# ANALYSIS OF STRAIN–PRESSURE FINITE ELEMENT METHODS FOR THE STOKES PROBLEM (†)

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**Abstract.** An analysis of some nonconforming approximations of the Stokes problem is presented. The approximations are based on a strain–pressure variational formulation. In particular, a convergence and stability result for a method recently proposed by Bathe and Pantuso is provided.

**Key words.** Stokes, enhanced strains, inf–sup condition.

## 1. Introduction

The purpose of this note is to provide an analysis of mixed finite element schemes for the stationary Stokes problem, using enhanced strain field as independent variable. It is well–known that finding good elements for the Stokes problem requires some care, basically because of the locking phenomenon and the possible appearance of spurious pressure modes (cf. [1], [6], for instance). In recent years, some optimal mixed methods, using the classical velocity–pressure variational formulation, have been presented (cf. [1], [6] and [9]) and mathematically studied (cf., for instance, [2]). Some of these are based on the enrichment of the velocity field by means of bubble functions in order to achieve stability. Bubble functions are, however, of high degree and they can therefore increase the computational cost in actual implementations. Instead, one can think of using different variational formulations of the problem. For instance, in [5] it was presented a formulation in which the strains, the augmented stresses and the displacements entered as independent unknowns. Again, in [4], a four–field formulation, in which also the pressure appeared, was studied and several finite element methods were shown to be stable and convergent. Even though in those papers the linear elasticity problem was considered, it is

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well–known that the same results can be applied for the stationary Stokes problem, by simply replacing the displacement field with the velocity field. In this paper we consider a variational formulation involving the strain field and the pressure field, as in [7]. For this formulation it is possible (cf. [7]) to select low order additional strain modes, capable to improve the stability features of the method. In a sense, these modes work as the classical bubble functions do. The only slight drawback is that choosing such low order modes leads to a non–conforming approximation scheme. As a consequence, one has to deal with consistency errors, which have to be controlled. We remark that the idea of enhancing the strain field was already employed in [8] and [9].

An outline of the paper is as follows. In Section 2 we present the Stokes problem in both the classical and the strain–pressure variational formulation. In Section 3 we discuss, in quite an abstract framework, the approximated problem involving an enhanced strain finite element space. Theorem 3.3 at the end of this Section is devoted to provide error estimates. In Section 4 we recall the method recently proposed by Bathe and Pantuso (cf. [7]) and we develop the error analysis in the case of rectangular elements only. The technique used for achieving a stability and a convergence result, consists of checking that the method falls into the class of those for which Theorem 3.3 can be applied. Combining this and standard interpolation theory, one comes to the desired error estimates of Theorem 4.3. In Section 5 we draw some concluding remarks. We have used, throughout the paper, the classical notation (cf. [2] and [3]). Furthermore, $c$ and $C$ will denote constants independent of $h$, which may differ in each occurence.

## 2. The Stokes problem: a strain–pressure variational formulation.

Let $\Omega$ be a regular open set in $\mathbf{R}^n$, with boundary $\Gamma$, $n \geq 2$. It is well–known that the Stokes problem, with homogeneous boundary conditions reads as follows:

$$\text{find } (\underline{u}, p) \text{ such that} \quad \begin{cases} -\nu\Delta\,\underline{u} + \nabla\, p = \underline{f} & \text{in } \Omega \\ \operatorname{div}\underline{u} = 0 & \text{in } \Omega \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\underline{u}$ is the velocity field, $p$ is the pressure field and $\underline{f}$ is the loading term. A standard variational formulation of problem (2.1) consists in finding $(\underline{u}, p) \in V \times Q = \big(\mathrm{H}^1_0(\Omega)\big)^n \times \mathrm{L}^2(\Omega)/\mathbf{R}$ which solves the system

$$\begin{cases} \nu \displaystyle\int_\Omega \nabla^S(\underline{u}) : \nabla^S(\underline{v})\, d\underline{x} - \int_\Omega p \,\operatorname{div} \underline{v} = \int_\Omega \underline{f} \cdot \underline{v} \qquad \forall \underline{v} \in V \\ \\ \displaystyle\int_\Omega q \,\operatorname{div} \underline{u} = 0 \qquad \forall q \in Q \ , \end{cases} \tag{2.2}$$

where $\nabla^S(\cdot)$ is the symmetric gradient operator acting on the velocity field. It is well–known that this problem is a well–posed one and it fits into the framework of the classical saddle–point theory, extensively studied in [2], for instance. In particular, the inf–sup condition

$$\inf_{q \in Q} \sup_{\underline{v} \in V} \frac{\int_\Omega q \,\operatorname{div} \underline{v}}{||\,\underline{v}\,||_V ||q||_Q} \geq \beta \tag{2.3}$$

holds, with $\beta > 0$ and for each $(\underline{v},\, q) \in V \times Q$. For the subsequent analysis of the finite element discretization it will be useful to introduce another variational formulation, involving this time the strain field $\boldsymbol{\varepsilon}$ and the pressure field $p$. In order to do so, let us first define

$$\Sigma = \nabla^S \left( \mathrm{H}^1_0(\Omega)^n \right) . \tag{2.4}$$

Hence $\Sigma$ is nothing but the space of symmetric gradients of functions in $\left( \mathrm{H}^1_0(\Omega)^n \right)$. Moreover, we will equip $\Sigma$ with the usual $\mathrm{L}^2$–norm. Secondly, let us introduce the linear operator

$$\begin{aligned} G &: \Sigma \rightarrow V \\ \boldsymbol{\tau} &\rightarrow \underline{v} \quad \text{with } \underline{v} \text{ such that } \nabla^S(\underline{v}) = \boldsymbol{\tau} \end{aligned} \tag{2.5}$$

Notice that $G$ is a well–defined bijective operator, which is also continuous due to Korn's inequality. Then, by the open mapping theorem, it follows that $G$ is indeed an isomorphism. This means that there exists a positive constant $\gamma$ such that for all $\boldsymbol{\tau} \in \Sigma$

$$\frac{1}{\gamma} ||\boldsymbol{\tau}||_\Sigma \leq ||G(\boldsymbol{\tau})||_V \leq \gamma ||\boldsymbol{\tau}||_\Sigma . \tag{2.6}$$

If we now set in equation (2.2)

$$\boldsymbol{\varepsilon} = \nabla^S \underline{u} \in \Sigma, \tag{2.7}$$

noting that the trace

$$\operatorname{tr}\boldsymbol{\varepsilon} = \operatorname{div} G(\boldsymbol{\varepsilon}) = \operatorname{div}\underline{u}, \tag{2.8}$$

we easily get to the following variational formulation of the Stokes problem: find $(\boldsymbol{\varepsilon}, p) \in \Sigma \times Q$ such that

$$\begin{cases} \nu \displaystyle\int_\Omega \boldsymbol{\varepsilon} : \boldsymbol{\tau}\, d\underline{x} - \int_\Omega p \operatorname{tr}\boldsymbol{\tau} = \int_\Omega \underline{f} \cdot G(\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \Sigma \\ \\ \displaystyle\int_\Omega q \operatorname{tr}\boldsymbol{\varepsilon} = 0 \qquad \forall q \in Q\ . \end{cases} \tag{2.9}$$

