

# A SHELL CLASSIFICATION BY INTERPOLATION

Claudio BAIOCCHI  
Dipartimento di Matematica “Guido Castelnuovo”  
Università di Roma “La Sapienza”, Italy

Carlo LOVADINA\*  
Dipartimento di Matematica  
Università di Pavia, Italy

## Abstract

The shell problem and its asymptotic are investigated. A connection between the asymptotic behavior and real Interpolation Theory is established. Thus, a detailed study of the cases when neither the bending energy nor the membrane energy dominate is provided. An application to a cylindrical shell is also detailed. Although only the Koiter shells have been considered, the same procedure can be used for other models, such as Naghdi’s one, for example.

## 1 Introduction

In treating the shell problem with thickness  $\varepsilon$  (cf. [10]), one is led to consider the problem

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{U} \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in \mathcal{U} . \end{cases} \quad (1)$$

Above  $a^m(\cdot, \cdot)$  is the membrane bilinear form,  $a^b(\cdot, \cdot)$  the bending bilinear form and  $\mathcal{U}$  is the admissible displacement space, which also takes into account the kinematical boundary conditions imposed to the structure. The different scaling of the forms involved (the first proportional to  $\varepsilon$  and the second proportional to  $\varepsilon^3$ ) causes great difficulties in studying the asymptotic behavior of the solution as  $\varepsilon \rightarrow 0$ . It has been soon realized that there are at least two main situations for

---

\*Corresponding author. Address:

Carlo Lovadina  
Dipartimento di Matematica, Università di Pavia  
Via Ferrata 1, I-27100, Italy.  
E-mail: lovadina@dimat.unipv.it

the shell response. An instance for the first case is provided by a cylindrical shell of semicircular cross-section, which is clamped along one rectilinear side and is loaded by a uniform radial pressure. Here, the bending energy is much greater than the membrane energy when the thickness is small compared to the other characteristic lengths of the shell, and the limit problem can be identified as a constrained problem, the constraint being that the limit solution has to be an inextensional displacement. Instead, if we consider a hemispherical shell loaded by a uniform radial pressure and clamped along the whole boundary, the membrane energy dominates upon the bending one, and the limit problem is a standard second order elliptic problem, related to a membrane-type problem. A first shell classification has been proposed by engineers: problem (1) is considered, then a comparison between the membrane energy  $\frac{\varepsilon}{2}a^m(u_\varepsilon, u_\varepsilon)$  and the bending energy  $\frac{\varepsilon^3}{2}a^b(u_\varepsilon, u_\varepsilon)$  is performed. The shell is called *membrane dominated* if  $\frac{\varepsilon}{2}a^m(u_\varepsilon, u_\varepsilon) \gg \frac{\varepsilon^3}{2}a^b(u_\varepsilon, u_\varepsilon)$ , while it is called *bending dominated* if  $\frac{\varepsilon^3}{2}a^b(u_\varepsilon, u_\varepsilon) \gg \frac{\varepsilon}{2}a^m(u_\varepsilon, u_\varepsilon)$ . Notice that this kind of shell classification is not rigorous, since the meaning of *much greater than* is rather cloudy. Another classification has been proposed by Ciarlet (cf. [10]). It makes an essential use of the inextensional displacement space defined by

$$\mathcal{U}_1 = \left\{ v \in \mathcal{U}, a^m(v, w) = 0, \quad \forall w \in \mathcal{U} \right\}. \quad (2)$$

More precisely, a shell is called *membrane dominated* if  $\mathcal{U}_1 = (0)$ , while *bending dominated* if  $\mathcal{U}_1 \neq (0)$ . This terminology is justified by the fact that

- if  $\mathcal{U}_1 = (0)$ , then typically (but not always)  $\frac{\varepsilon}{2}a^m(u_\varepsilon, u_\varepsilon) \gg \frac{\varepsilon^3}{2}a^b(u_\varepsilon, u_\varepsilon)$ , and the membrane-type limit problem is recovered by scaling the load  $f$  in a way proportional to  $\varepsilon$ ;
- if  $\mathcal{U}_1 \neq (0)$ , then typically (but not always)  $\frac{\varepsilon^3}{2}a^b(u_\varepsilon, u_\varepsilon) \gg \frac{\varepsilon}{2}a^m(u_\varepsilon, u_\varepsilon)$ , and the bending-type limit problem is recovered by scaling the load  $f$  in a way proportional to  $\varepsilon^3$ .

