}\, \underline{s}_M + \int_{\partial T_i} v_M\, \underline{s}_M \cdot \underline{n}_{T_i} \Big\} \tag{4.6}$$

Since equation (4.6) is true for every $(\underline{0}, v_M) \in V_{0,M}$ vanishing on the boundary of macroelement M, it follows that $\underline{s}_M \in N_M$ has to satisfy

$$\begin{cases} \mathrm{div}\, \underline{s}_M = 0 \qquad \text{in each } T_i \in M \\ \\ \underline{s}_M \cdot \underline{n} \text{ is continuous along the internal interfaces of } T_i \in M \end{cases} \tag{4.7}$$

Hence there exists a unique φ_M such that

$$\begin{cases} \varphi_M \in \mathcal{L}^1_{k+1}(M)/\mathbf{R} \\ \underline{s}_M = \underline{\mathrm{rot}}\, \varphi_M \end{cases} \tag{4.8}$$

where $\mathcal{L}^1_{k+1}(M)$ is the usual space of continuous functions which are polynomials of degree *k+1* on each T_i of M.

Let us now take $(\underline{\eta}_M, 0) \in V_{0,M}$. Since $\underline{s}_M \in N_M$, we have

$$\int_M \underline{\eta}_M \cdot \underline{s}_M = 0 \qquad \forall (\underline{\eta}_M, 0) \in V_{0,M}. \tag{4.9}$$

By (4.8) we obtain

$$\int_M \underline{\eta}_M \cdot \underline{\mathrm{rot}}\, \varphi_M = 0 \qquad \forall (\underline{\eta}_M, 0) \in V_{0,M} \tag{4.10}$$

from which we wish to deduce $\underline{\mathrm{rot}}\, \varphi_M = 0$.
To perform this step, consider on M the Stokes–like problem

given $\underline{\chi} \in (L^2(M))^2$, *find* $(\underline{u}_M, p_M) \in (\widetilde{\mathcal{L}}^1_{k+2}(M))^2 \times \mathcal{L}^1_{k+1}(M)/\mathbf{R}$ *such that*

$$\begin{cases} \displaystyle\int_M \underline{\underline{\nabla}}\, \underline{u}_M : \underline{\underline{\nabla}}\, \underline{v}_M - \int_M p_M \,\mathrm{rot}\, \underline{v}_M = \int_M \underline{\chi} \cdot \underline{v}_M & \forall \underline{v}_M \in (\widetilde{\mathcal{L}}^1_{k+2}(M))^2 \\ \displaystyle\int_M \underline{u}_M \cdot \underline{\mathrm{rot}}\, q_M = 0 & \forall q_M \in \mathcal{L}^1_{k+1}(M) \end{cases} \tag{4.11}$$

where $\widetilde{\mathcal{L}}^1_{k+2}(M) = \mathcal{L}^1_{k+2}(M) \cap H^1_0(M)$.

D. Boffi (cf. [5]) has shown that discretization of the Stokes problem by means of a standard P_{k+2} element for velocity and a standard continuous P_{k+1} element for pressure is stable for every $k \geq 0$, whenever in the mesh there are at least three elements. It's obvious that this is also true for problem (4.11). In particular, stability implies that there are not spurious pressure modes, i.e.

$$\Big\{ \int_M \underline{\mathrm{rot}}\, q_M \cdot \underline{v}_M = 0 \quad \forall \underline{v}_M \in (\widetilde{\mathcal{L}}^1_{k+2}(M))^2 \Big\} \Longrightarrow \underline{\mathrm{rot}}\, q_M = 0 \tag{4.12}$$

Compared to (4.10), implication (4.12) is exactly what we are looking for. Hence $\underline{s}_M \in N_M \Rightarrow \underline{s}_M = 0$ and the proof is now complete.
□

Remark. Note that it's always possible to derive, from a given regular family $\{\mathcal{T}_h\}_{h>0}$, a macroelement family $\{\mathcal{M}_h\}_{h>0}$ which fulfills the assumption of Proposition 4.1 and for which there are only a finite number of reference macroelements, provided in each $\mathcal{T}_h$ there are at least three triangles.

Recalling Proposition 2.2 and standard interpolation theory, it's straightforward to obtain the following error bound

PROPOSITION 4.2. *Let $(\underline{\theta}, w, \underline{\gamma})$ be the solution of problem (2.6) and $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$ be the solution of the discretized problem by means of the spaces (4.1), then there exists a constant C independent of h and t such that*

$$||\underline{\theta} - \underline{\theta}_h||_{1,\Omega} + ||w - w_h||_{1,\Omega} + |||\underline{\gamma} - \underline{\gamma}_h|||_t \leq \leq Ch^{k+1}\Big(||\underline{\theta}||_{k+2,\Omega} + ||w||_{k+2,\Omega} + ||\underline{\gamma}||_{k+1,\Omega}\Big) \tag{4.13}$$

