

**PARTIAL SELECTIVE REDUCED INTEGRATION SCHEMES  
AND KINEMATICALLY LINKED INTERPOLATIONS  
FOR PLATE BENDING PROBLEMS**

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Some finite elements for the approximation to the solution of the Reissner-Mindlin plate problem are presented. They all take advantage of a stabilization technique recently proposed by Arnold and Brezzi. Moreover, a Kinematically Linked Interpolation approach has been used to improve the convergence features. A general theoretical analysis of stability and convergence is also provided, together with extensive numerical tests.

## 1. Introduction

The development of finite element schemes to approximate the solution of the Reissner-Mindlin plate problem poses, as it is well-known, great difficulties. The reason stands in the shear energy term, which essentially imposes the Kirchhoff constraint when the thickness is numerically small. This constraint is in general too severe for low-order elements, so that the well-known *shear locking effect* may occur, thus compromising the accuracy of the discrete solution <sup>9,12,19</sup>.

In recent years, a lot of efforts have been spent to design performing finite element methods for the plate problem. Most of them arise from a mixed approach and they are based on a suitable reduction in the influence of the shear energy term <sup>1,3,9,10,13,14,19</sup>. But a naive reduction for the shear energy term could cause a lack of stability, exhibited by oscillations for the discrete vertical displacement solution <sup>9,19</sup>.

Recently, Arnold and Brezzi proposed a mixed formulation, capable to avoid spurious modes for any choice of elements <sup>2</sup>. The idea (called *Partial Selective Reduced Integration scheme*) consists first in splitting the shear energy term into two parts and then in integrating exactly one of the two parts (to prevent spurious modes) and reducing the second one (to prevent locking effects). Some finite

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elements using this strategy have been proposed and mathematically analysed in <sup>20,21</sup>. We remark that the principle of this technique was already introduced for the approximation of shell problems in <sup>25</sup>.

In this paper we use the Partial Selective Reduced Integration technique in connection with the Kinematically Linked Interpolation scheme. The latter strategy <sup>5,7,26,27</sup>, developed in order to avoid shear locking, essentially consists in a suitable enrichment of the vertical displacement degrees of freedom by means of the rotational degrees of freedom. Some elements using this technique have already been analysed in <sup>7,18,22,23</sup>. The paper is organized as follows.

Section 2 briefly recalls the Reissner-Mindlin plate problem along with the standard mixed formulation. Then the modified formulation, involving the splitting of the shear energy term by means of a *splitting* parameter, is presented.

In Section 3 the idea of Kinematically Linked Interpolations is discussed, in connection with the Partial Selective Reduced Integration technique. A general stability and error analysis for the scheme at hand is provided.

Section 4 gives some example of elements (one of them triangular and the other two quadrilateral) falling into the theory developed in the previous Section. Thus, a rigorous proof of stability and optimal convergence is provided for all the proposed elements.

Finally, Section 5 deals with numerical tests for all the presented elements. The numerical experiments confirm the theoretical results of Section 3 in terms of convergence rate. Furthermore, as the Partial Selective Reduced Integration scheme involves a splitting parameter  $\mu$ , numerical tests on the effects relative to different choices of  $\mu$  are also considered. The reason for performing a numerical investigation on a good choice of the splitting parameter is due to the absence of a theoretical analysis, so far. The results show that  $\mu$  should be chosen as  $1/h^2$ ,  $h$  being the meshsize.

In what follows, we will use standard notation <sup>17,28</sup>; furthermore, the constant  $C$  is independent of both  $h$  and  $t$ , and it is not necessarily the same at each occurrence.

## 2. The Reissner-Mindlin model and a mixed formulation

Let us denote with  $A = \Omega \times (-t/2, t/2)$  the region in  $\mathbf{R}^3$  occupied by an undeformed elastic plate of thickness  $t > 0$ . The Reissner-Mindlin plate model <sup>19</sup> describes the bending behavior of the plate in terms of the transverse displacement  $w(t)$  and of the rotations  $\underline{\theta}(t)$  of the fibers normal to the midplane  $\Omega$ .

In the case of a clamped plate, the stationary problem consists in finding the couple  $(\underline{\theta}(t), w(t))$  that minimizes the functional

$$\Pi_t(\underline{\theta}(t), w(t)) = \frac{1}{2} \int_{\Omega} \mathcal{C} \mathcal{E} \underline{\theta}(t) : \mathcal{E} \underline{\theta}(t) + \frac{\lambda t^{-2}}{2} \int_{\Omega} |\underline{\theta}(t) - \nabla w(t)|^2 - \int_{\Omega} f w(t) \, dx \, dy \quad (2.1)$$

over the space  $V = \Theta \times W = (H_0^1(\Omega))^2 \times H_0^1(\Omega)$ . In (2.1)  $\mathcal{C}$  is a positive-definite fourth order symmetric tensor in which Young's modulus  $E$  and Poisson's ratio  $\nu$

enter. More precisely,  $\mathcal{C}$  is defined by

$$\mathcal{C}\mathcal{T} = \frac{E}{12(1-\nu^2)}((1-\nu)\mathcal{T} + \nu \text{tr}(\mathcal{T})\mathcal{I}), \quad (2.2)$$

where  $\mathcal{T}$  is an arbitrary second order tensor,  $\text{tr}(\mathcal{T})$  is its trace, and  $\mathcal{I}$  is the identity second order tensor. Furthermore,  $\mathcal{E}\underline{\theta}(t)$  is the symmetric gradient of the field  $\underline{\theta}(t)$  and  $\lambda = Ek/2(1+\nu)$ , with  $k$  a shear correction factor (usually taken as  $5/6$ ). Accordingly, the first term in (2.1) is the plate bending energy, the second term is the shear energy, while the last one corresponds to the external energy. Korn's inequality assures that  $a(\cdot, \cdot) = \int_{\Omega} \mathcal{C}\mathcal{E}(\cdot) : \mathcal{E}(\cdot)$  is a coercive form over  $\Theta$  so that there exists a unique solution  $(\underline{\theta}(t), w(t))$  in  $V$  of the

**Problem  $P_t$ :** For  $t > 0$  fixed, find  $(\underline{\theta}(t), w(t))$  in  $V$  such that

$$a(\underline{\theta}(t), \underline{\eta}) + \lambda t^{-2} (\underline{\theta}(t) - \underline{\nabla} w(t), \underline{\eta} - \underline{\nabla} v) = \int_{\Omega} f v \quad \forall (\underline{\eta}, v) \in V. \quad (2.3)$$

It is well-known that finding a good finite element method for problem (2.3) is not at all a trivial task, because of the *shear locking* phenomenon<sup>9,12,19</sup>. In fact, as the thickness is numerically small, the shear energy term in (2.3) imposes the Kirchhoff constraint, which is too severe for low-order elements. Some ways to overcome this undesirable lack of convergence have been proposed and analysed in recent years<sup>1,3,9,10,12,13,14,24</sup>. Several of them are based on a mixed formulation of Problem (2.1). More precisely, let us introduce the scaled shear stress<sup>12</sup>

$$\underline{\xi} = \lambda t^{-2} (\underline{\theta} - \underline{\nabla} w) \quad (2.4)$$

as independent unknown. Thus, the solution of the plate problem turns out to be the *critical point* of the functional

$$\tilde{\Pi}_t(\underline{\theta}, w, \underline{\gamma}) = \frac{1}{2} a(\underline{\theta}, \underline{\theta}) - \frac{\lambda^{-1} t^2}{2} \|\underline{\xi}\|_{0,\Omega}^2 + (\underline{\xi}, \underline{\theta} - \underline{\nabla} w) - (f, w) \quad (2.5)$$

on  $V \times (L^2(\Omega))^2$ . Hence, the associated mixed variational plate problem can be written as:

**Problem  $\tilde{P}_t$ :** For  $t > 0$  fixed, find  $(\underline{\theta}(t), w(t), \underline{\xi}(t))$  in  $V \times (L^2(\Omega))^2$  such that

$$\begin{cases} a(\underline{\theta}(t), \underline{\eta}) + (\underline{\xi}(t), \underline{\eta} - \underline{\nabla} v) = (f, v) & \forall (\underline{\eta}, v) \in V \\ (\underline{s}, \underline{\theta}(t) - \underline{\nabla} w(t)) - \lambda^{-1} t^2 (\underline{\xi}(t), \underline{s}) = 0 & \forall \underline{s} \in (L^2(\Omega))^2 \end{cases} \quad (2.6)$$

Passing to a discretization of problem (2.6), it is easily seen that now the Kirchhoff constraint arising when the plate thickness is numerically small, is imposed *only in a weak sense*. Thus, the locking effect may not occur. Unfortunately, since we are dealing with the approximation of a mixed problem, another pathology might compromise the reliability of an approximation scheme: the occurrence of the so-called *spurious modes*, or *zero-energy modes*<sup>9,19</sup>. Mathematically, this arises as a

consequence of a lack of stability in the discretization procedure <sup>12</sup>. The reason essentially stands in the *bending energy term*  $a(\underline{\theta}, \underline{\theta})$ , which gives no control on the vertical displacement field. In the paper <sup>2</sup>, Arnold and Brezzi have proposed a new mixed variational formulation capable to avoid the occurrence of spurious modes for any choice of finite element spaces. The idea consists in setting a different Lagrange multiplier than the one described by (2.4). Namely, they set

$$\underline{\gamma} = \lambda(t^{-2} - \mu)(\underline{\theta} - \nabla w), \quad (2.7)$$

where  $\mu$  is such that  $0 < \mu < t^{-2}$ . Hence, it is not hard to show that the associated mixed variational plate problem can be written as

**Problem  $MP_t$ :** For  $t > 0$  fixed, find  $(\underline{\theta}(t), w(t), \underline{\gamma}(t))$  in  $V \times (L^2(\Omega))^2$  such that

$$\begin{cases} a(\underline{\theta}(t), \underline{\eta}) + \mu(\underline{\theta}(t) - \nabla w(t), \underline{\eta} - \nabla v) + (\underline{\gamma}(t), \underline{\eta} - \nabla v) = (f, v) & \forall (\underline{\eta}, v) \in V \\ (\underline{s}, \underline{\theta}(t) - \nabla w(t)) - \lambda^{-1}\tau^2(\underline{\gamma}(t), \underline{s}) = 0 & \forall \underline{s} \in (L^2(\Omega))^2 \end{cases} \quad (2.8)$$

Above, we have set

$$\tau^2 = \frac{t^2}{1 - \mu t^2}. \quad (2.9)$$

It is important to notice that now the quadratic form  $a(\underline{\theta}, \underline{\theta}) + \mu(\underline{\theta} - \nabla w, \underline{\theta} - \nabla w)$  gives control on the deflections also, thus preventing spurious modes to occur in the discretization procedure. As a consequence, any finite element scheme based on formulation (2.8) will be performant provided it is locking-free. In what follows we will then use system (2.8) as the starting point for our discretizations.