A third classification has been given by Sanchez-Palencia (cf. [15], [16] and [17]), where the spectral properties of the linear operator associated with the membrane bilinear form plays a crucial role. Recently, a different classification has been proposed by Blouza, Brezzi and Lovadina (cf. [5] and [6]). First, the *scaled* problem is considered

$$\begin{cases} \text{Find } u_\varepsilon(\beta) \in \mathcal{U} \text{ such that} \\ \varepsilon a^m(u_\varepsilon(\beta), v) + \varepsilon^3 a^b(u_\varepsilon(\beta), v) = \varepsilon^\beta \langle f, v \rangle \quad \forall v \in \mathcal{U}, \end{cases} \quad (3)$$

where  $\beta$  is a real parameter. Then, the *scaled* elastic energy, namely

$$E(\varepsilon, \beta) := \varepsilon^{1-\beta} a^m(u_\varepsilon(\beta), u_\varepsilon(\beta)) + \varepsilon^{3-\beta} a^b(u_\varepsilon(\beta), u_\varepsilon(\beta)) , \quad (4)$$

is investigated. The classification criterium is thus based on a sort of *proper choice of  $\beta$* , which leads to a nice behavior of the scaled elastic energy. We remark that in the cases when the right choice is  $\beta = 1$  (resp.:  $\beta = 3$ ), then the shell is membrane dominated (resp.: bending dominated) [referring to the engineering terminology]. A nice feature of this latter shell classification is that it is capable to capture the so called *intermediate states* (cf. [14]), i.e. those situations when neither the bending energy nor the membrane energy dominate, characterized by a proper choice of  $\beta$  with  $1 < \beta < 3$ .

The main aim of this paper is to go deeper into the classification based on the behavior of the scaled elastic energy, employing the real Interpolation Theory (cf. [2] and [12]). Furthermore, we will study the intermediate states in details. In particular, we will see that there is a strict relationship between the right choice of  $\beta$  and the regularity of the datum  $f$ , regularity which, in turn, can be measured by means of the real Interpolation Theory. The paper is organized as follows. In Section 2 we briefly recall the Koiter shell problem, we give the notation used in the sequel, and we introduce our classification criterium (based on the concept of *problem order*). In Section 3 we review some well-known results about the asymptotic of a shell. We remark that these are essentially the cases of a bending dominated state and a membrane dominated state. Section 4 is the core of the paper. In the first subsection, we prove a rather abstract result which establishes a connection between the asymptotics and the regularity of the datum. In the second subsection we apply this result to our classification criterium, thus recognizing that the problem order can be recovered by the feature of  $f$  to belong in some Interpolation spaces. Section 5 provides a result concerning the behavior of the percentage of the total elastic energy which asymptotically goes in the bending part, answering in a positive way a conjecture raised by Sanchez-Palencia. It appears that under some assumptions (cf. Theorem 5.1), the above-mentioned ratio is related to the problem order, and thus to the regularity of  $f$ . Finally, as an application of our theory, in Section 6 we provide a detailed analysis of a cylindrical shell of intermediate order, already considered in [14].

## 2 The shell problem and the classification basic criterium

Let  $(e_1, e_2, e_3)$  be the usual orthonormal basis for the Euclidean space  $\mathbf{R}^3$  equipped with the standard inner product. We denote by  $u \cdot v$  the inner product between two vectors in  $\mathbf{R}^3$ , by  $|u| = \sqrt{u \cdot u}$  the associated norm and by  $u \wedge v$  the exterior product between them.

In the sequel, Greek subscripts will take their values in the set  $\{1, 2\}$  while Latin subscripts will take their values in the set  $\{1, 2, 3\}$ . We will also employ the Einstein summation convention.

Let  $\omega$  be a Lipschitz domain in  $\mathbf{R}^2$ . We consider a shell with midsurface  $S = \varphi(\bar{\omega})$ , where  $\varphi \in W^{2,\infty}(\omega; \mathbf{R}^3)$  is an injective map such that the vectors  $a_\alpha = \partial_\alpha \varphi$  are linearly independent at each point of  $\bar{\omega}$ . We define the unit normal vector to the surface at point  $\varphi(x)$  as  $a_3 = a_1 \wedge a_2 / |a_1 \wedge a_2|$ .

We recall that the vectors  $a_i(x)$  define the covariant basis at  $\varphi(x)$ . The contravariant basis  $a^j$  is defined by the relations  $a_i \cdot a^j = \delta_i^j$  (note that  $a^3 = a_3$ ); one has  $a_i \in W^{1,\infty}(\omega; \mathbf{R}^3)$  and  $a^i \in W^{1,\infty}(\omega; \mathbf{R}^3)$ . We furthermore set  $a(x) = |a_1(x) \wedge a_2(x)|^2$ .