It is straightforward to check that there exist two positive constants $\alpha$, $\delta$ such that

$$\begin{aligned} &\nu \int_\Omega \boldsymbol{\tau} : \boldsymbol{\tau}\, d\underline{x} \geq \alpha ||\boldsymbol{\tau}||_\Sigma^2 && \forall \boldsymbol{\tau} \in \Sigma \\ &\inf_{q \in Q} \sup_{\boldsymbol{\tau} \in \Sigma} \frac{\int_\Omega q \operatorname{tr}\boldsymbol{\tau}}{||\boldsymbol{\tau}||_\Sigma ||q||_Q} \geq \delta && \forall (\boldsymbol{\tau},\, q) \in \Sigma \times Q. \end{aligned} \tag{2.10}$$

Note that the inf–sup condition in (2.10) follows directly from (2.3), (2.6) and (2.8). Hence, by the standard theory, problem (2.9) admits a unique solution $(\boldsymbol{\varepsilon},\, p) \in \Sigma \times Q$ and

$$||\boldsymbol{\varepsilon}||_\Sigma + ||p||_Q \leq C(\underline{f}). \tag{2.11}$$

## 3. Analysis of nonconforming approximations using enhanced strains.

We are now interested in the approximation of problem (2.9) of the previous Section by means of finite elements. Choosing a *conforming* approximation essentially means that one selects finite element spaces $\Sigma_h \subset \Sigma$, $Q_h \subset Q$ and searches for a pair $(\boldsymbol{\varepsilon}_h, p_h) \in \Sigma_h \times Q_h$ such that

$$\begin{cases} \nu \displaystyle\int_\Omega \boldsymbol{\varepsilon}_h : \boldsymbol{\tau}_h\, d\underline{x} - \int_\Omega p_h \operatorname{tr}\boldsymbol{\tau}_h = \int_\Omega \underline{f} \cdot G(\boldsymbol{\tau}_h) \qquad \forall \boldsymbol{\tau}_h \in \Sigma_h \\ \\ \displaystyle\int_\Omega q_h \operatorname{tr}\boldsymbol{\varepsilon}_h = 0 \qquad \forall q_h \in Q_h\ . \end{cases} \tag{3.1}$$

We wish to remark that the choice $\Sigma_h \subset \Sigma$ implies that every $\boldsymbol{\tau}_h \in \Sigma_h$ is actually the symmetric gradient of a piecewise polynomial continuous vector function.

It follows that any conforming discretization of problem (2.9) is definitely equivalent to some conforming approximation arising from the classical velocity–pressure variational formulation (2.2). We will therefore face the problem to study some kind of *nonconforming* approximation of the strain–pressure formulation. Hence, the first hypothesis we assume is that the space $\Sigma_h$ can be split into the direct sum

$$\Sigma_h = \Sigma_h^c \oplus \Sigma_h^e, \tag{3.2}$$

where $\Sigma_h^c \subset \Sigma$, but $\Sigma_h^e$ is made up by symmetric tensor functions which are *not* symmetric gradients of any vector function in $\left(\mathrm{H}_0^1(\Omega)\right)^n$. Therefore an arbitrary element $\boldsymbol{\tau}_h \in \Sigma_h$ can be written, in a unique fashion, as the sum

$$\boldsymbol{\tau}_h = \boldsymbol{\tau}_c + \boldsymbol{\tau}_e, \tag{3.3}$$

where $\boldsymbol{\tau}_c \in \Sigma_h^c$ and $\boldsymbol{\tau}_e \in \Sigma_h^e$. In a sense, $\Sigma_h^e$ can be considered as a sort of *strain enhancement space*. The hope now is that this enhancement space will be able to improve the convergence and stability features of the method in which only $\Sigma_h^c$ is involved. Furthermore, we will equip $\Sigma_h$ with the usual $\mathrm{L}^2$–norm, which will be denoted by $||\cdot||_\Sigma$, with a little abuse of notation.

A first glance to eqs. (3.1) shows that a difficulty arises. In fact, the linear operator $G$ is not defined on the whole $\Sigma_h$ if one follows the choice (3.2): $G$ is indeed well–defined (cf.(2.5)) only on the conforming space $\Sigma_h^c$.

A possible way out is to introduce a discrete operator $G_h$, defined by

$$G_h(\boldsymbol{\tau}_h) = G_h(\boldsymbol{\tau}_c + \boldsymbol{\tau}_e) = G(\boldsymbol{\tau}_c). \tag{3.4}$$

Hence, the discrete problem we are going to study consists in finding $(\boldsymbol{\varepsilon}_h, p_h) \in \Sigma_h \times Q_h$ such that

$$\begin{cases} a(\boldsymbol{\varepsilon}_h,\, \boldsymbol{\tau}_h) - (p_h,\, \mathrm{tr}\,\boldsymbol{\tau}_h) = (\underline{f},\, G(\boldsymbol{\tau}_c)) \qquad \forall \boldsymbol{\tau}_h \in \Sigma_h \\ \\ (q_h,\, \mathrm{tr}\,\boldsymbol{\varepsilon}_h) = 0 \qquad \forall q_h \in Q_h\ , \end{cases} \tag{3.5}$$

where

$$a(\boldsymbol{\varepsilon}_h,\, \boldsymbol{\tau}_h) = \nu \int_\Omega \boldsymbol{\varepsilon}_h : \boldsymbol{\tau}_h \, d\underline{x}\ . \tag{3.6}$$

*Remark.* Note that in the right–hand side of the first equation of (3.5), only the conforming part of the test function $\boldsymbol{\tau}_h$ enters. This means that at the discrete level the operator $G$ has been substituted by the operator $G_h$ (cf. (3.4)).

□

For problem (3.5) it is standard (cf. [2]) to obtain the following existence and uniqueness result.

**Proposition 3.1**: Assume that there exists a positive constant $\beta$ such that

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{\tau}_h \in \Sigma_h} \frac{(q_h, \operatorname{tr} \boldsymbol{\tau}_h)}{||q_h||_Q ||\boldsymbol{\tau}_h||_\Sigma} \geq \beta \tag{3.7}$$

Then problem (3.5) has a unique solution $(\boldsymbol{\varepsilon}_h, p_h) \in \Sigma_h \times Q_h$. Moreover, if $\beta$ is independent of $h$, then the approximation is stable.