### 3. Kinematically Linked Interpolation methods

From now on, for the sake of simplicity and without loss of generality, we choose  $\lambda = 1$ . Hence, the mixed variational problem reads as follows

**Problem  $MP_\tau$ :** For  $\tau > 0$  fixed, find  $(\underline{\theta}, w, \underline{\gamma})$  in  $V \times (L^2(\Omega))^2$  such that

$$\begin{cases} a(\underline{\theta}, w; \underline{\eta}, v) + (\underline{\gamma}, \underline{\eta} - \nabla v) = (f, v) & \forall (\underline{\eta}, v) \in V \\ (\underline{s}, \underline{\theta} - \nabla w) - \tau^2(\underline{\gamma}, \underline{s}) = 0 & \forall \underline{s} \in (L^2(\Omega))^2, \end{cases} \quad (3.10)$$

$\tau$  being defined by (2.9). In the first equation of (3.10) we have set

$$a(\underline{\theta}, w; \underline{\eta}, v) = a(\underline{\theta}, \underline{\eta}) + \mu(\underline{\theta} - \nabla w, \underline{\eta} - \nabla v). \quad (3.11)$$

Let us now introduce a sequence  $\{\mathcal{T}_h\}_{h>0}$  of partitionings of  $\Omega$  into elements  $K$  (triangles or quadrilaterals) whose diameter  $h_K$  is bounded by  $h$ . We also suppose the regularity <sup>17</sup> of  $\{\mathcal{T}_h\}_{h>0}$ , in the sense that there exists a constant  $\sigma > 0$  such that

$$h_K \leq \sigma \rho_K \quad \forall K \in \mathcal{T}_h \quad (3.12)$$

where  $\rho_K$  is the maximum diameter of the circles contained in  $K$ .

A standard discretization of Problem (3.10) consists in choosing finite dimensional spaces  $\Theta_h \subset \Theta$ ,  $W_h \subset W$  and  $\Gamma_h \subset (L^2(\Omega))^2$  and in considering the discrete problem

$$\begin{cases} \text{find } (\underline{\theta}_h, w_h, \underline{\gamma}_h) \text{ in } \Theta_h \times W_h \times \Gamma_h \text{ such that} \\ a(\underline{\theta}_h, w_h; \underline{\eta}, v) + (\underline{\gamma}_h, \underline{\eta} - \nabla v) = (f, v) \quad \forall (\underline{\eta}, v) \in \Theta_h \times W_h \\ (\underline{s}, \underline{\theta}_h - \nabla w_h) - \tau^2 (\underline{\gamma}_h, \underline{s}) = 0 \quad \forall \underline{s} \in \Gamma_h. \end{cases} \quad (3.13)$$

Eliminating the multiplier from system (3.13) and using (2.9) and (3.11), it is seen that our scheme is equivalent to the following displacement-based method

$$\begin{aligned} & \text{find } (\underline{\theta}_h, w_h) \text{ in } \Theta_h \times W_h \text{ such that} \\ & a(\underline{\theta}_h, \underline{\eta}) + \mu(\underline{\theta}_h - \nabla w_h, \underline{\eta} - \nabla v) \\ & + (t^{-2} - \mu)(P_h(\underline{\theta}_h - \nabla w_h), \underline{\eta} - \nabla v) = (f, v) \end{aligned} \quad (3.14)$$

$\forall (\underline{\eta}, v) \in \Theta_h \times W_h$ , where  $P_h$  is the  $L^2$ -projection operator onto the space  $\Gamma_h$ .

Hence, the scheme splits the shear energy term into two parts. The first one is simply

$$\frac{\mu}{2} \|\underline{\theta}_h - \nabla w_h\|_0^2,$$

while the second one is

$$\frac{t^{-2} - \mu}{2} \|P_h(\underline{\theta}_h - \nabla w_h)\|_0^2.$$

Therefore, a part of the shear energy is exactly integrated (to prevent spurious modes), while the remaining part is reduced by the projection operator. Thus, the scheme can be called ‘‘Partial Selective Reduced Integration Scheme’’. In connection with the above technique, a Kinematically Linked Interpolation approach can be used in the discretization procedure<sup>5,7,26,27</sup>. This means that we wish to ‘‘augment’’ the deflection space by means of the rotational degrees of freedom. More precisely, a suitable linear and bounded operator  $L : \Theta_h \rightarrow H_0^1(\Omega)$  is defined and the new finite element space

$$V_h = \left\{ (\underline{\eta}_h, v_h + L\underline{\eta}_h) : \underline{\eta}_h \in \Theta_h, v_h \in W_h \right\} \quad (3.15)$$

is selected to approximate the kinematic unknowns. Hence, the discretized problem turns out to be

$$\begin{aligned} & \text{find } (\underline{\theta}_h, w_h^*; \underline{\gamma}_h) \text{ in } V_h \times \Gamma_h \text{ such that} \\ & \begin{cases} a(\underline{\theta}_h, w_h^*; \underline{\eta}, v) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla} v) = (f, v) & \forall (\underline{\eta}, v) \in V_h \\ (\underline{s}, \underline{\theta}_h - \underline{\nabla} w_h^*) - \tau^2(\underline{\gamma}_h, \underline{s}) = 0 & \forall \underline{s} \in \Gamma_h, \end{cases} \end{aligned} \quad (3.16)$$

where  $w_h^* := w_h + L\underline{\theta}_h$ . Due to (3.15), Problem (3.16) is obviously equivalent to the following

$$\begin{aligned} & \text{find } (\underline{\theta}_h, w_h, \underline{\gamma}_h) \text{ in } \Theta_h \times W_h \times \Gamma_h \text{ such that} \\ & \begin{cases} a_h(\underline{\theta}_h, w_h; \underline{\eta}, v) + (\underline{\gamma}_h, \underline{\eta} - \underline{\nabla}(v + L\underline{\eta})) = (f, v + L\underline{\eta}) & \forall (\underline{\eta}, v) \in \Theta_h \times W_h \\ (\underline{s}, \underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h)) - \tau^2(\underline{\gamma}_h, \underline{s}) = 0 & \forall \underline{s} \in \Gamma_h, \end{cases} \end{aligned} \quad (3.17)$$

where

$$a_h(\underline{\theta}_h, w_h; \underline{\eta}, v) := a(\underline{\theta}_h, \underline{\eta}) + \mu(\underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h), \underline{\eta} - \underline{\nabla}(v + L\underline{\eta})). \quad (3.18)$$

Again, eliminating the multiplier from system (3.17), it is seen that our scheme is equivalent to the following displacement-based method

$$\begin{aligned} & \text{find } (\underline{\theta}_h, w_h) \text{ in } \Theta_h \times W_h \text{ such that} \\ & a(\underline{\theta}_h, \underline{\eta}) + \mu(\underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h), \underline{\eta} - \underline{\nabla}(v + L\underline{\eta})) \\ & + (t^{-2} - \mu)(P_h(\underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h)), \underline{\eta} - \underline{\nabla}(v + L\underline{\eta})) = (f, v + L\underline{\eta}) \end{aligned} \quad (3.19)$$

$\forall (\underline{\eta}, v) \in \Theta_h \times W_h$ , where  $P_h$  is the  $L^2$ -projection operator onto the space  $\Gamma_h$ .

Hence, the scheme splits the shear energy term into two parts. The first one is simply

$$\frac{\mu}{2} \|\underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h)\|_0^2, \quad (3.20)$$

while the second one is

$$\frac{t^{-2} - \mu}{2} \|P_h(\underline{\theta}_h - \underline{\nabla}(w_h + L\underline{\theta}_h))\|_0^2. \quad (3.21)$$

**Remark 3.1** Although the linking operator  $L$  is defined only on  $\Theta_h$ , a natural extension to  $\Theta$  is provided by any interpolation operator  $(\cdot)_{II} : \Theta \longrightarrow \Theta_h$ , simply by setting

$$L\underline{\eta} := L\underline{\eta}_{II} \quad \forall \underline{\eta} \in \Theta.$$

□

We now perform a general error analysis for a method based on formulation (3.17), by means of the following mesh-dependent norm for the Lagrange multiplier:

$$\|\underline{\xi}\|_h^2 := \sum_{K \in \mathcal{T}_h} h_K^2 \|\underline{\xi}\|_{0,K}^2 \quad \forall \underline{\xi} \in (L^2(\Omega))^2 \quad (3.22)$$

Thus, given  $(\underline{\theta}, w, \underline{\xi}) \in \Theta \times W \times (L^2(\Omega))^2$ , we set

$$\|(\underline{\theta}, w, \underline{\xi})\|^2 := \|\underline{\theta}\|_1^2 + \|w\|_1^2 + \|\underline{\xi}\|_h^2 + \tau^2 \|\underline{\xi}\|_0^2. \quad (3.23)$$

Furthermore, let us assume true the following hypothesis involving only the discretization spaces

$$\sup_{(\underline{\eta}, v)} \frac{(\underline{s}, \underline{\eta} - \nabla(v + L\underline{\eta}))}{\|\underline{\eta}\|_1 + \|v\|_1} \geq \beta \|\underline{s}\|_h \quad \forall \underline{s} \in \Gamma_h, \quad (3.24)$$

where  $\beta$  is a positive constant independent of  $h$  and the supremum is taken over the discrete space  $\Theta_h \times W_h$ . In this Section we always suppose that hypothesis (3.24) is fulfilled.

Moreover, let us set

$$\mathcal{A}(\underline{\theta}, w, \gamma; \underline{\eta}, v, \underline{s}) := a(\underline{\theta}, w; \underline{\eta}, v) + (\gamma, \underline{\eta} - \nabla v) - (\underline{s}, \underline{\theta} - \nabla w) + \tau^2 (\gamma, \underline{s}), \quad (3.25)$$

and

$$\begin{aligned} \mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_h, v_h, \underline{s}_h) : &= a_h(\underline{\theta}_h, w_h; \underline{\eta}_h, v_h) + (\underline{\gamma}_h, \underline{\eta}_h - \nabla(v_h + L\underline{\eta}_h)) \\ &- (\underline{s}_h, \underline{\theta}_h - \nabla(w_h + L\underline{\theta}_h)) + \tau^2 (\underline{\gamma}_h, \underline{s}_h). \end{aligned} \quad (3.26)$$

Hence, the continuous Problem (3.10) can be written as

find  $(\underline{\theta}, w, \gamma)$  in  $\Theta \times W \times (L^2(\Omega))^2$  such that

$$\mathcal{A}(\underline{\theta}, w, \gamma; \underline{\eta}, v, \underline{s}) = (f, v) \quad \forall (\underline{\eta}, v, \underline{s}) \in \Theta \times W \times (L^2(\Omega))^2, \quad (3.27)$$

while the discretized one reads as follows

find  $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$  in  $\Theta_h \times W_h \times \Gamma_h$  such that

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}, v, \underline{s}) = (f, v + L\underline{\eta}) \quad \forall (\underline{\eta}, v, \underline{s}) \in \Theta_h \times W_h \times \Gamma_h. \quad (3.28)$$

We are now ready to prove our stability result. The technique used is very similar to the one detailed in <sup>7,15</sup>.

**Proposition 3.1** *Given  $(\underline{\theta}_h, w_h, \underline{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h$ , there exists  $(\underline{\eta}_h, v_h, \underline{s}_h) \in \Theta_h \times W_h \times \Gamma_h$  such that*

$$\| \underline{\eta}_h, v_h, \underline{s}_h \| \leq C \| \underline{\theta}_h, w_h, \underline{\gamma}_h \| \quad (3.29)$$

and

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_h, v_h, \underline{s}_h) \geq C \| \underline{\theta}_h, w_h, \underline{\gamma}_h \|^2. \quad (3.30)$$

**Proof:** Let  $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$  be given in  $\Theta_h \times W_h \times \Gamma_h$ . The proof is performed in two steps.