The Christoffel symbols of the surface are given by  $\Gamma_{\alpha\beta}^\rho = \Gamma_{\beta\alpha}^\rho = a^\rho \cdot \partial_\beta a_\alpha$  and it holds  $\Gamma_{\alpha\beta}^\rho \in L^\infty(\omega)$ . We suppose that the shell is clamped along a part  $\gamma_0$  of its boundary  $\partial\omega$ .

Let us introduce the following Hilbert space (cf. [4]):

$$\mathcal{U} = \left\{ v \in H^1(\omega; \mathbf{R}^3), \partial_{\alpha\beta} v \cdot a_3 \in L^2(\omega), v = \partial_\alpha v \cdot a_3 = 0 \text{ on } \gamma_0 \right\} \quad (5)$$

equipped with the norm

$$\|v\|_{\mathcal{U}} = (\|v\|_{H^1(\omega; \mathbf{R}^3)}^2 + \sum_{\alpha, \beta} \|\partial_{\alpha\beta} v \cdot a_3\|_{L^2(\omega)}^2)^{\frac{1}{2}}. \quad (6)$$

The Koiter problem for a shell of thickness  $\varepsilon$  (cf. [4] and [10]) is

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{U} \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in \mathcal{U}, \end{cases} \quad (7)$$

where

$$a^m(u, v) = \int_\omega a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(u) \gamma_{\rho\sigma}(v) \sqrt{a} \, dx \quad (8)$$

and

$$a^b(u, v) = \frac{1}{12} \int_\omega a^{\alpha\beta\rho\sigma} \Upsilon_{\alpha\beta}(u) \Upsilon_{\rho\sigma}(v) \sqrt{a} \, dx. \quad (9)$$

In (8) and (9),  $a^{\alpha\beta\rho\sigma} \in W^{1,\infty}(\omega)$  is an elasticity tensor satisfying the usual symmetry and positive-definiteness conditions,

$$\gamma_{\alpha\beta}(u) = \frac{1}{2} (\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha) \in L^2(\omega) \quad (10)$$

is the deformation tensor, while

$$\Upsilon_{\alpha\beta}(u) = (\partial_{\alpha\beta} u - \Gamma_{\alpha\beta}^\rho \partial_\rho u) \cdot a_3 \in L^2(\omega) \quad (11)$$

is the change of curvature tensor, both given by their covariant components. Note that  $a^m(\cdot, \cdot)$  is related to the elastic membrane energy, while  $a^b(\cdot, \cdot)$  to the elastic bending energy. Finally,  $f \in \mathcal{U}'$ , the topological dual space of  $\mathcal{U}$ . In what follows, we will always suppose that  $f \neq 0$ . We also remark that if  $f$  is more regular, say  $f \in L^2(\omega)$ , then the duality pairing is an integral, i.e. it holds

$$\langle f, v \rangle = \int_\omega f \cdot v \sqrt{a} \, dx. \quad (12)$$

It is well known that the Koiter problem (7) has a unique solution  $u_\varepsilon \in \mathcal{U}$  (see [4] and [10]). We recall (see [16]) that

$$\mathcal{U}_1 = \left\{ v \in \mathcal{U}, a^m(v, w) = 0, \quad \forall w \in \mathcal{U} \right\} = \left\{ v \in \mathcal{U}, \gamma_{\alpha\beta}(v) = 0 \text{ in } \omega \right\} \quad (13)$$

is the inextensional displacements space of the midsurface  $S$ . Thanks to the continuity of the form  $a^m(\cdot, \cdot)$  on  $\mathcal{U}$ ,  $\mathcal{U}_1$  is a closed subspace of  $\mathcal{U}$ . Thus,  $\mathcal{U}_1$  is a Hilbert space. Let us denote by  $\mathcal{U}_1^0 \subset \mathcal{U}'$  the polar set of  $\mathcal{U}_1$ , i.e.

$$\mathcal{U}_1^0 = \left\{ f \in \mathcal{U}' \mid \langle f, v \rangle = 0 \quad \forall v \in \mathcal{U}_1 \right\}. \quad (14)$$

The orthogonal space of  $\mathcal{U}_1$  in  $\mathcal{U}$  is given by

$$V := \left\{ u \in \mathcal{U}, a^b(u, v) = 0, \quad \forall v \in \mathcal{U}_1 \right\}, \quad (15)$$

when  $\mathcal{U}$  is equipped with the inner product  $(a^m(u, v) + a^b(u, v))$ . Clearly,  $V$  is a Hilbert space, with the norm inherited by  $\mathcal{U}$ . Thus, we have

$$\mathcal{U} = \mathcal{U}_1 \oplus V. \quad (16)$$

In the sequel, we also need the following space

$$W = \text{the completion of } V \text{ with the norm } a^m(v, v)^{\frac{1}{2}} := \|v\|_W \quad (17)$$

Note that on  $V$   $\|v\|_W$  is indeed a norm, not only a seminorm. We also remark that  $a^b(\cdot, \cdot)$  is coercive on  $\mathcal{U}_1$ .