□

*In the remainder of this Section, the inf–sup condition (3.7) with $\beta$ independent of $h$, will be supposed to hold.*

Before developing the error analysis, we wish to introduce the following notation for the kernel of the discrete trace operator:

$$K_h = \{\boldsymbol{\tau}_h \in \Sigma_h : \ (\operatorname{tr} \boldsymbol{\tau}_h, q_h) = 0 \qquad \forall q_h \in Q_h\} . \tag{3.8}$$

We are now ready to perform our error analysis. First, let us notice that, if $(\boldsymbol{\varepsilon}, p)$ is the solution of the continuous problem (2.9), then we have

$$a(\boldsymbol{\varepsilon}, \boldsymbol{\tau}_h) - (p, \operatorname{tr} \boldsymbol{\tau}_h) = (\underline{f}, G(\boldsymbol{\tau}_c)) + a(\boldsymbol{\varepsilon}, \boldsymbol{\tau}_e) - (p, \operatorname{tr} \boldsymbol{\tau}_e) , \tag{3.9}$$

for each $\boldsymbol{\tau}_h = (\boldsymbol{\tau}_c + \boldsymbol{\tau}_e) \in \Sigma_h$.

Take now $\boldsymbol{\tau}_h \in K_h$. By coerciveness we have

$$\begin{aligned} \alpha ||\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h||_\Sigma^2 &\leq a(\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h, \boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h) \\ &\leq a(\boldsymbol{\varepsilon}_h - \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h) + a(\boldsymbol{\varepsilon} - \boldsymbol{\tau}_h, \boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h) , \qquad (3.10) \\ &= T_1 + T_2 \end{aligned}$$

where $\boldsymbol{\varepsilon}_h = (\boldsymbol{\varepsilon}_c + \boldsymbol{\varepsilon}_e) \in \Sigma_h$ is the first component of the solution of the discretized problem (3.5).

*Remark.* Since the bilinear form $a(\cdot, \cdot)$ is coercive on the whole space $\mathrm{L}^2$, estimate (3.10) holds for every $\boldsymbol{\tau}_h \in \Sigma_h$, not only for $\boldsymbol{\tau}_h \in K_h$. However, to get an error estimate, we will need to take $\boldsymbol{\tau}_h \in K_h$ (cf. (3.13) and (3.14)).

□

Let us estimate the two terms $T_1$ and $T_2$. We begin with estimating $T_2$. By continuity ($M = \nu$) and little algebra we have

$$T_2 \leq \frac{M}{2\delta_2}||\boldsymbol{\varepsilon} - \boldsymbol{\tau}_h||^2_{\Sigma} + \frac{M\delta_2}{2}||\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h||^2_{\Sigma} \tag{3.11}$$

with $\delta_2$ positive to be chosen.

Let us come to $T_1$. By using (3.9) and the first equation of (3.5) we easily obtain

$$T_1 = (p_h - p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h)) - a(\boldsymbol{\varepsilon}, \,\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e) + (p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e)). \tag{3.12}$$

Since $(\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h) \in K_h$ (cf. second equation of (3.5)), we have

$$(p_h - p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h)) = (q_h - p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h)) \qquad \forall q_h \in Q_h. \tag{3.13}$$

It follows that, for each $\boldsymbol{\tau}_h \in K_h$ and $q_h \in Q_h$,

$$T_1 = (q_h - p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h)) - a(\boldsymbol{\varepsilon}, \,\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e) + (p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e)). \tag{3.14}$$

Now, by continuity of the trace operator we obtain

$$(q_h - p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h)) \leq \frac{C}{2\delta_3}||q_h - p||^2_Q + \frac{C\delta_3}{2}||\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h||^2_{\Sigma}, \tag{3.15}$$

with $\delta_3$ positive to be chosen. Therefore it remains to treat

$$-a(\boldsymbol{\varepsilon}, \,\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e) + (p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e)). \tag{3.16}$$

To this end, take $\tilde{\varepsilon}$, $\mathrm{L}^2$–orthogonal to the space $\Sigma^e_h$; moreover take $\tilde{p}$, $\mathrm{L}^2$–orthogonal to the space $\mathrm{tr}(\Sigma^e_h)$. Is is easy to see that (cf. (3.6))

$$a(\boldsymbol{\varepsilon}, \,\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e) = a(\boldsymbol{\varepsilon} - \tilde{\varepsilon}, \,\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e) \tag{3.17}$$

and

$$(p, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e)) = (p - \tilde{p}, \,\mathrm{tr}\,(\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e)). \tag{3.18}$$

Hence we have

$$-a(\boldsymbol{\varepsilon}, \,\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e) \leq \frac{M}{2\delta_4}||\boldsymbol{\varepsilon} - \tilde{\varepsilon}||^2_{\Sigma} + \frac{M\delta_4}{2}||\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e||^2_{\Sigma} \tag{3.19}$$

and

$$(p,\, \mathrm{tr}\,(\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e)) \leq \frac{C}{2\delta_5}||p-\tilde{p}||_Q^2 + \frac{C\delta_5}{2}||\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e||_\Sigma^2 \tag{3.20}$$

with $\delta_4$ and $\delta_5$ positive to be chosen.

*Remark.* An inequality of the type (3.19) can be achieved even for an *arbitrary* continuous, symmetric and coercive bilinear form $a(\cdot,\, \cdot)$. To see this, notice that such a bilinear form defines an inner product on $\Sigma$, let us say $(\cdot,\, \cdot)_a$, which is indeed equivalent to the $\mathrm{L}^2$ one. Hence, choosing this time $\tilde{\boldsymbol{\varepsilon}}$ orthogonal to $\Sigma_h^e$ with respect to the inner product $(\cdot,\, \cdot)_a$, we have

$$\begin{aligned} a(\boldsymbol{\varepsilon},\, \boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e) &= a(\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}},\, \boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e) \\ &\leq M||\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}||_\Sigma ||\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e||_\Sigma \end{aligned} \tag{3.21}$$

and the desired inequality easily follows.

□

Choosing $\delta_2$, $\delta_3$ sufficiently small, it follows from (3.10), (3.11), (3.15), (3.19) and (3.20) that

$$\begin{aligned} ||\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h||_\Sigma^2 &\leq c\left(||q_h - p||_Q^2 + ||\boldsymbol{\varepsilon} - \boldsymbol{\tau}_h||_\Sigma^2\right) + \\ &\frac{M}{2\delta_4}||\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}||_\Sigma^2 + \frac{M\delta_4}{2}||\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e||_\Sigma^2 + \frac{C}{2\delta_5}||p - \tilde{p}||_Q^2 + \frac{C\delta_5}{2}||\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e||_\Sigma^2 \end{aligned} \tag{3.22}$$

with $\delta_4$ and $\delta_5$ still to be chosen.

Therefore, it appears that a drawback arises. In fact, it happens that the terms

$$\frac{M\delta_4}{2}||\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e||_\Sigma^2$$

and

$$\frac{C\delta_5}{2}||\boldsymbol{\varepsilon}_e - \boldsymbol{\tau}_e||_\Sigma^2$$

cannot be absorbed in the left–hand side of (3.22) unless we have an inequality of the type

$$||\boldsymbol{\sigma}_e||_\Sigma \leq C_2||\boldsymbol{\sigma}_h||_\Sigma \tag{3.23}$$

for each $\boldsymbol{\sigma}_h = (\boldsymbol{\sigma}_c + \boldsymbol{\sigma}_e) \in \Sigma_h$, with $C_2$ positive and independent of $h$.