*i)* Let us first choose  $(\underline{\eta}_1, v_1, \underline{s}_1) \in \Theta_h \times W_h \times \Gamma_h$  such that  $\underline{\eta}_1 = \underline{\theta}_h$ ,  $v_1 = w_h$  and  $\underline{s}_1 = \underline{\gamma}_h$ . It is obvious that

$$\| \underline{\eta}_1, v_1, \underline{s}_1 \| = \| \underline{\theta}_h, w_h, \underline{\gamma}_h \|. \quad (3.31)$$

Furthermore, it holds

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_1, v_1, \underline{s}_1) = a(\underline{\theta}_h, \underline{\theta}_h) + \mu \| \underline{\theta}_h - \nabla(w_h + L\underline{\theta}_h) \|_0^2 + \tau^2 \| \underline{\gamma}_h \|_0^2. \quad (3.32)$$

By Korn's inequality, Poincaré's inequality and a little algebra it follows that

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_1, v_1, \underline{s}_1) \geq C_1 \left( \| \underline{\theta}_h \|_1^2 + \| w_h \|_1^2 + \tau^2 \| \underline{\gamma}_h \|_0^2 \right). \quad (3.33)$$

*ii)* Let us first notice that from (3.24) it follows that there exists  $(\underline{\eta}_2, v_2) \in \Theta_h \times W_h$  such that

$$\| \underline{\eta}_2 \|_1 + \| v_2 \|_1 \leq C \| \underline{\gamma}_h \|_h \quad (3.34)$$

and

$$(\underline{\gamma}_h, \underline{\eta}_2 - \nabla(v_2 + L\underline{\eta}_2)) = \| \underline{\gamma}_h \|_h^2. \quad (3.35)$$

Choose now  $(\underline{\eta}_2, v_2, \underline{s}_2) \in \Theta_h \times W_h \times \Gamma_h$ , such that  $\underline{s}_2 = 0$ . On one hand, due to estimate (3.34), the following inequality holds

$$\| \underline{\eta}_2, v_2, \underline{s}_2 \| \leq C \| \underline{\theta}_h, w_h, \underline{\gamma}_h \|. \quad (3.36)$$

On the other hand

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_2, v_2, \underline{\mathfrak{s}}_2) = a_h(\underline{\theta}_h, w_h; \underline{\eta}_2, v_2) + (\underline{\gamma}_h, \underline{\eta}_2 - \nabla(v_2 + L\underline{\eta}_2)), \quad (3.37)$$

so that, due to (3.35)

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_2, v_2, \underline{\mathfrak{s}}_2) = a_h(\underline{\theta}_h, w_h; \underline{\eta}_2, v_2) + \|\underline{\gamma}_h\|_h^2. \quad (3.38)$$

To control the first term in the right-hand side of equation (3.38), we note that from (3.34)

$$\begin{aligned} a_h(\underline{\theta}_h, w_h; \underline{\eta}_2, v_2) &\geq -\frac{M}{2\delta} (\|\underline{\theta}_h\|_1^2 + \|w_h\|_1^2) - \frac{\delta M}{2} (\|\underline{\eta}_2\|_1^2 + \|v_2\|_1^2) \\ &\geq -\frac{M}{2\delta} (\|\underline{\theta}_h\|_1^2 + \|w_h\|_1^2) - \frac{\delta CM}{2} \|\underline{\gamma}_h\|_h^2. \end{aligned}$$

Taking  $\delta$  sufficiently small, we get

$$\mathcal{A}_h(\underline{\theta}_h, w_h, \underline{\gamma}_h; \underline{\eta}_2, v_2, \underline{\mathfrak{s}}_2) \geq C_2 \|\underline{\gamma}_h\|_h^2 - C_3 (\|\underline{\theta}_h\|_1^2 + \|w_h\|_1^2). \quad (3.39)$$

Now it only suffices to take a suitable linear combination of  $\{(\underline{\eta}_i, v_i, \underline{\mathfrak{s}}_i)\}_{i=1}^2$  so that by (3.31), (3.33), (3.36) and (3.39) it follows that (3.29) and (3.30) hold. The proof is then complete.  $\square$

The stability result just established gives the error estimate as in the following proposition.

**Proposition 3.2** *Let  $(\underline{\theta}, w, \underline{\gamma})$  be the solution of Problem (3.27). Let  $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$  be the solution of the discretized Problem (3.28). Given  $(\underline{\theta}_{II}, w_I, \underline{\gamma}^*) \in \Theta_h \times W_h \times \Gamma_h$  the following error estimate holds*

$$\begin{aligned} &\|\underline{\theta} - \underline{\theta}_h\|_1 + \|w - w_h\|_1 + \|\underline{\gamma} - \underline{\gamma}_h\|_h + \tau \|\underline{\gamma} - \underline{\gamma}_h\|_0 \\ &\leq C (\|\underline{\theta} - \underline{\theta}_{II}\|_1 + \|w - w_I\|_1 + \|\underline{\gamma} - \underline{\gamma}^*\|_{\Gamma'} + \|\underline{\gamma} - \underline{\gamma}^*\|_h + \tau \|\underline{\gamma} - \underline{\gamma}^*\|_0) \\ &+ C \sup_{\underline{\mathfrak{s}} \in \Gamma_h} \frac{(\underline{\mathfrak{s}}, \underline{\theta} - \underline{\theta}_{II} - \nabla(w - w_I - L\underline{\theta}_{II}))}{\|\underline{\mathfrak{s}}\|_h + \tau \|\underline{\mathfrak{s}}\|_0} \\ &+ \sup_{(\underline{\eta}, v) \in \Theta_h \times W_h} \frac{(\underline{\theta} - \underline{\theta}_{II} - \nabla(w - w_I - L\underline{\theta}_{II}), \underline{\eta} - \nabla(v + L\underline{\eta}))}{\|\underline{\eta}\|_1 + \|v\|_1}. \end{aligned} \quad (3.40)$$

In (3.40),  $\Gamma'$  is the dual space of  $\Gamma = H_0(\text{rot}; \Omega)$  <sup>12</sup>.

**Proof:** By Proposition 3.1, given  $(\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*) \in \Theta_h \times W_h \times \Gamma_h$ , there exists  $(\underline{\eta}_h, v_h, \underline{\mathfrak{s}}_h) \in \Theta_h \times W_h \times \Gamma_h$  such that

$$\| \|\underline{\eta}_h, v_h, \underline{\mathfrak{s}}_h\| \| \leq C \| \|\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*\| \| \quad (3.41)$$

and

$$\mathcal{A}_h(\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*; \underline{\eta}_h, v_h, \underline{\mathfrak{s}}_h) \geq C \| \|\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*\| \|^2. \quad (3.42)$$

Hence

$$\begin{aligned} & C \| \|\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*\| \|^2 \\ & \leq a_h(\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I; \underline{\eta}_h, v_h) + (\underline{\gamma}_h - \underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\ & \quad - (\underline{\mathfrak{s}}_h, \underline{\theta}_h - \underline{\theta}_{II} - \underline{\nabla}(w_h - w_I + L(\underline{\theta}_h - \underline{\theta}_{II}))) + \tau^2(\underline{\gamma}_h - \underline{\gamma}^*, \underline{\mathfrak{s}}_h) \\ & = (f, v_h + L\underline{\eta}_h) - a_h(\underline{\theta}_{II}, w_I; \underline{\eta}_h, v_h) - (\underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\ & \quad - (\underline{\mathfrak{s}}_h, -\underline{\theta}_{II} + \underline{\nabla}(w_I + L\underline{\theta}_{II})) - \tau^2(\underline{\gamma}^*, \underline{\mathfrak{s}}_h). \end{aligned} \quad (3.43)$$

But since

$$(f, v_h + L\underline{\eta}_h) = \mathcal{A}(\underline{\theta}, w, \underline{\gamma}; \underline{\eta}_h, v_h + L\underline{\eta}_h, \underline{\mathfrak{s}}_h) \quad (3.44)$$

we obtain

$$\begin{aligned} & C \| \|\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*\| \|^2 \leq a(\underline{\theta} - \underline{\theta}_{II}, \underline{\eta}_h) \\ & \quad + \mu(\underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}), \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\ & \quad + (\underline{\gamma} - \underline{\gamma}^*, \underline{\eta}_h - \underline{\nabla}(v_h + L\underline{\eta}_h)) \\ & \quad - (\underline{\mathfrak{s}}_h, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I + L\underline{\theta}_{II})) + \tau^2(\underline{\gamma} - \underline{\gamma}^*, \underline{\mathfrak{s}}_h). \end{aligned} \quad (3.45)$$

By estimate (3.45) it follows

$$\begin{aligned} & C \| \|\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^*\| \|^2 \\ & \leq C (\|\underline{\theta} - \underline{\theta}_{II}\|_1 + \|\underline{\gamma} - \underline{\gamma}^*\|_{\Gamma'} + \tau\|\underline{\gamma} - \underline{\gamma}^*\|_0) \| \|\underline{\eta}_h, v_h, \underline{\mathfrak{s}}_h\| \| \end{aligned}$$

$$\begin{aligned}
 & + C \|\underline{\eta}_h, v_h, \underline{s}_h\| \left( \sup_{\underline{s} \in \Gamma_h} \frac{(\underline{s}, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}))}{\|\underline{s}\|_h + \tau \|\underline{s}\|_0} \right. \\
 & \left. + \sup_{(\underline{\eta}, v) \in \Theta_h \times W_h} \frac{(\underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}), \underline{\eta} - \underline{\nabla}(v + L\underline{\eta}))}{\|\underline{\eta}\|_1 + \|v\|_1} \right). \quad (3.46)
 \end{aligned}$$

Therefore, from (3.41) one obtains

$$\begin{aligned}
 & \| \|\underline{\theta}_h - \underline{\theta}_{II}, w_h - w_I, \underline{\gamma}_h - \underline{\gamma}^* \| \| \\
 & \leq C (\|\underline{\theta} - \underline{\theta}_{II}\|_1 + \|\underline{\gamma} - \underline{\gamma}^*\|_{\Gamma'} + \tau \|\underline{\gamma} - \underline{\gamma}^*\|_0) \\
 & + C \sup_{\underline{s} \in \Gamma_h} \frac{(\underline{s}, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}))}{\|\underline{s}\|_h + \tau \|\underline{s}\|_0} \\
 & + C \sup_{(\underline{\eta}, v) \in \Theta_h \times W_h} \frac{(\underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}), \underline{\eta} - \underline{\nabla}(v + L\underline{\eta}))}{\|\underline{\eta}\|_1 + \|v\|_1}, \quad (3.47)
 \end{aligned}$$

and estimate (3.40) follows from the triangle inequality.  $\square$

Let us now see how the first supremum in the right-hand side of (3.40) can be treated. Suppose that the continuous solution is sufficiently regular.