We are now ready to introduce our basic criterium which will lead us to classify the possible behavior of a general shell. We introduce the *energy functional* defined by

$$E(\varepsilon, v) := \varepsilon a^m(v, v) + \varepsilon^3 a^b(v, v) \quad \forall v \in \mathcal{U}. \quad (18)$$

Our aim is to study the asymptotic behavior (as  $\varepsilon \rightarrow 0$ ) of the energy functional, when it is tested on the solution  $u_\varepsilon \in \mathcal{U}$  of problem (7). We then set

$$E(\varepsilon) := E(\varepsilon, u_\varepsilon) = \varepsilon a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^3 a^b(u_\varepsilon, u_\varepsilon). \quad (19)$$

To continue, we consider the *energy functional of order  $\beta$*  defined by

$$E(\varepsilon, v; \beta) := \varepsilon^\beta E(\varepsilon, v) \quad \forall v \in \mathcal{U}. \quad (20)$$

When the energy functional of order  $\beta$  is tested on  $u_\varepsilon$ , we will simply set  $E(\varepsilon, \beta)$ , i.e.

$$E(\varepsilon, \beta) := \varepsilon^\beta E(\varepsilon) = \varepsilon^{\beta+1} a^m(u_\varepsilon, u_\varepsilon) + \varepsilon^{\beta+3} a^b(u_\varepsilon, u_\varepsilon). \quad (21)$$

We notice that, by linearity, the definition given above is consistent with that introduced in (4) for problem (3).

Moreover, from (7) we see that

$$E(\varepsilon, \beta) = \varepsilon^\beta \langle f, u_\varepsilon \rangle . \quad (22)$$

It is not hard to show (cf. [5]) that, independently of  $f \in \mathcal{U}'$ , one has for problem (7):

- if  $\beta < 1$ , then  $E(\varepsilon, \beta) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ ;
- if  $\beta > 3$ , then  $E(\varepsilon, \beta) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Our idea consists in classifying the asymptotic shell behaviors by means of a sort of *proper choice* of the exponent  $\beta$  entering in (21). More precisely, we introduce the following

**Definition 2.1** *Given  $f \in \mathcal{U}'$ , we say that the Koiter problem (7) is of order  $\alpha$  if*

$$\alpha = \inf \left\{ \beta \mid E(\varepsilon, \beta) \in L^\infty(0, 1) \right\} . \quad (23)$$

□

By the discussion above it is straightforward to obtain that

- if the Koiter problem (7) is of order  $\alpha$ , then  $1 \leq \alpha \leq 3$ .

We will see in the next sections that the problem order is strictly linked to the regularity of the datum  $f \in \mathcal{U}'$ . We conclude this section by noting that if  $f \in \mathcal{U}_1^0$  (cf. (14)), then the Koiter problem (7) can be equivalently formulated as

$$\left\{ \begin{array}{l} \text{Find } u_\varepsilon \in V \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in V , \end{array} \right. \quad (24)$$

where  $V$  is defined by (15).

As a consequence, in the sequel we will refer to problem (7) if  $f \notin \mathcal{U}_1^0$ , and to problem (24) if  $f \in \mathcal{U}_1^0$ .

### 3 Known results: the extremal cases $\alpha = 3$ and $\alpha = 1$

In this section we will review some known results, which indeed cover only the cases when the order of the shell problem is  $\alpha = 3$  or  $\alpha = 1$ . We remark that the case  $\alpha = 3$  (resp.  $\alpha = 1$ ) corresponds in the engineering literature to the *bending dominated state* (resp. *membrane dominated state*).

### 3.1 The case $f \notin \mathcal{U}_1^0$

This is certainly the easiest situation to deal with. In fact, it is well known that the following result holds true (cf. [9])

**Theorem 3.1** *Fix  $f \notin \mathcal{U}_1^0$  and consider the problem*

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{U} \text{ such that} \\ \varepsilon^3 a^b(u_\varepsilon, v) + \varepsilon a^m(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in \mathcal{U} . \end{cases} \quad (25)$$

*Then problem (25) is of order  $\alpha = 3$ . Furthermore, let  $u_0 \in \mathcal{U}_1$  be the solution of the bending-type problem*

$$\begin{cases} \text{Find } u_0 \in \mathcal{U}_1 \text{ such that} \\ a^b(u_0, v_0) = \langle f, v_0 \rangle \quad \forall v_0 \in \mathcal{U}_1 . \end{cases} \quad (26)$$