The matter now is to establish whether assuming (3.23) to be true is a very restrictive assumption or, on the contrary, is reasonable in actual situations. Notice that we obviously have

$$||\boldsymbol{\sigma}_h||^2_\Sigma = ||\boldsymbol{\sigma}_e + \boldsymbol{\sigma}_c||^2_\Sigma = ||\boldsymbol{\sigma}_e||^2_\Sigma + ||\boldsymbol{\sigma}_c||^2_\Sigma + 2(\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_c)\,. \tag{3.24}$$

It is clear that the best for having (3.23) is that

$$(\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_c) = 0 \qquad\qquad \forall \boldsymbol{\sigma}_e \in \Sigma^e_h,\ \forall \boldsymbol{\sigma}_c \in \Sigma^c_h\,, \tag{3.25}$$

i.e. $\Sigma^e_h$ and $\Sigma^c_h$ are $\mathrm{L}^2$–orthogonal each other. Condition (3.25) is too restrictive, but, fortunately, not necessary at all. What one really needs is that there exists a constant $\theta$, independent of $h$, such that

$$\sup_{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_c} \frac{(\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_c)}{||\boldsymbol{\sigma}_e||_\Sigma ||\boldsymbol{\sigma}_c||_\Sigma} \le \theta < 1, \tag{3.26}$$

or, which is equivalent,

$$\inf_{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_c} \frac{(\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_c)}{||\boldsymbol{\sigma}_e||_\Sigma ||\boldsymbol{\sigma}_c||_\Sigma} \ge -\theta > -1. \tag{3.27}$$

Having (3.27), one gets from (3.24):

$$\begin{aligned} ||\boldsymbol{\sigma}_h||^2_\Sigma &\ge ||\boldsymbol{\sigma}_e||^2_\Sigma + ||\boldsymbol{\sigma}_c||^2_\Sigma - 2\theta ||\boldsymbol{\sigma}_e||_\Sigma ||\boldsymbol{\sigma}_c||_\Sigma \\ &\ge ||\boldsymbol{\sigma}_e||^2_\Sigma - \theta^2 ||\boldsymbol{\sigma}_e||^2_\Sigma = (1-\theta^2)||\boldsymbol{\sigma}_e||^2_\Sigma \end{aligned} \tag{3.28}$$

and then (3.23) follows.

*Remark.* Condition (3.26) essentially means that any two $\boldsymbol{\sigma}_e \in \Sigma^e_h$ and $\boldsymbol{\sigma}_c \in \Sigma^c_h$ are far from being parallel, uniformly with respect to $h$. In a sense, it is a sort of *minimum angle condition* for the spaces $\Sigma^e_h$ and $\Sigma^c_h$. The same condition, written in a different way, has been recognized to be crucial for the analysis developed in [8]. It should be clear (cf. also (3.7)) that such a condition is quite naturally fulfilled if one hopes to gain stability in enhancing $\Sigma^c_h$ with $\Sigma^e_h$. In the remainder of this Section we thus assume (3.23) to hold.

□

Hence, if (3.26) holds, from (3.22) we have, choosing $\delta_4$ and $\delta_5$ sufficiently small,

$$||\boldsymbol{\varepsilon}_h - \boldsymbol{\tau}_h||_\Sigma \le c\big(||q_h - p||_Q + ||p - \tilde{p}||_Q + ||\boldsymbol{\varepsilon} - \boldsymbol{\tau}_h||_\Sigma + ||\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}||_\Sigma\big). \tag{3.29}$$

From the triangle inequality and (3.29) one gets

$$||\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h||_\Sigma \leq c\Big(\inf_{q_h\in Q_h}||p-q_h||_Q+\inf_{\boldsymbol{\tau}_h\in K_h}||\boldsymbol{\varepsilon}-\boldsymbol{\tau}_h||_\Sigma+ \\ ||p-\tilde{p}||_Q+||\boldsymbol{\varepsilon}-\tilde{\boldsymbol{\varepsilon}}||_\Sigma\Big). \tag{3.30}$$

We need the following lemma, whose proof can be found, for instance, in [2].

**Lemma 3.2**: Assume that there exists a positive constant $\beta$, independent of $h$, such that

$$\inf_{q_h\in Q_h}\sup_{\boldsymbol{\tau}_h\in\Sigma_h}\frac{(q_h,\,\mathrm{tr}\,\boldsymbol{\tau}_h)}{||q_h||_Q||\boldsymbol{\tau}_h||_\Sigma}\geq\beta. \tag{3.31}$$

Then there exists a positive constant $c_1$, independent of $h$, such that

$$\inf_{\boldsymbol{\tau}_h\in K_h}||\boldsymbol{\varepsilon}-\boldsymbol{\tau}_h||_\Sigma\leq c_1\inf_{\boldsymbol{\sigma}_h\in\Sigma_h}||\boldsymbol{\varepsilon}-\boldsymbol{\sigma}_h||_\Sigma \tag{3.32}$$

□

Hence, the inf on $K_h$ in the right–hand side of (3.30) can be replaced, thanks to Lemma (3.2), by an inf on $\Sigma_h$, to finally obtain

$$||\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h||_\Sigma \leq c\Big(\inf_{q_h\in Q_h}||p-q_h||_Q+\inf_{\boldsymbol{\sigma}_h\in \Sigma_h}||\boldsymbol{\varepsilon}-\boldsymbol{\sigma}_h||_\Sigma+ \\ ||p-\tilde{p}||_Q+||\boldsymbol{\varepsilon}-\tilde{\boldsymbol{\varepsilon}}||_\Sigma\Big). \tag{3.33}$$

Let us now perform an error analysis for the pressure field. For each $q_h\in Q_h$, thanks to the inf–sup condition, we have

$$\beta||p_h-q_h||_Q\leq\sup_{\boldsymbol{\sigma}_h\in\Sigma_h}\frac{(\mathrm{tr}\,\boldsymbol{\sigma}_h,\,p_h-q_h)}{||\boldsymbol{\sigma}_h||_\Sigma}. \tag{3.34}$$

Recalling (3.9) one gets

$$a(\boldsymbol{\varepsilon}_h-\boldsymbol{\varepsilon},\,\boldsymbol{\sigma}_h)=(\mathrm{tr}\,\boldsymbol{\sigma}_h,\,p_h-p)-a(\boldsymbol{\varepsilon},\,\boldsymbol{\sigma}_e)+(\mathrm{tr}\,\boldsymbol{\sigma}_e,\,p). \tag{3.35}$$