Since

$$\tau^2 \underline{\gamma} = \underline{\theta} - \underline{\nabla} w \quad (3.48)$$

we have that

$$\underline{\nabla} L\underline{\theta}_{II} = \tau^2 \underline{\nabla} L\underline{\gamma}_{II} + \underline{\nabla} L(\underline{\nabla} w)_{II}. \quad (3.49)$$

Hence,

$$(\underline{s}, -\underline{\nabla}(w - w_I - L\underline{\theta}_{II})) = (\underline{s}, -\underline{\nabla}(w - w_I - L(\underline{\nabla} w)_{II})) + \tau^2 (\underline{s}, \underline{\nabla} L\underline{\gamma}_{II}). \quad (3.50)$$

It follows that

$$\begin{aligned}
 & (\underline{s}, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II})) \\
 & \leq C \left( h^{-1} \|\underline{\theta} - \underline{\theta}_{II}\|_0 + h^{-1} \|w - \Pi w\|_1 + \tau \|\underline{\nabla} L\underline{\gamma}_{II}\|_0 \right) \\
 & \times (\|\underline{s}\|_h + \tau \|\underline{s}\|_0), \quad (3.51)
 \end{aligned}$$

where we have set

$$\Pi w := w_I + L(\underline{\nabla} w)_{II}. \quad (3.52)$$

By estimate 3.51 we obtain

$$\begin{aligned} & \sup_{\underline{s} \in \Gamma_h} \frac{(\underline{s}, \underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}))}{\|\underline{s}\|_h + \tau \|\underline{s}\|_0} \leq \\ C & \left( h^{-1} \|\underline{\theta} - \underline{\theta}_{II}\|_0 + h^{-1} \|w - \Pi w\|_1 + \tau \|\underline{\nabla} L \underline{\gamma}_{II}\|_0 \right). \end{aligned} \quad (3.53)$$

**Remark 3.2** *The term  $h^{-1} \|\underline{\theta} - \underline{\theta}_{II}\|_0$  causes no trouble, since the  $L^2$ -norm, and not the  $H^1$ -norm, is involved.*

□

The second supremum in (3.40) can be treated by using a similar technique. Let us set

$$\underline{\xi} = \underline{\eta} - \underline{\nabla}(v + L\underline{\eta}). \quad (3.54)$$

By equation (3.49) we obtain

$$(\underline{\xi}, -\underline{\nabla}(w - w_I - L\underline{\theta}_{II})) = (\underline{\xi}, -\underline{\nabla}(w - w_I - L(\underline{\nabla} w)_{II})) + \tau^2 (\underline{\xi}, \underline{\nabla} L \underline{\gamma}_{II}). \quad (3.55)$$

It follows that

$$(\underline{\xi}, -\underline{\nabla}(w - w_I - L\underline{\theta}_{II})) \leq C \left( \|\underline{\nabla}(w - \Pi w)\|_0 + \tau^2 \|\underline{\nabla} L \underline{\gamma}_{II}\|_0 \right) \|\underline{\xi}\|_0. \quad (3.56)$$

Since  $\tau$  is bounded,  $\tau^2 \leq C\tau$  and hence, recalling also (3.54), from (3.56) it follows that

$$\begin{aligned} & \sup_{(\underline{\eta}, v) \in \Theta_h \times W_h} \frac{(\underline{\theta} - \underline{\theta}_{II} - \underline{\nabla}(w - w_I - L\underline{\theta}_{II}), \underline{\eta} - \underline{\nabla}(v + L\underline{\eta}))}{\|\underline{\eta}\|_1 + \|v\|_1} \leq \\ C & \left( \|\underline{\theta} - \underline{\theta}_{II}\|_0 + \|w - \Pi w\|_1 + \tau \|L \underline{\gamma}_{II}\|_1 \right). \end{aligned} \quad (3.57)$$

Finally, supposing we have the necessary regularity, we can conclude that the following Proposition holds

**Proposition 3.3** *Let  $(\underline{\theta}, w, \underline{\gamma})$  be the solution of Problem (3.27). Let  $(\underline{\theta}_h, w_h, \underline{\gamma}_h)$  be the solution of the discretized Problem (3.28). Suppose that condition (3.24) holds. Suppose moreover to have a scalar interpolating operator  $(\cdot)_I$  taking values on  $W_h$ , a vectorial interpolating operator  $(\cdot)_{II}$  taking values on  $\Theta_h$ , and, finally, a vectorial operator  $(\cdot)^*$  taking values on  $\Gamma_h$ . Setting  $\Pi w := w_I + L(\underline{\nabla} w)_{II}$  (cf. (3.52)), the following error estimate holds*

$$\begin{aligned}
 & \|\underline{\theta} - \underline{\theta}_h\|_1 + \|w - w_h\|_1 + \|\underline{\gamma} - \underline{\gamma}_h\|_h + \tau\|\underline{\gamma} - \underline{\gamma}_h\|_0 \\
 \leq & C \left( \|\underline{\theta} - \underline{\theta}_{II}\|_1 + h^{-1}\|\underline{\theta} - \underline{\theta}_{II}\|_0 + \|w - w_I\|_1 + h^{-1}\|w - \Pi w\|_1 \right. \\
 & \left. + \|\underline{\gamma} - \underline{\gamma}^*\|_{\Gamma'} + \|\underline{\gamma} - \underline{\gamma}^*\|_h + \tau(\|\underline{\gamma} - \underline{\gamma}^*\|_0 + |L\underline{\gamma}_{II}|_1) \right). \tag{3.58}
 \end{aligned}$$

□

From estimate (3.58), it is easily seen that the key point stands in the term  $h^{-1}\|w - \Pi w\|_1$ . In fact, just to fix ideas, let us suppose that both  $(\cdot)_{II}$  and  $(\cdot)_I$  are the standard Lagrange interpolation operators onto the continuous piecewise linear functions. Moreover, consider the limit case  $\tau = 0$  (which means also  $t = 0$ ). Then estimate (3.58) provides an  $O(h)$  error bound for the kinematic unknowns if the term  $\|w - \Pi w\|_1$  is  $O(h^2)$ . Hence, we need better approximation features for the operator  $\Pi$  than those met by the Lagrange operator. More precisely, it is natural to require that

$\Pi$  is a  $P_2$ -invariant operator,

so that by Bramble-Hilbert lemma it holds

$$h^{-1}\|w - \Pi w\|_1 \leq Ch|w|_3. \tag{3.59}$$

In general, if  $\|v - v_I\|_1 = O(h^k)$ , we wish to select an operator  $L$  in such a way that  $\|v - \Pi v\|_1 = O(h^{k+1})$ , where  $\Pi$  is defined by (3.52). This is the case of all the elements based on the linking philosophy described in the literature.

#### 4. Some example of elements

The aim of this Section is to present and study some elements designed following the Kinematically Linked Interpolation technique, in connection with the Partial Selective Reduced Integration scheme. We wish to remark that the analyses for the quadrilateral elements are performed only in the rectangular case. Moreover, we will suppose to have enough regularity for the continuous solution.

##### 4.1. A low-order quadrilateral element: Q4-LIMS

The Q4-LIMS element arises from the following selection of discretization spaces:

$$\begin{aligned}
 \Theta_h &= \{ \underline{\eta}_h \in \Theta : \underline{\eta}_{h|K} \in (Q_1(K) \oplus B_4(K))^2 \quad \forall K \in \mathcal{T}_h \} \\
 W_h &= \{ v_h \in W : v_{h|K} \in Q_1(K) \quad \forall K \in \mathcal{T}_h \} \\
 \Gamma_h &= \{ \underline{s}_h \in (L^2(\Omega))^2 : \underline{s}_{h|K} \in (P_0(K))^2 \quad \forall K \in \mathcal{T}_h \}, \tag{4.60}
 \end{aligned}$$

where  $P_0(K)$  is the space of constant functions defined on  $K$ ,  $Q_1(K)$  is the classical isoparametric bilinear function space and  $B_4(K)$  is the space of quartic bubbles defined on  $K$ .

To define the linear operator  $L$ , let us introduce for each  $K \in \mathcal{T}_h$  the functions

$$\varphi_i = \lambda_j \lambda_k \lambda_m. \quad (4.61)$$

In (4.61)  $\{\lambda_i\}_{1 \leq i \leq 4}$  are the equations of the sides of  $K$  and the indices  $(i, j, k, m)$  form a permutation of the set  $(1, 2, 3, 4)$ . We notice that the function  $\varphi_i$  is a sort of edge bubble relatively to the edge  $e_i$  of  $K$ . Let us now set

$$EB(K) = \text{Span} \{\varphi_i\}_{1 \leq i \leq 4}. \quad (4.62)$$

The operator  $L$  is locally defined <sup>6</sup> as

$$(L\underline{\eta}_h)|_K = \sum_{i=1}^4 \alpha_i \varphi_i \in EB(K), \quad (4.63)$$

by requiring that

$$(\underline{\eta}_h - \nabla L\underline{\eta}_h) \cdot \underline{\tau}_i = \text{constant along each } e_i, \quad (4.64)$$

where  $\tau_i$  is the standard unit tangent to the edge  $e_i$ .

It is interesting to observe that discretization spaces adopted in the Q4-LIMS element coincide with the ones used in <sup>29</sup>. However, we remark that in the mentioned reference no Partial Selective Reduced Integration approach has been used, so that the scheme suffers from a spurious zero energy mode, leading to a lack in robustness (cf. also Section 5.3).

We now check that the inf-sup condition (3.24) holds. Given  $\underline{s} \in \Gamma_h$ , let us choose  $\underline{\eta}_h$  such that its restriction on  $K$  is  $(\underline{\eta}_h)|_K = h_K^2 b_{K\underline{s}}$  and select  $v_h = 0$  ( $b_K$  being the standard quartic bubble defined on  $K$ ). Note that this is admissible due to choice (4.60). Since  $L\underline{\eta}_h = 0$  (because  $\underline{\eta}_h$  vanishes along each edge of  $K$ ), it is straightforward to have

$$(\underline{s}, \underline{\eta}_h - \nabla(v_h + L\underline{\eta}_h)) = (\underline{s}, \underline{\eta}_h) \geq C \|\underline{s}\|_h^2 \quad (4.65)$$

and, by a scaling argument,

$$\|\underline{\eta}_h\|_1 \leq C \|\underline{s}\|_h, \quad (4.66)$$

from which the inf-sup condition easily follows.

Let us now study the features of the linear operator  $L$ . Choose both  $(\cdot)_I$  and  $(\cdot)_{II}$  as the usual standard Lagrange interpolation operators onto the continuous piecewise bilinear functions. Using the technique of <sup>22</sup>, it is easily seen that it holds

$$|L\underline{\eta}|_1 \leq Ch |\underline{\eta}|_1 \quad \forall \underline{\eta} \in \Theta_h. \quad (4.67)$$

It follows that

$$|L\underline{\gamma}_{II}|_1 \leq Ch|\underline{\gamma}_{II}|_1 \leq Ch|\underline{\gamma}|_1. \quad (4.68)$$

Moreover, setting

$$\Pi v = v_I + L(\underline{\nabla} v)_{II},$$

we wish to show that the linear operator  $\Pi$  is  $P_2$ -invariant. This allows us to conclude that  $h^{-1}\|w - \Pi w\|_1 = O(h)$  in estimate (3.58). We will actually show a stronger result, proceeding, as usual, locally on each element  $K$ . Let us define the space  $W(K)$  as

$$W(K) = Q_1(K) \oplus EB(K). \quad (4.69)$$

We now check that  $\Pi|_K$  is  $W(K)$ -invariant, and since

$$P_2(K) \subset W(K),$$

our goal will be achieved ( $P_2(K)$  being the space of quadratic functions defined on  $K$ ).

First, take  $v \in Q_1(K)$ ; we have  $v_I = v$ . Hence, for this case we wish to have  $L(\underline{\nabla} v)_{II} = 0$ . Notice that, since  $(\underline{\nabla} v)_{II} \in (Q_1(K))^2$ ,

$$(\underline{\nabla} v)_{II} = \underline{\nabla} v.$$

Moreover, it holds

$$(\underline{\nabla} v)_{II} \cdot \underline{\tau}_i = \text{constant along each } e_i,$$

so that, by unicity, the equation

$$((\underline{\nabla} v)_{II} - \underline{\nabla} L(\underline{\nabla} v)_{II}) \cdot \underline{\tau}_i = \text{constant along each } e_i$$

has the solution  $L(\underline{\nabla} v)_{II} = 0$ .