*Then, as  $\varepsilon \rightarrow 0$ ,*

$$\|\varepsilon^3 u_\varepsilon - u_0\|_{\mathcal{U}} \rightarrow 0 \quad (27)$$

*and*

$$\frac{\varepsilon^3 a^b(u_\varepsilon, u_\varepsilon)}{E(\varepsilon)} \rightarrow 1. \quad (28)$$

*Moreover,  $u_0 \neq 0$ .*

□

### 3.2 The case $f \in \mathcal{U}_1^0$ and $f \in W'$

As already noticed, for this case we refer to problem (24). We have the following (cf. [16])

**Theorem 3.2** *Fix  $f \in \mathcal{U}_1^0$  and suppose moreover that  $f \in W'$ , where  $W'$  is the dual space of  $W$  (cf. 17). Consider the problem*

$$\begin{cases} \text{Find } u_\varepsilon \in V \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in V , \end{cases} \quad (29)$$

*Then problem (29) is of order  $\alpha = 1$ . Furthermore, let  $u_0 \in W$  be the solution of the membrane-type problem*

$$\begin{cases} \text{Find } u_0 \in W \text{ such that} \\ a^m(u_0, v_0) = \langle f, v_0 \rangle \quad \forall v_0 \in W . \end{cases} \quad (30)$$

Then

$$\|\varepsilon u_\varepsilon - u_0\|_W \rightarrow 0 \quad (31)$$

and

$$\frac{\varepsilon^3 a^b(u_\varepsilon, u_\varepsilon)}{E(\varepsilon)} \rightarrow 0 \quad (32)$$

as  $\varepsilon \rightarrow 0$ . Moreover,  $u_0 \neq 0$ .

□

## 4 Intermediate orders

These are surely the most subtle cases to treat. From the results of the previous section, we can suppose that  $f \in \mathcal{U}_1^0$  (so that we will refer to problem (24)) and at the same time  $f \notin W'$ . As we will see (cf. Theorem 4.3), the order of the problem can be related to the regularity of  $f$ , regularity which is measured by some Interpolation spaces between  $W'$  and  $V'$ . For the notation and results in Interpolation theory, we refer to [2] and [12]. We remark that the theory developed in the next subsection essentially follows the guidelines of [1].

### 4.1 Asymptotic behavior and regularity of the datum

We begin by setting some notations. In the sequel, we will denote by  $A^b$  the (linear and continuous) operator

$$\begin{aligned} A^b &: V \longrightarrow V' \\ v' &\langle A^b u, v \rangle_V = a^b(u, v) \quad \forall u, v \in V . \end{aligned} \quad (33)$$

Similarly, we define the (linear and continuous) operator  $A^m$  as

$$\begin{aligned} A^m &: V \longrightarrow V' \\ v' &\langle A^m u, v \rangle_V = a^m(u, v) \quad \forall u, v \in V . \end{aligned} \quad (34)$$

By the definition of the space  $W$  (cf. (17)), the form  $a^m(\cdot, \cdot)$  can be uniquely extended to a bilinear and continuous form  $\tilde{a}^m(\cdot, \cdot)$  on  $W$ . Hence we can define the (linear and continuous) operator  $\tilde{A}^m$  by

$$\tilde{A}^m : W \longrightarrow W'$$

$${}_W \langle \tilde{A}^m u, v \rangle_W = a^m(u, v) \quad \forall u, v \in W, \quad (35)$$

whose restriction on  $V$  is obviously  $A^m$ . With a little abuse of notation, also  $\tilde{A}^m$  will be denoted by  $A^m$  in the sequel.

We now prove a preliminary result, useful in what follows. We have

**Lemma 4.1** *Let us define  $V_m \subset W'$  by*

$$V_m = \{f \in W' : f = A^m v, \quad v \in V\}. \quad (36)$$

*Equipped with the norm  $\|f\|_{V_m} := \|v\|_V$  (where  $v \in V$  is the only element in  $V$  such that  $f = A^m v$ ), the space  $V_m$  is a Hilbert space. Moreover, for each  $1/2 < \sigma < 1$  and each  $1 \leq p \leq +\infty$ , it holds*

$$(V_m, V')_{\sigma, p} \subset (W', V')_{2\sigma-1, p}, \quad (37)$$

*with continuous inclusion.*

**Proof:** Since  $A^m$  is injective, for each  $f \in V_m$  there is only one  $v \in V$  such that  $A^m v = f$ . It is straightforward to obtain that  $V_m$  is complete and the norm  $\|\cdot\|_{V_m}$  is hilbertian. Furthermore, we have for each  $f \in V_m$

$$\|f\|_{W'}^2 \leq \|f\|_{V_m} \|f\|_{V'}. \quad (38)$$

Setting  $f = A^m v$ , with  $v \in V$ , the above estimate can be established by noting that it holds

$$\|f\|_{W'} = \|v\|_W \quad (39)$$

and

$$\|v\|_W^2 = a^m(v, v) = {}_V \langle A^m v, v \rangle_V \leq \|A^m v\|_{V'} \|v\|_V \leq \|f\|_{V'} \|f\|_{V_m}. \quad (40)$$