It follows that

$$(\mathrm{tr}\,\boldsymbol{\sigma}_h,\,p_h-q_h)=(\mathrm{tr}\,\boldsymbol{\sigma}_h,\,p_h-p)+(\mathrm{tr}\,\boldsymbol{\sigma}_h,\,p-q_h)= \\ a(\boldsymbol{\varepsilon}_h-\boldsymbol{\varepsilon},\,\boldsymbol{\sigma}_h)+a(\boldsymbol{\varepsilon},\,\boldsymbol{\sigma}_e)-(\mathrm{tr}\,\boldsymbol{\sigma}_e,\,p)+(\mathrm{tr}\,\boldsymbol{\sigma}_h,\,p-q_h). \tag{3.36}$$

By continuity, (3.23) and (3.36), from (3.34) one gets

$$||p_h - q_h||_Q \leq c\big(||\boldsymbol{\varepsilon}_h - \boldsymbol{\varepsilon}||_\Sigma + ||\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}||_\Sigma + ||p - q_h||_Q + ||p - \tilde{p}||_Q\big), \tag{3.37}$$

and finally, using also the triangle inequality and (3.33),

$$||p - p_h||_Q \leq C\left(\inf_{\boldsymbol{\sigma}_h \in \Sigma_h} ||\boldsymbol{\varepsilon} - \boldsymbol{\sigma}_h||_\Sigma + ||\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}||_\Sigma + \inf_{q_h \in Q_h} ||p - q_h||_Q + ||p - \tilde{p}||_Q\right). \tag{3.38}$$

What we have done so far can be summarized in the following

**Theorem 3.3**: Let $(\boldsymbol{\varepsilon}, p) \in \Sigma \times Q$ be the solution of system (2.9). Choose $\Sigma_h = \Sigma_h^c \oplus \Sigma_h^e$ as in (3.2) and $Q_h \subset Q$. Assume that there exists a positive constant $\beta$ such that

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{\tau}_h \in \Sigma_h} \frac{(q_h, \operatorname{tr} \boldsymbol{\tau}_h)}{||q_h||_Q ||\boldsymbol{\tau}_h||_\Sigma} \geq \beta. \tag{3.39}$$

Let $(\boldsymbol{\varepsilon}_h, p_h) \in \Sigma_h \times Q_h$ be the solution of the discretized problem (3.5). Assume moreover that

$$\sup_{\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_c} \frac{(\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_c)}{||\boldsymbol{\sigma}_e||_\Sigma ||\boldsymbol{\sigma}_c||_\Sigma} \leq \theta < 1, \tag{3.40}$$

for each $\boldsymbol{\sigma}_c \in \Sigma_h^c$ and $\boldsymbol{\sigma}_e \in \Sigma_h^e$, with $\theta$ independent of $h$. Finally, take any $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{p}$ such that

$$\begin{cases} a(\tilde{\boldsymbol{\varepsilon}}, \boldsymbol{\sigma}_e) = 0 \qquad \forall \boldsymbol{\sigma}_e \in \Sigma_h^e \\ \\ (\tilde{p}, \operatorname{tr} \boldsymbol{\sigma}_e) = 0 \qquad \forall \boldsymbol{\sigma}_e \in \Sigma_h^e. \end{cases} \tag{3.41}$$

Then the following error estimates hold

$$||\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h||_\Sigma \leq C\Big(\inf_{\boldsymbol{\sigma}_h \in \Sigma_h} ||\boldsymbol{\varepsilon} - \boldsymbol{\sigma}_h||_\Sigma + \inf_{q_h \in Q_h} ||p - q_h||_Q + ||\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}||_\Sigma + ||p - \tilde{p}||_Q\Big), \tag{3.42}$$

$$||p - p_h||_Q \leq C\big(\inf_{\boldsymbol{\sigma}_h \in \Sigma_h} ||\boldsymbol{\varepsilon} - \boldsymbol{\sigma}_h||_\Sigma + \inf_{q_h \in Q_h} ||p - q_h||_Q + ||\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}||_\Sigma + ||p - \tilde{p}||_Q\big). \tag{3.43}$$

□

*Remark.* If there exist $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{p}$, satisfying (3.41), which are also "good approximations" of $\boldsymbol{\varepsilon}$ and $p$, then (3.42) and (3.43) provide convergence. This is the case of the method described in the next Section.

☐

*Remark.* We wish to recall that condition (3.40) is sufficient for having estimate (3.23).

☐

## 4. The Bathe–Pantuso method: a simplified analysis.

Aim of this Section is to provide a first analysis of a method recently proposed by Bathe and Pantuso (cf. [7]) using the results of Section 3. We will only treat the two–dimensional case of rectangular elements. The case of general quadrilaterals will be the topic of future communications. Thus, let $\Omega$ be a rectangular open set in $\mathbf{R}^2$ and $\{\mathcal{T}_h\}$ be a regular family of meshes, of meshsize $h$, made up by rectangular elements $K$ with sides parallel to the coordinate axes. With $(x_K,\, y_K)$ we will denote the coordinates of the center of $K$, while with $h_x^K$ and $h_y^K$ we will denote the lengths of the $x$–sides and of the $y$–sides, respectively. Following Bathe–Pantuso (cf. [7]), let $\Sigma_h^c$ be the space

$$\Sigma_h^c = \nabla^S\left((\mathrm{H}_0^1(\Omega)\cap\mathcal{L}_{[1]}^1(\Omega;\mathcal{T}_h))^2\right) \tag{4.1}$$

where $\mathcal{L}_{[1]}^1(\Omega;\mathcal{T}_h)$ is the space of continuous functions which are bilinear in each $K\in\mathcal{T}_h$.

Furthermore, let $Q_h$ be the space

$$Q_h = \mathcal{L}_{[1]}^1(\Omega;\mathcal{T}_h)/\mathbf{R}. \tag{4.2}$$

Note that also the pressure field, following this choice, will be approximated by means of continuous functions. It remains to choose $\Sigma_h^e$ to finally construct

$$\Sigma_h = \Sigma_h^c \oplus \Sigma_h^e. \tag{4.3}$$

Let us first introduce the following notation: given a symmetric tensor $\boldsymbol{\tau}$, we will identify it with the triple of its cartesian conponents. Thus

$$\boldsymbol{\tau} = (\tau_{11},\, \tau_{22},\, \tau_{12}). \tag{4.4}$$

Fix now $K\in\mathcal{T}_h$. Let us consider the tensor functions defined on $K$

$$\begin{cases} \boldsymbol{\tau}_1^K = (x - x_K,\, 0,\, 0) & \boldsymbol{\tau}_4^K = (0,\, 0,\, y - y_K) \\ \boldsymbol{\tau}_2^K = (0,\, y - y_K,\, 0) & \boldsymbol{\tau}_5^K = ((x - x_K)(y - y_K),\, 0,\, 0) \\ \boldsymbol{\tau}_3^K = (0,\, 0,\, x - x_K) & \boldsymbol{\tau}_6^K = (0,\, (x - x_K)(y - y_K),\, 0), \end{cases} \tag{4.5}$$

and set

$$E(K) = \text{span}\left\{\boldsymbol{\tau}_i^K \,:\, i = 1, 2, ..., 6\right\}. \tag{4.6}$$

We are now ready to introduce the space $\Sigma_h^e$ as suggested in [7], setting

$$\Sigma_h^e = \left\{\boldsymbol{\tau}_e \in (\mathrm{L}^2(\Omega))_S^4 \,:\, \boldsymbol{\tau}_{e|K} \in E(K) \qquad \forall K \in \mathcal{T}_h\right\}. \tag{4.7}$$

We remark that $\Sigma_h^e \cap \Sigma_h^c = \{0\}$, so that one is allowed to consider $\Sigma_h$ as in (4.3) using (4.1) and (4.7). Let us try to apply the results of the previous Section. We will proceed into three steps.