Next, take  $v \in EB(K)$ . Since  $v$  vanishes at the vertices of  $K$ , we have  $v_I = 0$ . We thus wish to have  $L(\underline{\nabla} v)_{II} = v$  for this second case. It obviously suffices to show that

$$L(\underline{\nabla} \varphi_i)_{II} = \varphi_i. \quad (4.70)$$

Then, suppose to take  $\varphi_i$  such that  $\underline{\tau}_i$  is aligned with the  $x$  axis (the other case can be treated by the same technique). Then  $\underline{\nabla} \varphi_i$  does *not* belong to  $(Q_1(K))^2$ , but we still have

$$\underline{\nabla} \varphi_i \in Q_1(K) \times P_2(K).$$

Hence, even though  $\underline{\nabla} \varphi_i \neq (\underline{\nabla} \varphi_i)_{II}$ , we have

$$(\nabla \varphi_i - (\nabla \varphi_i)_{II}) \cdot \tau_i = 0. \quad (4.71)$$

Consider now the equation in the unknown  $L(\nabla \varphi_i)_{II}$

$$((\nabla \varphi_i)_{II} - \nabla L(\nabla \varphi_i)_{II}) \cdot \tau_i = \text{constant along } e_i. \quad (4.72)$$

On one hand, equation (4.72) has a unique solution. On the other hand, due to equation (4.71), it is equivalent to the equation

$$(\nabla \varphi_i - \nabla L(\nabla \varphi_i)_{II}) \cdot \tau_i = \text{constant along } e_i, \quad (4.73)$$

which has the obvious unique solution  $L(\nabla \varphi_i)_{II} = \varphi_i$ . It follows that also equation (4.72) has the same solution  $L(\nabla \varphi_i)_{II} = \varphi_i$ , and then  $\Pi v = v$  for this second case, too. Thus, we can conclude that  $\Pi$  is actually  $W(K)$ -invariant. Recalling estimate (4.68), by standard interpolation theory and by Proposition 3.3, we have that the method under consideration is first order convergent for the kinematic unknowns, uniformly in the thickness.

#### 4.2. A higher-order quadrilateral element: Q8-LIMS

The Q8-LIMS element arises from the following selection of discretization spaces

$$\begin{aligned} \Theta_h &= \{ \underline{\eta}_h \in \Theta : \underline{\eta}_{h|K} \in (Q_2^r(K))^2 \oplus (P_1(K))^2 b_K \quad \forall K \in \mathcal{T}_h \} \\ W_h &= \{ v_h \in W : v_{h|K} \in Q_2^r(K) \quad \forall K \in \mathcal{T}_h \} \\ \Gamma_h &= \{ \underline{s}_h \in (L^2(\Omega))^2 : \underline{s}_{h|K} \in (P_1(K))^2 \quad \forall K \in \mathcal{T}_h \}, \end{aligned} \quad (4.74)$$

where  $P_1(K)$  is the space of linear functions defined on  $K$ ,  $Q_2^r(K)$  is the classical isoparametric serendipity space and  $b_K$  is the standard bubble function defined on  $K$ .

To define the linear operator  $L$ , let us introduce for each  $K \in \mathcal{T}_h$  the functions

$$\varphi_i = \lambda_{i+1} \lambda_{i+2} \lambda_{i+3} \left( \frac{\lambda_{i+1} - \lambda_{i+3}}{2} \right). \quad (4.75)$$

In (4.75)  $\{\lambda_i\}_{1 \leq i \leq 4}$  are the equations of the sides of  $K$  and the additions within indices should be understood modulus 4. Again, the function  $\varphi_i$  is a sort of edge bubble (this time of fourth degree) relatively to the edge  $e_i$  of  $K$ . Moreover, notice that the function  $\varphi_i$  vanishes at all vertices and midpoints of  $K$  (i.e. at every point associated with the standard Lagrangian basis of the serendipity element). Let us now set

$$EB_4(K) = \text{Span} \{ \varphi_i \}_{1 \leq i \leq 4}. \quad (4.76)$$

The operator  $L$  is locally defined as

$$(L\underline{\eta}_h)|_K = \sum_{i=1}^4 \alpha_i \varphi_i \in EB_4(K), \quad (4.77)$$

by requiring that

$$(\underline{\eta}_h - \underline{\nabla} L\underline{\eta}_h) \cdot \underline{\tau}_i = \text{linear along each } e_i. \quad (4.78)$$

We proceed by studying the features of the operator  $L$ . First, it is seen that the operator  $L$  is well-defined by condition (4.78). In fact, let us notice that each  $\underline{\eta}_h \in \Theta_h$  can be locally (on each rectangle  $K$ ) uniquely split into

$$\underline{\eta}_h = \underline{\eta}_1 + \underline{\eta}_2 + \underline{\eta}_b, \quad (4.79)$$

where  $\underline{\eta}_1$  is the ‘‘bilinear part’’ of  $\underline{\eta}_h$ ,  $\underline{\eta}_b$  is a bubble-type function, while  $\underline{\eta}_2$  is a vector function vanishing at the vertices of  $K$ . Hence, since  $\underline{\eta}_1 \cdot \underline{\tau}_i$  is already linear and  $\underline{\eta}_b \cdot \underline{\tau}_i = 0$ , condition (4.78) leads to solve

$$(\underline{\eta}_2 - \alpha_i \underline{\nabla} \varphi_i) \cdot \underline{\tau}_i = b_i + c_i s \quad \text{along each } e_i, \quad (4.80)$$

where  $s$  is a local coordinate on  $e_i$  running from  $-1$  up to  $1$ . Integrating along  $e_i$  one obtains

$$\int_{e_i} \underline{\eta}_2 \cdot \underline{\tau}_i = b_i |e_i| \quad \text{along each } e_i, \quad (4.81)$$

where  $|e_i|$  is the length of the edge. From (4.81) we get  $b_i$ :

$$b_i = \frac{1}{|e_i|} \int_{e_i} \underline{\eta}_2 \cdot \underline{\tau}_i := \overline{\underline{\eta}_2 \cdot \underline{\tau}_i}. \quad (4.82)$$

To continue, multiply equation (4.80) by  $s$  and integrate along the side. It is easy to see that one has

$$c_i = 0.$$

Therefore

$$\alpha_i \underline{\nabla} \varphi_i \cdot \underline{\tau}_i = \underline{\eta}_2 \cdot \underline{\tau}_i - \overline{\underline{\eta}_2 \cdot \underline{\tau}_i}. \quad (4.83)$$

Notice that equation (4.83) is uniquely solvable in the unknown  $\alpha_i$ . In fact,  $\underline{\nabla} \varphi_i \cdot \underline{\tau}_i$  is an even quadratic polynomial with respect to the local coordinate  $s$  and it has zero mean value on  $e_i$ . Furthermore,  $\underline{\eta}_2 \cdot \underline{\tau}_i$  is an even quadratic polynomial in  $s$ , so that  $\underline{\eta}_2 \cdot \underline{\tau}_i - \overline{\underline{\eta}_2 \cdot \underline{\tau}_i}$  is even in  $s$  and it has zero mean value on  $e_i$ . It follows that  $\underline{\nabla} \varphi_i \cdot \underline{\tau}_i$  and  $\underline{\eta}_2 \cdot \underline{\tau}_i - \overline{\underline{\eta}_2 \cdot \underline{\tau}_i}$  differ by a multiplicative constant  $\alpha_i$ . Moreover, notice that  $L\underline{\eta}_h$  is defined by  $\underline{\eta}_2$  only. Finally, from (4.83) by a scaling argument <sup>22</sup> we get

$$|\alpha_i| \|\underline{\nabla} \varphi_i \cdot \underline{\mathcal{T}}_i\|_{L^\infty(e_i)} \leq C |\underline{\eta}_2|_{1,K}, \quad (4.84)$$

so that

$$|\alpha_i| \leq Ch_K |\underline{\eta}_2|_{1,K}. \quad (4.85)$$

Hence,

$$|L\underline{\eta}_h|_{1,K} \leq C \sum_{i=1}^3 |\alpha_i| \leq Ch_K |\underline{\eta}_2|_{1,K} \leq Ch_K |\underline{\eta}_h - \underline{\eta}_1 - \underline{\eta}_b|_{1,K} \leq Ch_K^2 |\underline{\eta}_h|_{2,K}. \quad (4.86)$$

**Remark 4.3** *The last inequality in (4.86) follows from the following argument. Consider the linear operator  $I_h : \Theta_{h|K} \rightarrow (Q_1(K))^2 \oplus (P_1(K))^2 b_K$ , defined by*

$$I_h \underline{\eta}_h = \underline{\eta}_1 + \underline{\eta}_b, \quad (4.87)$$

where  $\underline{\eta}_h = \underline{\eta}_1 + \underline{\eta}_2 + \underline{\eta}_b$  (cf. (4.79)). It is easily seen that  $I_h$  is  $Q_1(K)$ -invariant, so that the estimate

$$|\underline{\eta}_h - \underline{\eta}_1 - \underline{\eta}_b|_{1,K} \leq Ch_K |\underline{\eta}_h|_{2,K} \quad (4.88)$$

follows from the standard approximation theory.

□

As far as the inf-sup condition (3.24) is concerned, it is easily seen that it is fulfilled, by taking suitable bubble functions on the rotations and using the same strategy as in the previous subsection. It remains to treat the term  $\|w - \Pi w\|_1$ . Let us choose  $(\cdot)_I$  and  $(\cdot)_{II}$  as the standard Lagrange interpolation operators over the continuous piecewise serendipity functions. Setting

$$\Pi v = v_I + L(\underline{\nabla} v)_{II}$$

we wish to have that  $\Pi_K$  is  $W_4(K)$ -invariant, where

$$W_4(K) = Q_2^r(K) \oplus EB_4(K). \quad (4.89)$$

Note that it holds

$$P_3(K) \subset W_4(K),$$

where  $P_3(K)$  is the space polynomials of degree at most three defined on  $K$ .

To begin, choose  $v \in Q_2^r(K)$ ; then  $v_I = v$  and

$$L(\underline{\nabla} v)_{II} = L(\underline{\nabla} v) = 0,$$

since  $\underline{\nabla} v \in ((Q_2^r(K))^2)$  and  $\underline{\nabla} v \cdot \underline{\mathcal{T}}_i$  is already linear. Hence,  $\Pi v = v$  in this case.

Take now  $v \in EB_4(K)$ . We have  $v_I = 0$ . We thus wish to have  $L(\underline{\nabla} v)_{II} = v$  for this second case. We proceed as in the previous subsection. It obviously suffices to show that

$$L(\underline{\nabla} \varphi_i)_{II} = \varphi_i. \quad (4.90)$$

Then, suppose to take  $\varphi_i$  such that  $\underline{\tau}_i$  is aligned with the  $x$  axis (the other case can be treated by the same technique). Then  $\underline{\nabla} \varphi_i$  does *not* belong to  $(Q_2^r(K))^2$ , but we still have

$$\underline{\nabla} \varphi_i \in Q_2^r(K) \times P_3(K).$$

Hence, even though  $\underline{\nabla} \varphi_i \neq (\underline{\nabla} \varphi_i)_{II}$ , we have

$$(\underline{\nabla} \varphi_i - (\underline{\nabla} \varphi_i)_{II}) \cdot \underline{\tau}_i = 0. \quad (4.91)$$

Consider now the equation in the unknown  $(\underline{\nabla} \varphi_i)_{II}$

$$((\underline{\nabla} \varphi_i)_{II} - \underline{\nabla} L(\underline{\nabla} \varphi_i)_{II}) \cdot \underline{\tau}_i = \text{constant along } e_i. \quad (4.92)$$

On one hand, equation (4.92) has a unique solution. On the other hand, due to equation (4.91), it is equivalent to

$$(\underline{\nabla} \varphi_i - \underline{\nabla} L(\underline{\nabla} \varphi_i)_{II}) \cdot \underline{\tau}_i = \text{constant along } e_i, \quad (4.93)$$

which has the obvious unique solution  $L(\underline{\nabla} \varphi_i)_{II} = \varphi_i$ . It follows that also equation (4.92) has the same solution  $L(\underline{\nabla} \varphi_i)_{II} = \varphi_i$ , and then  $\Pi v = v$  for this second case, too. We thus can conclude that  $\Pi$  is actually  $W_4(K)$ -invariant, hence  $P_3(K)$ -invariant also (since  $P_3(K) \subset W_4(K)$ ). Finally, by estimate (4.86)

$$|L\gamma_{II}|_1 \leq Ch^2 |\gamma|_2. \quad (4.94)$$

Recalling estimate (4.68), by standard approximation theory and by Proposition (3.3), we have that the method under consideration is  $O(h^2)$  for the kinematic unknowns, uniformly in the thickness.