Let then  $f \in (V_m, V')_{\sigma, p}$  and  $f(t) \in W(p, \sigma, V_m; p, \sigma - 1, V')$  such that

$$f = \int_{-\infty}^{+\infty} f(t) dt. \quad (41)$$

We thus have

$$e^{t\sigma} f(t) \in L^p(-\infty, +\infty; V_m) ; \quad e^{t(\sigma-1)} f(t) \in L^p(-\infty, +\infty; V') . \quad (42)$$

From inequality (38) we get

$$\begin{aligned} e^{t(\sigma-1/2)} \|f(t)\|_{W'} &\leq \left( e^{t(\sigma/2)} \|f(t)\|_{V_m}^{1/2} \cdot e^{t(\sigma/2-1/2)} \|f(t)\|_{V'}^{1/2} \right) \\ &\leq C \left( e^{t\sigma} \|f(t)\|_{V_m} + e^{t(\sigma-1)} \|f(t)\|_{V'} \right) , \end{aligned} \quad (43)$$

so that

$$e^{t(\sigma-1/2)} f(t) \in L^p(-\infty, +\infty; W') . \quad (44)$$

If  $1/2 < \sigma < 1$ , combining (44) with the second relation of (42) and using the homogeneity property, we obtain

$$f \in (W', V')_{2\sigma-1, p} . \quad (45)$$

Then poof is then complete. □

We are now ready to establish the following

**Theorem 4.1** *Given  $f \in V'$ , consider the problem*

$$\begin{cases} \text{Find } u_\varepsilon \in V \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in V . \end{cases} \quad (46)$$

*For each  $0 < \theta < 1$  and each  $1 \leq p \leq +\infty$  consider the following conditions*

$$\text{C1} \quad \varepsilon^{(\theta+1)} u_\varepsilon \in L_*^p(0, 1; W) ; \quad (47)$$

$$\text{C2} \quad \varepsilon^{(\theta+2)} u_\varepsilon \in L_*^p(0, 1; V) ; \quad (48)$$

$$\text{C3} \quad f \in (W', V')_{\theta, p} . \quad (49)$$

*Then conditions C2 and C3 are equivalent. Moreover, C3 (or C2, of course) implies C1.*

**Proof:** By the change of variable  $t = \ln(\varepsilon)$  and setting  $u(t) = u_\varepsilon|_{\varepsilon=e^t}$  we have that problem (46) is equivalent to

$$\begin{cases} \text{Find } u(t) \in V \text{ such that} \\ e^t a^m(u(t), v) + e^{3t} a^b(u(t), v) = \langle f, v \rangle \quad \forall v \in V . \end{cases} \quad (50)$$

As an equation in  $V'$  it is equivalent to

$$\begin{cases} \text{Find } u(t) \in V \text{ such that} \\ e^t A^m u(t) + e^{3t} A^b u(t) = f . \end{cases} \quad (51)$$

1. [C2 implies C3] Let  $\varepsilon^{(\theta+2)} u_\varepsilon \in L_*^p(0, 1; V)$ . We thus have that

$$e^{t(\theta+2)} u(t) \in L^p(-\infty, 0; V) . \quad (52)$$

Let us now split the datum  $f \in V'$  as

$$f = f_0(t) + f_1(t) \quad (53)$$

where (cf. (51))

$$f_0(t) = e^t A^m u(t) \in V_m , \quad f_1(t) = e^{3t} A^b u(t) \in V' \quad \forall t \in (-\infty, 0] , \quad (54)$$

and

$$f_0(t) = 0 \in V_m , \quad f_1(t) = f \in V' \quad \forall t \in (0, +\infty) . \quad (55)$$

We notice that we obviously have (cf. (55))

$$e^{t(\theta+1)} f_0(t) \in L^p(0, +\infty; V_m) \quad \forall \theta \in (0, 1) . \quad (56)$$

Furthermore, still from (55) it holds (with the usual modification when  $p = +\infty$ )

$$\int_0^{+\infty} \left( e^{t(\theta-1)} \|f_1(t)\|_{V'} \right)^p dt = \|f\|_{V'}^p \int_0^{+\infty} e^{t(\theta-1)p} dt < +\infty \quad \forall \theta \in (0, 1) , \quad (57)$$

so that

$$e^{t(\theta-1)} f_1(t) \in L^p(0, +\infty; V') \quad \forall \theta \in (0, 1) . \quad (58)$$