*First step: finding suitable $\tilde{\boldsymbol{\varepsilon}}$ and $\tilde{p}$.* We begin by noting that each $\boldsymbol{\sigma}_e \in \Sigma_h^e$, and also its trace, has zero mean value in each $K \in \mathcal{T}_h$. Hence, choosing $\tilde{\boldsymbol{\varepsilon}}$ (resp. $\tilde{p}$) as the $\mathrm{L}^2$–orthogonal projection of $\boldsymbol{\varepsilon}$ (resp. $p$) onto the piecewise constant function space, one can easily see that (3.41) are fulfilled. Recalling that both $\Sigma$ and $Q$ are equipped with the $\mathrm{L}^2$–norm, from standard approximation theory (cf. [3], for instance) it holds

$$\begin{cases} ||\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}||_\Sigma \leq ch||\boldsymbol{\varepsilon}||_{1,\Omega} \\ ||p - \tilde{p}||_Q \leq ch||p||_{1,\Omega}. \end{cases} \tag{4.8}$$

*Remark.* The zero mean property of the functions in the enhanced strain space was already recognized to be crucial for the convergence analysis developed in [8]. □

*Second step: checking* (3.40). Let us come to check (3.40). Let $K \in \bigcup_{h>0} \mathcal{T}_h$ be fixed. By (4.1) it is easy to realize that

$$\boldsymbol{\sigma}_{c|K} \in \text{span}\left\{\boldsymbol{\sigma}_i^K \,:\, i = 1, 2, ..., 6\right\} \tag{4.9}$$

where

$$\left\{\begin{array}{ll} \boldsymbol{\sigma}_1^K = (1,\, 0,\, 0) & \boldsymbol{\sigma}_4^K = (0,\, 1,\, 0) \\ \boldsymbol{\sigma}_2^K = (0,\, 0,\, 1/2) & \boldsymbol{\sigma}_5^K = (0,\, x - x_K,\, 1/2(y - y_K)). \\ \boldsymbol{\sigma}_3^K = (y - y_K,\, 0,\, 1/2(x - x_K)) & \end{array}\right. \tag{4.10}$$

Hence

$$\boldsymbol{\sigma}_{c|K} = \sum_{i=1}^{5} \beta_i \boldsymbol{\sigma}_i^K \qquad \text{with } \beta_i \in \mathbf{R}. \tag{4.11}$$

Take now an arbitrary element $\boldsymbol{\sigma}_e \in E(K)$, i.e.

$$\boldsymbol{\sigma}_e = \sum_{j=1}^{6} \alpha_j \boldsymbol{\tau}_j^K \qquad \text{with } \alpha_j \in \mathbf{R}. \tag{4.12}$$

We have

$$(\boldsymbol{\sigma}_c,\, \boldsymbol{\sigma}_e)_K = \sum_{i=1}^{5} \sum_{j=1}^{6} \beta_i \alpha_j (\boldsymbol{\sigma}_i^K,\, \boldsymbol{\tau}_j^K)_K. \tag{4.13}$$

Note (cf. (4.5) and (4.10)) that

$$(\boldsymbol{\sigma}_i^K,\, \boldsymbol{\tau}_j^K)_K = 0 \qquad \text{if } (i,\, j) \neq (3,\, 3),\, (5,\, 4). \tag{4.14}$$

It follows from (4.13) and (4.14) that

$$(\boldsymbol{\sigma}_c,\, \boldsymbol{\sigma}_e)_K = \beta_3 \alpha_3 (\boldsymbol{\sigma}_3^K,\, \boldsymbol{\tau}_3^K)_K + \beta_5 \alpha_4 (\boldsymbol{\sigma}_5^K,\, \boldsymbol{\tau}_4^K)_K. \tag{4.15}$$

We need the following lemma.

**Lemma 4.1**: There exists a postive constant $\theta < 1$, independent of $K$, such that

$$(\boldsymbol{\sigma}_3^K,\, \boldsymbol{\tau}_3^K)_K \leq \theta ||\boldsymbol{\sigma}_3^K||_{0,K} ||\boldsymbol{\tau}_3^K||_{0,K} \tag{4.16}$$

and

$$(\boldsymbol{\sigma}_5^K,\, \boldsymbol{\tau}_4^K)_K \leq \theta ||\boldsymbol{\sigma}_5^K||_{0,K} ||\boldsymbol{\tau}_4^K||_{0,K}. \tag{4.17}$$

*Proof.* We will develop the proof of (4.16) only, since the other one can be treated using the same technique. Since (cf. (4.5) and (4.10))

$$\boldsymbol{\sigma}_3^K = \frac{1}{2}\boldsymbol{\tau}_3^K + \tilde{\boldsymbol{\tau}}^K \qquad \text{with } \tilde{\boldsymbol{\tau}}^K = (y - y_K,\, 0,\, 0), \tag{4.18}$$

we wish to find a positive constant $\theta < 1$ such that

$$\left(\frac{1}{2}\boldsymbol{\tau}_3^K + \tilde{\boldsymbol{\tau}}^K,\, \boldsymbol{\tau}_3^K\right)_K^2 \leq \theta^2 ||\frac{1}{2}\boldsymbol{\tau}_3^K + \tilde{\boldsymbol{\tau}}^K||_{0,K}^2 ||\boldsymbol{\tau}_3^K||_{0,K}^2. \tag{4.19}$$

Since $\tilde{\boldsymbol{\tau}}^K$ and $\boldsymbol{\tau}_3^K$ are orthogonal, (4.19) becomes

$$\frac{1}{4}||\boldsymbol{\tau}_3^K||_{0,K}^4 \leq \theta^2 \left(\frac{1}{4}||\boldsymbol{\tau}_3^K||_{0,K}^2 + ||\tilde{\boldsymbol{\tau}}^K||_{0,K}^2\right) ||\boldsymbol{\tau}_3^K||_{0,K}^2 \tag{4.20}$$

and hence

$$\frac{1}{4}||\boldsymbol{\tau}_3^K||_{0,K}^2 \leq \theta^2 \left(\frac{1}{4}||\boldsymbol{\tau}_3^K||_{0,K}^2 + ||\tilde{\boldsymbol{\tau}}^K||_{0,K}^2\right). \tag{4.21}$$