### 4.3. A quadratic triangular element: T9-LIMS

The T9-LIMS element arises from the following selection of discretization spaces

$$\begin{aligned} \Theta_h &= \{ \underline{\eta}_h \in \Theta : \underline{\eta}_{h|T} \in (P_2(T))^2 \oplus (P_1(T))^2 b_T \quad \forall T \in \mathcal{T}_h \} \\ W_h &= \{ v_h \in W : v_{h|T} \in P_2(T) \oplus B_3(T) \quad \forall T \in \mathcal{T}_h \} \\ \Gamma_h &= \{ \underline{s}_h \in (L^2(\Omega))^2 : \underline{s}_{h|T} \in (P_1(T))^2 \quad \forall T \in \mathcal{T}_h \}, \end{aligned} \quad (4.95)$$

where  $b_T = 27\lambda_1\lambda_2\lambda_3$ ,  $\lambda_i$ 's being the usual barycentric coordinates on  $T$ . To define the linear operator  $L$ , let us first introduce for each  $T \in \mathcal{T}_h$  the functions

$$\varphi_i = \lambda_j\lambda_k(\lambda_k - \lambda_j), \quad (4.96)$$

where the indices  $(i, j, k)$  form a permutation of the set  $(1, 2, 3)$ .

Let us now set

$$EB_3(T) = \text{Span} \{\varphi_i\}_{1 \leq i \leq 3}. \quad (4.97)$$

The operator  $L$  is locally defined as

$$(L\underline{\eta}_h)|_K = \sum_{i=1}^3 \alpha_i \varphi_i \in EB_3(T), \quad (4.98)$$

by requiring that

$$(\underline{\eta}_h - \underline{\nabla} L\underline{\eta}_h) \cdot \underline{\tau}_i = \text{linear along each } e_i. \quad (4.99)$$

Let us now turn our attention to the features of the operator  $L$ . In a recent paper <sup>7</sup> it has been established the following.

- The linear operator  $L$  is well-defined by condition (4.99) and it is continuous.
- Let us choose as  $(\cdot)_{II}$  the standard Lagrange interpolation operator over the continuous piecewise quadratic functions. Moreover, let  $(\cdot)_I$  be the Lagrange interpolation operator over the space  $W_h$ . Setting

$$\Pi v = v_I + L(\underline{\nabla} v)_{II},$$

we have that  $\Pi$  is  $P_3$ -invariant. Therefore, it holds

$$\|w - \Pi w\|_1 \leq Ch^3|w|_4.$$

Moreover, one has the following estimate

$$|L\underline{\eta}_{II}|_1 \leq Ch^2|\underline{\eta}|_2,$$

for every sufficiently regular vector function  $\underline{\eta}$ .

- The inf-sup condition (3.24) holds, by simply taking suitable bubble functions.

From the standard approximation theory and Proposition 3.3 we can then conclude that for the method under consideration we have the estimate

$$\|\underline{\theta} - \underline{\theta}_h\|_1 + \|w - w_h\|_1 + \|\underline{\gamma} - \underline{\gamma}_h\|_h + \tau\|\underline{\gamma} - \underline{\gamma}_h\|_0 \leq Ch^2, \quad (4.100)$$

where the constant  $C$  above depends on  $|\underline{\theta}|_3$ ,  $|w|_4$  and  $|\underline{\gamma}|_2$ .  $\square$

## 5. Numerical tests

The aim of this Section is to investigate the numerical performances of the interpolating schemes previously described. In particular, the numerical tests considered are designed to:

- investigate the influence of the element thickness  $t$  and of the element dimension  $h$  on the “optimal” splitting factor,
- check the rate of convergence on a model problem,
- show the non-applicability of interpolating schemes presenting zero energy modes through the study of a specific problem, i.e. a unilateral contact problem.

The interpolating schemes have been implemented into the Finite Element Analysis Program (FEAP) <sup>28</sup>.

### 5.1. The choice of the splitting factor

We wish to numerically investigate the sensitivity of the proposed methods with respect to the splitting parameter  $\mu$ . We recall that our methods are based on the minimization of the functional

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \mathcal{C} \mathcal{E} \underline{\theta}_h : \mathcal{E} \underline{\theta}_h + \frac{\lambda \mu}{2} \int_{\Omega} |\underline{\theta}_h - \nabla (w_h + L \underline{\theta}_h)|^2 \\ & + \frac{\lambda(t^{-2} - \mu)}{2} \int_{\Omega} |P_h(\underline{\theta}_h - \nabla (w_h + L \underline{\theta}_h))|^2 - \int_{\Omega} f(w_h + L \underline{\theta}_h) \end{aligned} \quad (5.101)$$

over the space  $V_h = \Theta_h \times W_h$ . Furthermore, we recall that in (5.101)  $P_h$  is the  $L^2$ -projection operator on the space  $\Gamma_h$ . Notice that all the methods presented in this paper take advantage of bubble functions for the rotational field. This means that it is possible to split  $\underline{\theta}_h$  as

$$\underline{\theta}_h = \underline{\theta}_1 + \underline{\theta}_b, \quad (5.102)$$

where  $\underline{\theta}_b$  is a bubble-type function and  $\underline{\theta}_1$  is determined by the values it takes at the global interpolation nodes of the mesh. Using the procedure of static condensation <sup>8,24</sup> for  $\underline{\theta}_b$ , it is possible to give an equivalent formulation of the plate problem in which only  $\underline{\theta}_1$  appears in the energy functional, as far as the approximated rotations are concerned. Although we will not detail the cumbersome calculations of such a procedure, we wish to notice that the main effect of the static condensation is to change the local shear energy term

$$\frac{\lambda(t^{-2} - \mu)}{2} \int_K |P_h(\underline{\theta}_h - \nabla (w_h + L \underline{\theta}_h))|^2 = \frac{\lambda \tau^{-2}}{2} \int_K |P_h(\underline{\theta}_h - \nabla (w_h + L \underline{\theta}_h))|^2 \quad (5.103)$$

into a term whose structure is essentially the following

$$\frac{\lambda}{2} \frac{1}{(\tau^2 + C_K h_K^2)} \int_K |P_h(\underline{\theta}_1 - \nabla(w_h + L\underline{\theta}_1))|^2, \quad (5.104)$$

$C_K$  being a positive constant. Looking at (5.104), it is seen that a consequence of the static condensation is that the choice  $\tau = 0$  (which also means  $t = 0$ ) is allowable.

Turning back to functional (5.101), it should be clear that if one chooses  $\mu$  numerically small, the effect of the Partial Selective Reduced Integration tends to vanish, leading to possible undesirable zero energy modes. On the other hand, choosing  $\mu$  close to  $t^{-2}$  essentially means exactly integrating the whole shear energy, so that the methods might become too stiff. We define the optimal splitting factor,  $\mu^{opt}$ , as the splitting factor which minimizes the condition number  $\kappa$ , where:

$$\kappa = \lambda_{max}/\lambda_{min}, \quad (5.105)$$

with  $\lambda_{max}$  and  $\lambda_{min}$  the maximum and the minimum eigenvalues of a single-element stiffness matrix of the equivalent displacement methods after condensation of the bubble-type rotations and after elimination of the rigid body modes. In practice we take as representative element a square (or an equilateral triangular) element with sides of length  $h_m$ , where  $h_m$  is a sort of ‘‘average size’’ of the elements of the given decomposition. This makes sense if the decomposition consists of elements essentially of the same order of magnitude, as it is the case in the experiments presented in this paper. Otherwise, one should think of using different  $\mu$ 's in different regions of the plate.

*We remark that the definition of the optimal splitting parameter given above is just motivated by a heuristic conjecture, only supported by the idea that a good condition number for the stiffness matrix is an important feature in the discretization procedure. A completely satisfactory justification (and comprehension) on how the optimal splitting parameter behaves is still missing.*

The element geometry is shown in Figure 1, while the elastic material properties are set equal to  $E = 10.92$  and  $\nu = 0.25$ .

Figures 2-4 show the dependency of  $\kappa$  on the plate thickness while keeping constant the element dimension ( $h_m = 1$ ), choosing  $t \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 0\}$ . Figures 5-7 show the dependency of  $\kappa$  on the element dimension while keeping constant the plate thickness ( $t = 10^{-3}$ ), choosing  $h_m \in \{1, 10^{-1}, 10^{-2}, 10^{-3}\}$ . For all the interpolation schemes considered the following observations can be made:

- For low values of the splitting factor ( $\mu \rightarrow 0$ ) the condition number  $\kappa$  grows. In fact, the formulation reduces the shear energy term too much and  $\lambda_{min} \rightarrow 0$ , showing the presence of at least one eigenvalue associated to a zero energy mode and clearly indicating a lack of robustness (see also Section ).
- For high values of the splitting factor ( $\mu \rightarrow t^{-2}$ ) the condition number  $\kappa$  grows. In fact, the formulation tends to reproduce a standard displacement based approach for which  $\lambda_{max}$  grows, due to the presence of locking effects.

- For intermediate values of the splitting factor it is possible to control the zero energy modes and at the same time to avoid locking, resulting in an acceptable condition number  $\kappa$ .
- The optimal splitting factor does not depend on the plate thickness while keeping fixed the element dimension. In particular, for all the interpolating schemes investigated herein  $\mu_{h_m=1}^{opt} \approx 10$ .
- The optimal splitting factor does depend on the element dimension while keeping fixed the plate thickness. In particular, for all the interpolating schemes investigated herein  $\mu_{t=10^{-3}}^{opt}$  seems to vary as  $1/h_m^2$ . We remark that choosing the splitting parameter  $\mu$  dependent on  $h_m$  (and hence on the meshsize  $h$ ) modifies the mathematical analysis of Section 3. However, an analysis (developed in an abstract framework) for the case  $\mu = \mu(h)$  has been presented in the paper <sup>11</sup>.