To continue, we consider  $t \in (-\infty, 0]$ . From the first equation of (54), we get that condition (52) holds if and only if

$$e^{t(\theta+1)} f_0(t) \in L^p(-\infty, 0; V_m) . \quad (59)$$

Furthermore, from the second equation of (54), condition (52) implies

$$e^{t(\theta-1)} f_1(t) \in L^p(-\infty, 0; V') . \quad (60)$$

Taking into account (56) and (58), it follows from (59) and (60) that

$$e^{t(\theta+1)} f_0(t) \in L^p(-\infty, +\infty; V_m) , \quad (61)$$

and

$$e^{t(\theta-1)} f_1(t) \in L^p(-\infty, +\infty; V') . \quad (62)$$

Hence, by the homogeneity property, we have that condition (52) implies

$$f \in (V_m, V')_{\frac{\theta+1}{2}, p} . \quad (63)$$

Referring to Lemma 4.1 with  $\sigma = \frac{\theta+1}{2}$ , we get from (63) that

$$f \in (W', V')_{\theta, p} . \quad (64)$$

2. [C3 implies C2] Let now  $f \in (W', V')_{\theta, p}$ . Then there exists a decomposition

$$f = f_{0,\varepsilon} + f_{1,\varepsilon} \quad (65)$$

such that

$$\varepsilon^\theta f_{0,\varepsilon} \in L_*^p(0, +\infty; W') ; \quad \varepsilon^{(\theta-1)} f_{1,\varepsilon} \in L_*^p(0, +\infty; V') . \quad (66)$$

This implies, in particular, that

$$\varepsilon^\theta f_{0,\varepsilon} \in L_*^p(0, 1; W') ; \quad \varepsilon^{(\theta-1)} f_{1,\varepsilon} \in L_*^p(0, 1; V') . \quad (67)$$

For  $j = 0, 1$  consider the problems (for  $0 < \varepsilon \leq 1$ )

$$\left\{ \begin{array}{l} \text{Find } u_{j,\varepsilon} \in V \text{ such that} \\ \varepsilon a^m(u_{j,\varepsilon}, v) + \varepsilon^3 a^b(u_{j,\varepsilon}, v) = \langle f_{j,\varepsilon}, v \rangle \quad \forall v \in V . \end{array} \right. \quad (68)$$

By the linearity of the problems we obviously have that the solution  $u_\varepsilon \in V$  of problem (46) can be split into

$$u_\varepsilon = u_{0,\varepsilon} + u_{1,\varepsilon} . \quad (69)$$

Choosing  $j = 0$ , from (68) we get the estimate

$$\varepsilon \|u_{0,\varepsilon}\|_W^2 + \varepsilon^3 \|u_{0,\varepsilon}\|_V^2 \leq C \|f_{0,\varepsilon}\|_{W'} \|u_{0,\varepsilon}\|_W, \quad (70)$$

which is true for  $0 < \varepsilon \leq 1$ .

Hence it follows

$$\|u_{0,\varepsilon}\|_W \leq C \varepsilon^{-1} \|f_{0,\varepsilon}\|_{W'} \quad (71)$$

so that from (70) we get

$$\|u_{0,\varepsilon}\|_V^2 \leq C \varepsilon^{-4} \|f_{0,\varepsilon}\|_{W'}^2. \quad (72)$$

Thus, we have the estimate

$$\|u_{0,\varepsilon}\|_V \leq C \varepsilon^{-2} \|f_{0,\varepsilon}\|_{W'}, \quad (73)$$

by which one obtains

$$\varepsilon^{(\theta+2)} \|u_{0,\varepsilon}\|_V \leq C \varepsilon^\theta \|f_{0,\varepsilon}\|_{W'}. \quad (74)$$

Similarly, choosing  $j = 1$  in (68) we may get

$$\varepsilon^{(\theta+2)} \|u_{1,\varepsilon}\|_V \leq C \varepsilon^{(\theta-1)} \|f_{1,\varepsilon}\|_{V'}. \quad (75)$$

Collecting estimates (74) and (75), using triangle inequality, from (69) we have

$$\varepsilon^{(\theta+2)} \|u_\varepsilon\|_V \leq C \left( \varepsilon^\theta \|f_{0,\varepsilon}\|_{W'} + \varepsilon^{(\theta-1)} \|f_{1,\varepsilon}\|_{V'} \right). \quad (76)$$