Before proceeding, recall that, as the family of meshes is assumed to be regular, there exists a positive constant $\rho$ such that

$$\frac{1}{\rho} \leq \frac{h_x^K}{h_y^K} \leq \rho \qquad \forall K \in \bigcup_{h>0} \mathcal{T}_h. \tag{4.22}$$

We need now to explicitely compute $||\boldsymbol{\tau}_3^K||_{0,K}^2$ and $||\tilde{\boldsymbol{\tau}}^K||_{0,K}^2$. It is easily seen that

$$\begin{cases} ||\boldsymbol{\tau}_3^K||_{0,K}^2 = \dfrac{1}{6}h_y^K(h_x^K)^3 \\[2ex] ||\tilde{\boldsymbol{\tau}}^K||_{0,K}^2 = \dfrac{1}{12}h_x^K(h_y^K)^3. \end{cases} \tag{4.23}$$

Therefore, from (4.22) and (4.23) it follows that

$$||\boldsymbol{\tau}_3^K||_{0,K}^2 \leq 2\rho^2||\tilde{\boldsymbol{\tau}}^K||_{0,K}^2. \tag{4.24}$$

Eventually, if we choose

$$1 > \theta > \left(1 + \frac{2}{\rho^2}\right)^{-1/2}, \tag{4.25}$$

the inequality (4.21) easily follows. By remarking that

$$\gamma = \left(1 + \frac{2}{\rho^2}\right)^{-1/2} < 1$$

is independent on $K$, condition (4.16) holds. The lemma is proved.

$\square$

By Lemma 4.1 it follows from (4.15)

$$(\boldsymbol{\sigma}_c,\, \boldsymbol{\sigma}_e)_K \leq \theta \left(\beta_3\alpha_3 ||\boldsymbol{\sigma}_3^K||_{0,K} ||\boldsymbol{\tau}_3^K||_{0,K} + \beta_5\alpha_4 ||\boldsymbol{\sigma}_5^K||_{0,K} ||\boldsymbol{\tau}_4^K||_{0,K}\right). \tag{4.26}$$

Hence we get

$$(\boldsymbol{\sigma}_c,\, \boldsymbol{\sigma}_e)_K \leq \theta \left(\beta_3^2 ||\boldsymbol{\sigma}_3^K||_{0,K}^2 + \beta_5^2 ||\boldsymbol{\sigma}_5^K||_{0,K}^2\right)^{1/2} \left(\alpha_3^2 ||\boldsymbol{\tau}_3^K||_{0,K}^2 + \alpha_4^2 ||\boldsymbol{\tau}_4^K||_{0,K}^2\right)^{1/2}. \tag{4.27}$$

Furthermore, we obviously have

$$\left(\beta_3^2 ||\boldsymbol{\sigma}_3^K||_{0,K}^2 + \beta_5^2 ||\boldsymbol{\sigma}_5^K||_{0,K}^2\right)^{1/2} \left(\alpha_3^2 ||\boldsymbol{\tau}_3^K||_{0,K}^2 + \alpha_4^2 ||\boldsymbol{\tau}_4^K||_{0,K}^2\right)^{1/2} \leq$$
$$\left(\sum_{i=1}^{5} \beta_i^2 ||\boldsymbol{\sigma}_i^K||_{0,K}^2\right)^{1/2} \left(\sum_{j=1}^{6} \alpha_j^2 ||\boldsymbol{\tau}_j^K||_{0,K}^2\right)^{1/2}. \tag{4.28}$$

Notice now that the following orthogonality conditions hold (cf. (4.5) and (4.10))

$$\begin{cases} (\boldsymbol{\sigma}_i^K,\, \boldsymbol{\sigma}_j^K)_K = 0 \qquad \forall\, i,j = 1,2,...,5; \qquad i \neq j, \\ \\ (\boldsymbol{\tau}_i^K,\, \boldsymbol{\tau}_j^K)_K = 0 \qquad \forall\, i,j = 1,2,...,6; \qquad i \neq j. \end{cases} \tag{4.29}$$

Hence we obtain

$$\left(\sum_{i=1}^{5} \beta_i^2 ||\boldsymbol{\sigma}_i^K||_{0,K}^2\right)^{1/2} \left(\sum_{j=1}^{6} \alpha_j^2 ||\boldsymbol{\tau}_j^K||_{0,K}^2\right)^{1/2} = ||\boldsymbol{\sigma}_c||_{0,K} ||\boldsymbol{\sigma}_e||_{0,K}. \tag{4.30}$$

From (4.27), (4.28) and (4.30) we then have

$$(\boldsymbol{\sigma}_c,\, \boldsymbol{\sigma}_e)_K \le \theta ||\boldsymbol{\sigma}_c||_{0,K} ||\boldsymbol{\sigma}_e||_{0,K}. \tag{4.31}$$

Summing up over $K \in \mathcal{T}_h$, we finally get

$$\begin{aligned}(\boldsymbol{\sigma}_c,\, \boldsymbol{\sigma}_e) = \sum_{K\in\mathcal{T}_h} (\boldsymbol{\sigma}_c,\, \boldsymbol{\sigma}_e)_K \le \theta \sum_{K\in\mathcal{T}_h} ||\boldsymbol{\sigma}_c||_{0,K} ||\boldsymbol{\sigma}_e||_{0,K} \\ \le \theta ||\boldsymbol{\sigma}_c||_{0,\Omega} ||\boldsymbol{\sigma}_e||_{0,\Omega}.\end{aligned} \tag{4.32}$$

and condition (3.40) follows.

*Third step: checking stability condition.* The last step consists in checking the inf–sup condition (3.39). In order to do so, we will employ Fortin's trick, recalled by the next lemma (cf. [2]).

**Lemma 4.1**: Suppose there exists a linear operator

$$\Pi_h : \Sigma \longrightarrow \Sigma_h$$

such that

$$\begin{cases} ||\Pi_h \boldsymbol{\tau}||_\Sigma \le c ||\boldsymbol{\tau}||_\Sigma & \forall \boldsymbol{\tau} \in \Sigma \\ \\ (\mathrm{tr}\, \boldsymbol{\tau} - \mathrm{tr}\, \Pi_h \boldsymbol{\tau},\, q_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{4.33}$$

Then the inf–sup condition (3.39) holds.