## 5.2. Rate of convergence

The convergence rate of the proposed schemes is now checked on a model problem for which the exact solution is explicitly known <sup>16</sup>. The model problem consists in a clamped square plate  $\Omega = (0, 1) \times (0, 1)$ , subject to the transverse load

$$f(x, y) = \frac{E}{12(1-\nu^2)} \left[ \begin{aligned} &12y(y-1)(5x^2-5x+1) \\ &\left(2y^2(y-1)^2 + x(x-1)(5y^2-5y+1)\right) \\ &+12x(x-1)(5y^2-5y+1) \\ &\left(2x^2(x-1)^2 + y(y-1)(5x^2-5x+1)\right) \end{aligned} \right]. \quad (5.106)$$

The exact solution is given by

$$\theta_1(x, y) = y^3(y-1)^3 x^2(x-1)^2(2x-1), \quad (5.107)$$

$$\theta_2(x, y) = x^3(x-1)^3 y^2(y-1)^2(2y-1), \quad (5.108)$$

$$w(x, y) = \frac{1}{3}x^3(x-1)^3 y^3(y-1)^3 \quad (5.109)$$

$$-\frac{2t^2}{5(1-\nu)} \left[ \begin{aligned} &y^3(y-1)^3 x(x-1)(5x^2-5x+1) \\ &+x^3(x-1)^3 y(y-1)(5y^2-5y+1) \end{aligned} \right].$$

The error of a discrete solution is measured through the discrete relative rotation error  $E_\theta$  and the discrete relative displacement error  $E_w$ , defined as

$$E_\theta^2 = \frac{\sum_{N_i} \left[ (\theta_{h1}(N_i) - \theta_1(N_i))^2 + (\theta_{h2}(N_i) - \theta_2(N_i))^2 \right]}{\sum_{N_i} \left[ (\theta_1(N_i))^2 + (\theta_2(N_i))^2 \right]}, \quad (5.110)$$

$$E_w^2 = \frac{\sum_{N_i} (w_h(N_i) - w(N_i))^2}{\sum_{N_i} w(N_i)^2}. \quad (5.111)$$

For simplicity, the summations are performed on all the nodes  $N_i$  relative to global interpolation parameters (that is, the internal parameters associated with bubble functions are neglected). The above error measures can also be seen as discrete  $L^2$ -type errors and a  $h^{k+1}$  convergence rate in  $L^2$  norm actually means a  $h^k$  convergence rate in the  $H^1$  energy-type norm.

The analyses are performed using regular meshes and discretizing only one quarter of the plate, due to symmetry considerations. Different values of thickness are considered, i.e.  $t \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-5}, 0\}$ .

Figures 8-13 show the relative rotation error  $E_\theta$  and the relative displacement error  $E_w$  versus the number of nodes per side for the three interpolation schemes considered using a constant splitting factor  $\mu = 10$ .

It is interesting to observe that:

- All the methods show the appropriate convergence rate for both rotations and vertical displacements (slopes showing the  $L^2$  interpolation errors are also reported in the figures).
- All the methods are almost insensitive to the thickness, in such a way that the error graphs for different choices of  $t$  are very close to each other. As a consequence, all the proposed elements are actually locking-free and they can be used for both thick and thin plate problems.

### 5.3. Inadequacy of schemes showing zero energy modes

A square plate supported on a unilateral simply supported boundary (i.e.  $w \leq 0$ ) is now considered. The unilateral condition may be imposed for example using an augmented lagrangian formulation, as discussed in <sup>4</sup>; in particular, the attention is focused on the first solution step, which basically coincide with the case of a bilateral support imposed through a penalty method.

The plate has a side  $L = 1$ , it is subjected to a pointwise load  $F = 1$  applied at the plate center. The material properties are:  $E = 10.92$ ,  $\nu = 0.25$ . Due to symmetry, only a quarter of the plate is considered.

The computations are performed using the Q4-LIMS scheme with  $\mu = 10$  and with  $\mu = 0$ . The latter case coincides with the element proposed by Zienkiewicz et al. in <sup>29</sup> and shows a zero energy mode. For comparison purposes, the problem is solved also using the Q4-LIM element <sup>6</sup>, which is a performing and stable element.

Figure 14 reports the transversal displacement along the boundary. The oscillating solution obtained by the interpolation scheme presenting a zero energy mode can be noted; this clearly implies the impossibility of using such a scheme in many problems and it is a clear sign of non robustness. The smooth solution obtained through the splitting approach and its good match with the one obtained through the Q4-LIM scheme are also noted.

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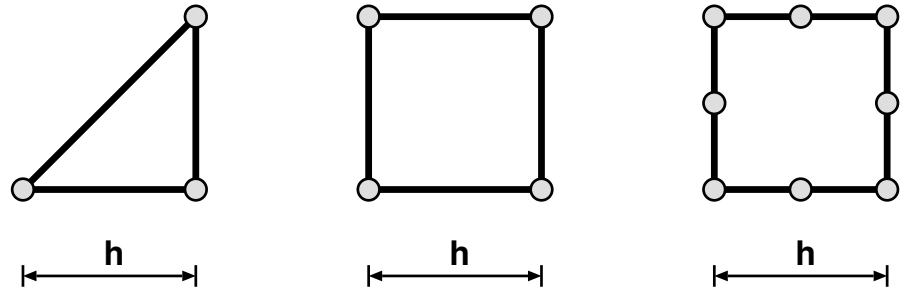


Figure 1: Element geometry.

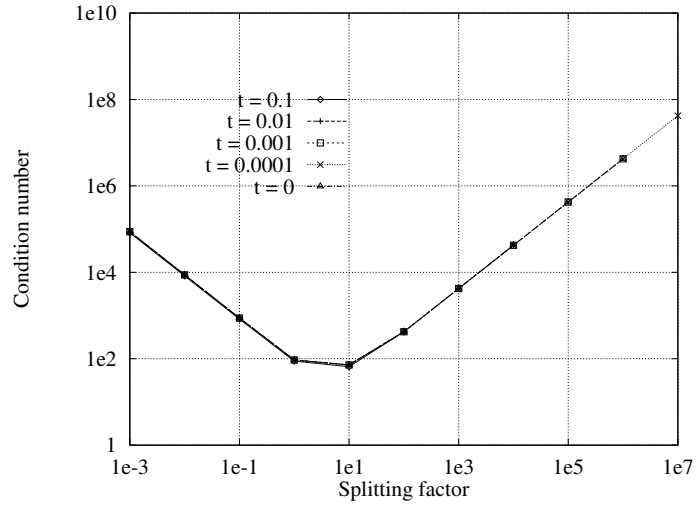


Figure 2: Q4-LIMS element. Condition number ( $\kappa = \lambda_{max}/\lambda_{min}$ ) versus splitting factor ( $\mu$ ) for different thicknesses ( $t$ ) and constant element dimension ( $h_m = 1$ ). It can be observed that the optimal splitting factor does not depend on the thickness.

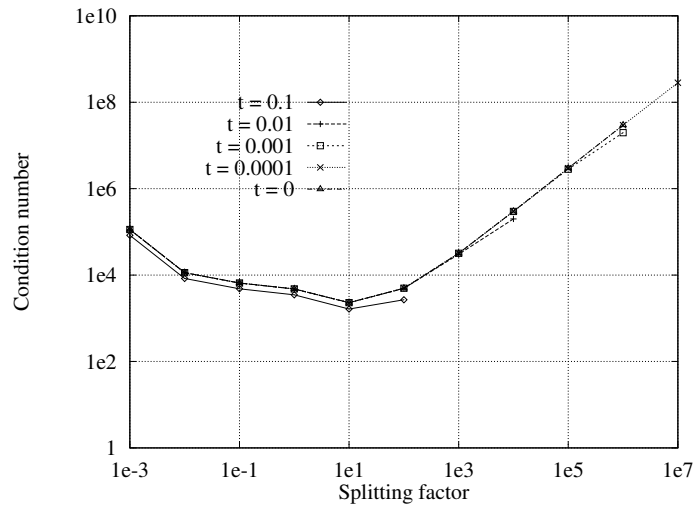


Figure 3: T9-LIMS element. Condition number ( $\kappa = \lambda_{max}/\lambda_{min}$ ) versus splitting factor ( $\mu$ ) for different thicknesses ( $t$ ) and constant element dimension ( $h_m = 1$ ). It can be observed that the optimal splitting factor does not depend on the thickness.

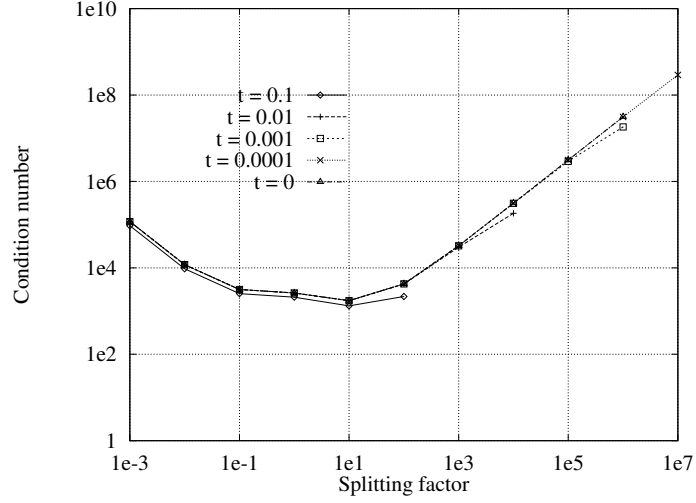


Figure 4: Q8-LIMS element. Condition number ( $\kappa = \lambda_{max}/\lambda_{min}$ ) versus splitting factor ( $\mu$ ) for different thicknesses ( $t$ ) and constant element dimension ( $h_m = 1$ ). It can be observed that the optimal splitting factor does not depend on the thickness.

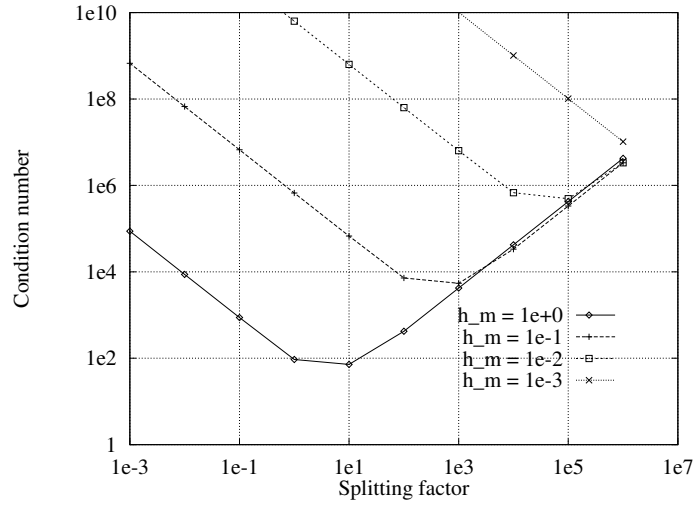


Figure 5: Q4-LIMS element. Condition number ( $\kappa = \lambda_{max}/\lambda_{min}$ ) versus splitting factor ( $\mu$ ) for different element dimensions ( $h_m$ ) and constant thickness ( $t = 10^{-3}$ ). It can be observed that the optimal splitting factor depends on the element dimension and it varies as  $1/h_m^2$ .

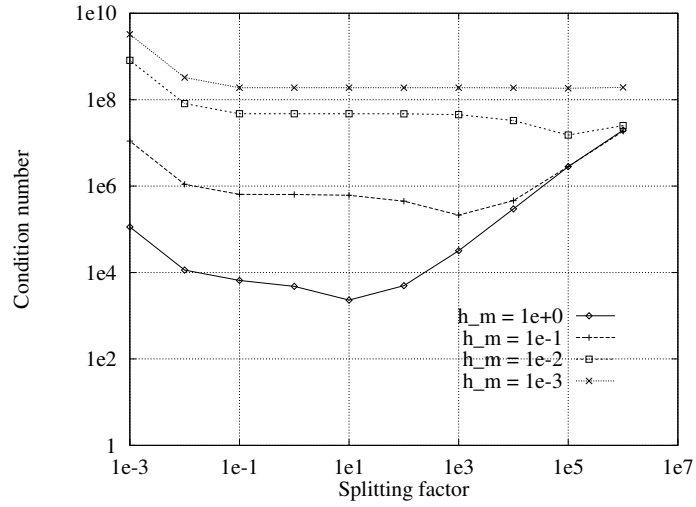


Figure 6: T9-LIMS element. Condition number ( $\kappa = \lambda_{max}/\lambda_{min}$ ) versus splitting factor ( $\mu$ ) for different element dimensions ( $h_m$ ) and constant thickness ( $t = 10^{-3}$ ). It can be observed that the optimal splitting factor depends on the element dimension and it varies as  $1/h_m^2$ .