5. Appendix. A brief discussion about implementation of the methods.

Let us start by recalling the modified mixed formulation (2.6):

given $f \in L^2(\Omega)$ and $t > 0$, find $(\underline{\theta}, w, \underline{\gamma}) \in V \times (L^2(\Omega))^2$ such that

$$\begin{cases} \tilde{a}(\underline{\theta}, w; \underline{\eta}, v) + (\underline{\gamma}, \underline{\eta} - \underline{\nabla}\, v) = (f, v) & \forall (\underline{\eta}, v) \in V \\ (\underline{s}, \underline{\theta} - \underline{\nabla}\, w) - \dfrac{t^2}{\lambda(1-\alpha t^2)}(\underline{\gamma}, \underline{s}) = 0 & \forall \underline{s} \in (L^2(\Omega))^2 \end{cases} \tag{A.1}$$

and its discrete counterpart

find $(\underline{\theta}_h, w_h, \underline{\gamma}_h) \in V_h \times \Gamma_h$ such that

$$\begin{cases} \tilde{a}(\underline{\theta}_h, w_h; \underline{\eta}_h, v_h) + (\underline{\gamma}_h, \underline{\eta}_h - \underline{\nabla}\, v_h) = (f, v_h) & \forall (\underline{\eta}_h, v_h) \in V_h \\ (\underline{s}_h, \underline{\theta}_h - \underline{\nabla}\, w_h) - \dfrac{t^2}{\lambda(1-\alpha t^2)}(\underline{\gamma}_h, \underline{s}_h) = 0 & \forall \underline{s}_h \in \Gamma_h \end{cases} \tag{A.2}$$

We have already noticed that α, which has the units of reciprocal of the square of length, is a parameter with $0 < \alpha < t^{-2}$. If we now choose the spaces Θ_h , W_h , Γ_h as in (4.1), from the second equation of (A.2) we deduce that

$$\underline{\gamma}_h = \lambda(t^{-2} - \alpha)P_h\,(\underline{\theta}_h - \underline{\nabla}\, w_h) \tag{A.3}$$

where P_h denotes the L^2–orthogonal projection operator onto Γ_h.

The variable $\underline{\gamma}_h$ can be eliminated from equations (A.2) and it follows that $(\underline{\theta}_h, w_h) \in V_h$ is the minimizer over V_h of the functional

$$E_h\,(\underline{\eta}, v) = \frac{1}{2}a(\underline{\eta}, \underline{\eta}) + \frac{\lambda\alpha}{2}||\,\underline{\eta} - \underline{\nabla}\, v\,||^2_{0,\Omega} + \frac{\lambda}{2}(t^{-2} - \alpha)||\,P_h\,(\underline{\eta} - \underline{\nabla}\, v)\,||^2_{0,\Omega} - (f, v) \tag{A.4}$$

From a computational point of view is much more convenient to implement the Euler–Lagrange equations arising from functional (A.4) instead of solving equations (A.2).

Let us now turn our attention on the parameter α. Looking at (A.4), we find that choosing a very small α means essentially that we have decided to treat the shear energy term by a reduced integration procedure. On the other hand, a choice of α close to t^{-2} means that the shear energy term is exactly integrated.

Therefore, we expect that a very small α may lead to spurious modes, while the discretization may exhibit a severe locking whenever α is too large. Hence a suitable choice of the parameter has to be done in order to achieve acceptable convergence results. Our guess is that the parameter α may be chosen so that the first two terms in the right hand side of (A.4) are well balanced.

Numerical experiments for the case $P_2 - P_2 - P_0$, i.e. the lowest order scheme of the family described in section 4, show that such a behaviour happens, indeed. Just to give an idea, we report here some numerical results concerning the discretization of a clamped square plate of unit length side and of thickness $t = 10^{-3}$. We used a uniform decomposition into 64 equal subsquares each of one divided into two triangles by a diagonal, thus leading to a triangular mesh of 128 elements. Due to symmetry, only one quadrant has been discretized in actual computations. Young's modulus and Poisson's ratio have been chosen as $E = 10.92 \cdot 10^6$ and $\nu = 0.3$, respectively. Finally, two different loads have been selected. The first one constists in choosing a uniform load $f = -1$. It turns out that exact solution of vertical displacement variable at the center of the plate is (cf. [14]):

$w_c = -12.6 \cdot 10^{-10}$.

We have obtained, for different values of α, the following results:

$$\begin{cases} \alpha = 10^5 & w_c = -47.3069 \cdot 10^{-12} \\ \alpha = 10^4 & w_c = -90.3253 \cdot 10^{-11} \\ \alpha = 10^3 & w_c = -11.6465 \cdot 10^{-11} \\ \alpha = 10^2 & w_c = -12.4892 \cdot 10^{-10} \\ \alpha = 10 & w_c = -12.5483 \cdot 10^{-10} \\ \alpha = 1 & w_c = -11.7878 \cdot 10^{-10} \\ \alpha = 10^{-5} & w_c = +85.9992 \cdot 10^{-8} \end{cases}$$

The second load is of purely academic interest: it has been designed in order to have the exact solution explicitly known in the whole square domain. This choice allows to compute the discrete L^2-norm relative error for both rotations and deflections (cf. [9]), as displayed below:

$$\begin{cases} \alpha = 10^5 & E_\theta = 0.693048 & E_w = 0.693494 \\ \alpha = 10^4 & E_\theta = 0.234855 & E_w = 0.220497 \\ \alpha = 10^3 & E_\theta = 0.062991 & E_w = 0.046621 \\ \alpha = 10^2 & E_\theta = 0.025429 & E_w = 0.093592 \\ \alpha = 10 & E_\theta = 0.037866 & E_w = 0.052546 \\ \alpha = 1 & E_\theta = 0.0421804 & E_w = 0.525834 \\ \alpha = 10^{-5} & E_\theta = 0.042767 & E_w = 52601.5 \end{cases}$$

Note that, within a reasonable range of values, the choice of α is not so crucial, showing that the method promises to be robust. For further numerical results about schemes based on the modified formulation used in the present work, we refer to [9]; in particular more detailed recipes for a good choice of the parameter α have been therein suggested.

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