□

First, let us consider the Clément operator $\Pi_c$ (cf. [2] and [3]). This operator induces a linear operator

$$\Pi_1 : \Sigma \longrightarrow \Sigma_h^c$$

defined by

$$\Pi_1 \boldsymbol{\tau} = \nabla^S\, \Pi_c G(\boldsymbol{\tau}). \tag{4.34}$$

This means that we have (cf. (2.5))

$$G(\boldsymbol{\tau} - \Pi_1 \boldsymbol{\tau}) = G(\boldsymbol{\tau}) - \Pi_c G(\boldsymbol{\tau}) \tag{4.35}$$

Furthermore, from the basic features of Clément operator (cf. [2]) it holds

$$\sum_{K \in \mathcal{T}_h} h_K^{2s-2} |\, G(\boldsymbol{\tau}) - \Pi_c G(\boldsymbol{\tau}) \,|_{s,K}^2 \leq c \,||\, G(\boldsymbol{\tau}) \,||_{1,\Omega}^2 \qquad\qquad s = 0\,,\, 1. \tag{4.36}$$

Secondly, let us consider the following problem. Given $\boldsymbol{\tau} \in \Sigma$, find $\Pi_2 \boldsymbol{\tau} \in \Sigma_h^e$ such that

$$(\operatorname{tr} \Pi_2 \boldsymbol{\tau},\, q_h) = (\operatorname{tr} \boldsymbol{\tau},\, q_h) \qquad \forall q_h \in Q_h. \tag{4.37}$$

Recalling that $\operatorname{tr} \boldsymbol{\tau} = \operatorname{div} G(\boldsymbol{\tau})$ and integrating by parts, one obtains that (4.37) is equivalent to

$$(\operatorname{tr} \Pi_2 \boldsymbol{\tau},\, q_h) = -(G(\boldsymbol{\tau}),\, \nabla\, q_h) \qquad \forall q_h \in Q_h. \tag{4.38}$$

Let us focus the problem locally, on each $K \in \mathcal{T}_h$. Then, we wish to find $\Pi_2^K \boldsymbol{\tau} \in E(K)$ (cf. (4.6) and (4.7)) such that

$$(\operatorname{tr} \Pi_2^K \boldsymbol{\tau},\, q_h)_K = -(G(\boldsymbol{\tau}),\, \nabla\, q_h)_K \qquad \forall q_h \in Q_h. \tag{4.39}$$

It is easy to see (cf. remark below) that problem (4.39) is solvable. Let us take the solution of minimum norm. A scaling argument shows that

$$|\Pi_2^K \boldsymbol{\tau}|_{0,K} \leq c h^{-1} |G(\boldsymbol{\tau})|_{0,K}. \tag{4.40}$$

Setting

$$\Pi_2 \boldsymbol{\tau} = \sum_{K \in \mathcal{T}_h} \chi_K \Pi_2^K \boldsymbol{\tau} \tag{4.41}$$

where $\chi_K$ is the characteristic function of $K$, we have thus defined a linear operator

$$\Pi_2 : \Sigma \longrightarrow \Sigma_h^e$$

satisfying

$$(\operatorname{tr} \Pi_2 \boldsymbol{\tau},\, q_h) = (\operatorname{tr} \boldsymbol{\tau},\, q_h) \qquad \forall q_h \in Q_h. \tag{4.42}$$

Finally, $Id$ being the identity operator, choose $\Pi_h$ as

$$\Pi_h = \Pi_2 (Id - \Pi_1) + \Pi_1. \tag{4.43}$$

Combining (4.35), (4.36) and (4.42), one can easily see that conditions (4.33) are fulfilled. Hence, by lemma 4.2, the inf–sup condition (3.39) holds.

*Remark.* Note that problem (4.39) is solvable if

$$\{q_h : (q_h, \operatorname{tr} \boldsymbol{\tau}_e)_K = 0 \qquad \forall \boldsymbol{\tau}_h \in E(K)\} \Longrightarrow \left\{q_{h|K} = \text{constant}\right\}, \tag{4.44}$$

where $E(K)$ is defined by (4.6).

But, due to the choice (4.2), it holds

$$(q_h, \operatorname{tr} \boldsymbol{\tau}_e)_K = \left( \sum_{i=0}^{3} q_i^K \varphi_i^K, \operatorname{tr} \boldsymbol{\tau}_e \right)_K \tag{4.45}$$

where $q_i^K \in \mathbf{R}$ and $\varphi_i^K$ are functions defined on $K$ by

$$\begin{cases} \varphi_0^K = 1 & \varphi_2^K = y - y_K \\ \\ \varphi_1^K = x - x_K & \varphi_3^K = (x - x_K)(y - y_K). \end{cases} \tag{4.46}$$

Recalling (4.5) it is easily seen that (4.44) follows.

$\square$

Since $\Sigma_h^c \subset \Sigma_h$ we obviously have

$$\inf_{\boldsymbol{\sigma}_h \in \Sigma_h} ||\boldsymbol{\varepsilon} - \boldsymbol{\sigma}_h||_\Sigma \leq \inf_{\boldsymbol{\tau}_c \in \Sigma_h^c} ||\boldsymbol{\varepsilon} - \boldsymbol{\tau}_c||_\Sigma. \tag{4.47}$$

Hence we get, by standard interpolation theory (cf. [3]) and Korn's inequality,

$$\begin{cases} \displaystyle\inf_{\boldsymbol{\sigma}_h \in \Sigma_h} ||\boldsymbol{\varepsilon} - \boldsymbol{\sigma}_h||_\Sigma \leq ch||G(\boldsymbol{\varepsilon})||_{2,\Omega} \\ \\ \displaystyle\inf_{q_h \in Q_h} ||p - q_h||_Q \leq ch||p||_{1,\Omega}. \end{cases} \tag{4.48}$$

Hence, applying Theorem 3.3, what we have developed in this Section leads to our main result, i.e. the

**Theorem 4.3**: Let $(\boldsymbol{\varepsilon}, p)$ be the solution of problem (2.9). Choose $\Sigma_h$ and $Q_h$ as shown by (4.1), (4.2), (4.3) and (4.4). Let $(\boldsymbol{\varepsilon}_h, p_h) \in \Sigma_h \times Q_h$ be the solution of problem (3.5), choosing $\Sigma_h$ and $Q_h$ as shown by (4.1), (4.2), (4.3) and (4.4). The following error estimates hold

$$\begin{cases} ||\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h||_\Sigma \leq Ch \\ \\ ||p - p_h||_Q \leq Ch \end{cases} \tag{4.49}$$

□

## 5. Conclusions.

A strain–pressure variational formulation for the stationary Stokes problem has been presented. Based on this formulation, an approximation technique by means of finite elements with enhanced strain field has been considered and analyzed. As an example of application of our results, we have considered a quadrilateral method recently proposed by Bathe and Pantuso (cf. [7]). We have restricted, as a first step of research, the analysis to the case of rectangular elements only. We have thus been able to obtain an error analysis which shows that the method is stable and convergent (cf. Theorem 4.3). Some possible topics for future communications are as follows. First, the case of arbitrary quadrilaterals has to be considered. Moreover, we wish to investigate the relationships between enriching the velocity field by bubble functions and enhancing the strain field. Again, a comparison between the Hughes–Franca stabilization technique (cf. [4] and [5]) and the enhanced strain method should be of some interest.

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