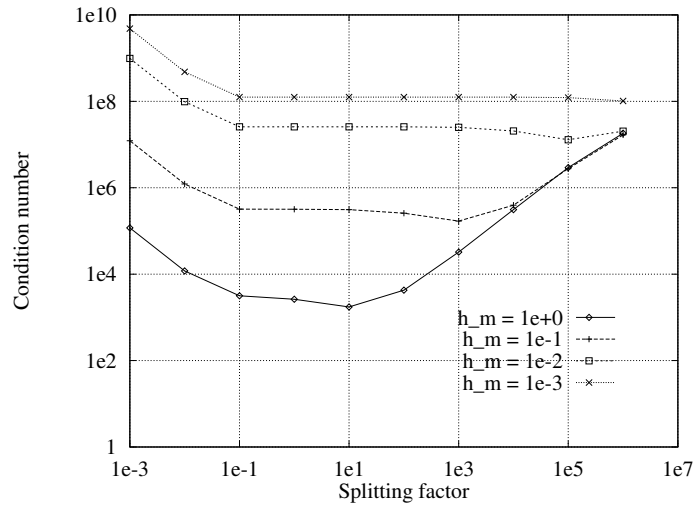


Figure 7: Q8-LIMS element. Condition number ( $\kappa = \lambda_{max}/\lambda_{min}$ ) versus splitting factor ( $\mu$ ) for different element dimensions ( $h_m$ ) and constant thickness ( $t = 10^{-3}$ ). It can be observed that the optimal splitting factor depends on the element dimension and it varies as  $1/h_m^2$ .

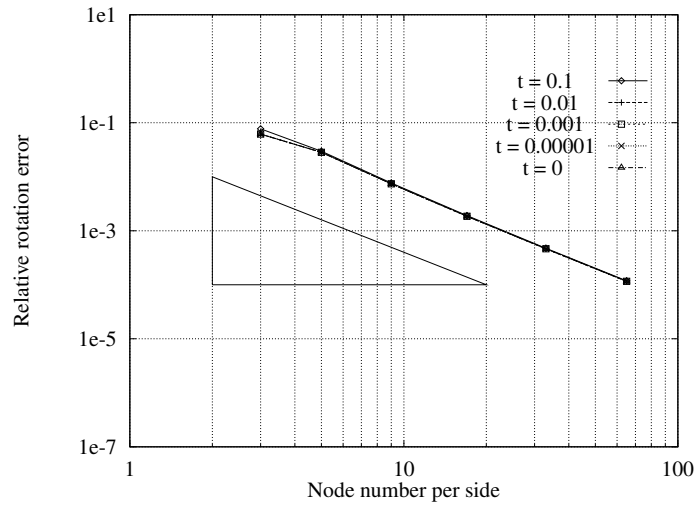


Figure 8: Q4-LIMS element. Relative rotation error versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^2$  convergence rate in the  $L^2$  error norm, corresponding to a  $h$  convergence rate in the  $H^1$  energy-type norm.

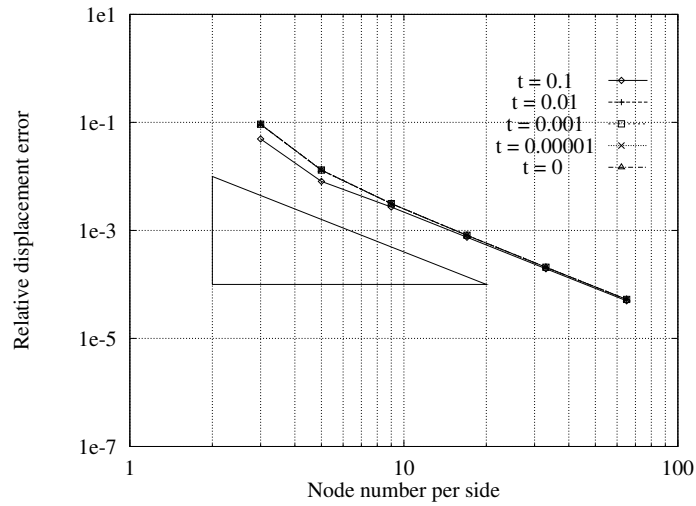


Figure 9: Q4-LIMS element. Relative deflection error versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^2$  convergence rate in the  $L^2$  error norm, corresponding to a  $h$  convergence rate in the  $H^1$  energy-type norm.

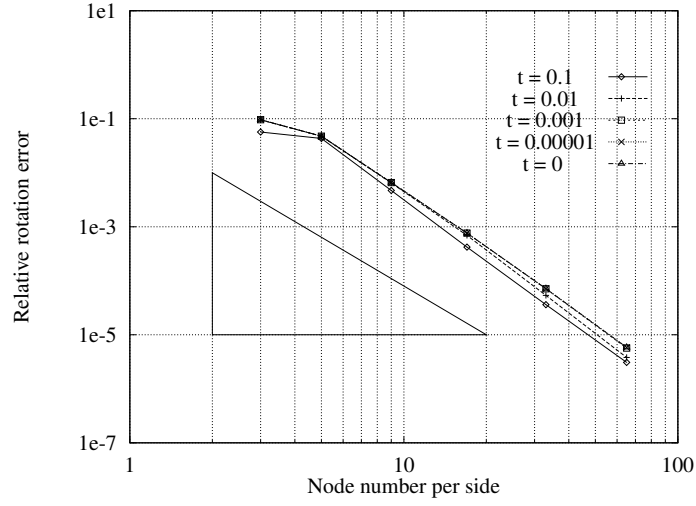


Figure 10: T9-LIMS element. Relative rotation error versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^3$  convergence rate in the  $L^2$  error norm, corresponding to a  $h^2$  convergence rate in the  $H^1$  energy-type norm.

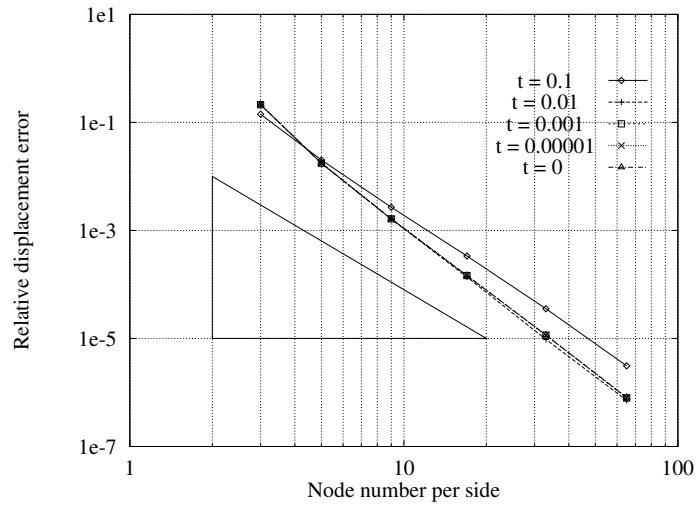


Figure 11: T9-LIMS element. Relative deflection error versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^3$  convergence rate in the  $L^2$  error norm, corresponding to a  $h^2$  convergence rate in the  $H^1$  energy-type norm.

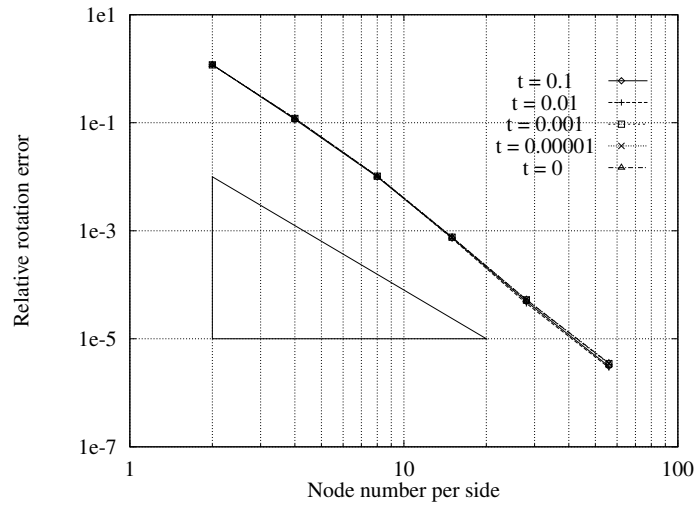


Figure 12: Q8-LIMS element. Relative rotation error versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^3$  convergence rate in the  $L^2$  error norm, corresponding to a  $h^2$  convergence rate in the  $H^1$  energy-type norm.

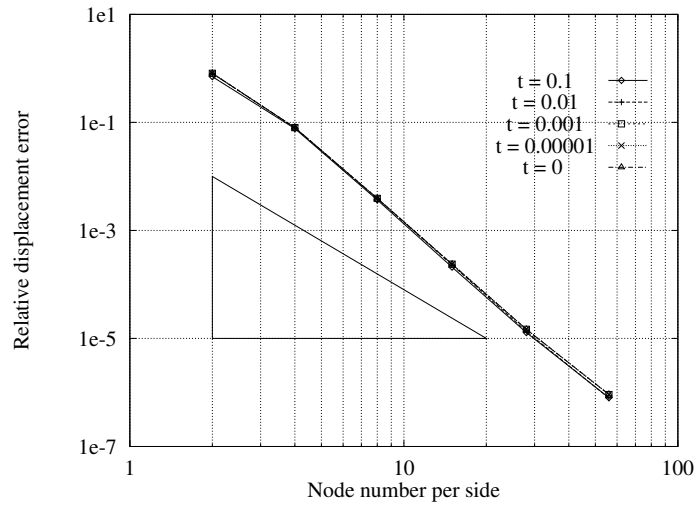


Figure 13: Q8-LIMS element. Relative deflection error versus number of nodes per side for different values of thickness. It can be observed the attainment of the  $h^3$  convergence rate in the  $L^2$  error norm, corresponding to a  $h^2$  convergence rate in the  $H^1$  energy-type norm.

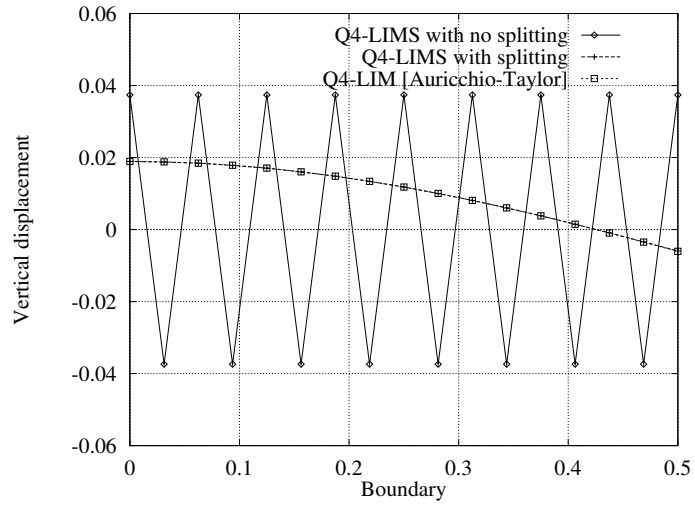


Figure 14: Unilateral contact problem. Displacement along the contact boundary. It can be observed that the presence of zero energy modes show up making impossible the solution search.