From (67) and (76) we thus obtain

$$\varepsilon^{(\theta+2)} u_\varepsilon \in L_*^p(0, 1; V). \quad (77)$$

3. [C3 implies C1] From (68) we get the estimate

$$\varepsilon \|u_{0,\varepsilon}\|_W^2 + \varepsilon^3 \|u_{0,\varepsilon}\|_V^2 \leq C \|f_{0,\varepsilon}\|_{W'} \|u_{0,\varepsilon}\|_W, \quad (78)$$

verified for  $0 < \varepsilon \leq 1$ . Hence it follows

$$\|u_{0,\varepsilon}\|_W \leq C \varepsilon^{-1} \|f_{0,\varepsilon}\|_{W'}. \quad (79)$$

Thus, we have the estimate

$$\varepsilon^{(\theta+1)} \|u_{0,\varepsilon}\|_W \leq C \varepsilon^\theta \|f_{0,\varepsilon}\|_{W'}. \quad (80)$$

Choosing  $j = 1$  in (68) we get

$$\varepsilon \|u_{1,\varepsilon}\|_W^2 + \varepsilon^3 \|u_{1,\varepsilon}\|_V^2 \leq C \|f_{1,\varepsilon}\|_{V'} \|u_{1,\varepsilon}\|_V . \quad (81)$$

It follows that

$$\varepsilon^3 \|u_{1,\varepsilon}\|_V \leq C \|f_{1,\varepsilon}\|_{V'} . \quad (82)$$

Inserting this estimate in (81) we obtain

$$\|u_{1,\varepsilon}\|_W \leq C \varepsilon^{-2} \|f_{1,\varepsilon}\|_{V'} , \quad (83)$$

so that

$$\varepsilon^{(\theta+1)} \|u_{1,\varepsilon}\|_W \leq C \varepsilon^{(\theta-1)} \|f_{1,\varepsilon}\|_{V'} . \quad (84)$$

By estimates (80) and (84), using triangle inequality, we have

$$\varepsilon^{(\theta+1)} \|u_\varepsilon\|_W \leq C \left( \varepsilon^\theta \|f_{0,\varepsilon}\|_{W'} + \varepsilon^{(\theta-1)} \|f_{1,\varepsilon}\|_{V'} \right) , \quad (85)$$

which implies (cf. (67))

$$\varepsilon^{(\theta+1)} u_\varepsilon \in L_*^p(0, 1; W) . \quad (86)$$

The proof of the theorem is thus complete.  $\square$

By recalling (19), a consequence of Theorem 4.1 is the following

**Proposition 4.1** *Given  $f \in V'$ , consider Problem (46). Then*

$$f \in (W', V')_{\theta,p} \quad (87)$$

*if and only if*

$$\varepsilon^{\theta+1/2} E(\varepsilon)^{1/2} \in L_*^p(0, 1) . \quad (88)$$

**Proof:** Suppose first that  $f \in (W', V')_{\theta,p}$ . Hence, by Theorem 4.1 it follows

$$\varepsilon^{(\theta+1)} \|u_\varepsilon\|_W \in L_*^p(0, 1) ; \quad \varepsilon^{(\theta+2)} \|u_\varepsilon\|_V \in L_*^p(0, 1) . \quad (89)$$

From (19) we easily get

$$E(\varepsilon)^{1/2} \leq C(\varepsilon^{1/2} \|u_\varepsilon\|_W + \varepsilon^{3/2} \|u_\varepsilon\|_V) , \quad (90)$$

so that

$$\varepsilon^{\theta+1/2} E(\varepsilon)^{1/2} \leq C(\varepsilon^{(\theta+1)} \|u_\varepsilon\|_W + \varepsilon^{(\theta+2)} \|u_\varepsilon\|_V) . \quad (91)$$

This implies (cf. (89))

$$\varepsilon^{\theta+1/2} E(\varepsilon)^{1/2} \in L_*^p(0, 1) . \quad (92)$$

Conversely, suppose

$$\varepsilon^{\theta+1/2} E(\varepsilon)^{1/2} \in L_*^p(0, 1) . \quad (93)$$

Since by coercivity we have (for  $0 < \varepsilon < 1$ )

$$E(\varepsilon)^{1/2} \geq C\varepsilon^{3/2} \|u_\varepsilon\|_V , \quad (94)$$

it follows from (93) that

$$\varepsilon^{(\theta+2)} \|u_\varepsilon\|_V \in L_*^p(0, 1) . \quad (95)$$

Now, from Theorem 4.1 we deduce

$$f \in (W', V')_{\theta, p} . \quad (96)$$

The proof is complete.  $\square$

Moreover, by setting  $p = +\infty$  in Theorem 4.1 and in Proposition 4.1, we get the following

**Corollary 4.1** *Given  $f \in V'$ , consider the problem*

$$\begin{cases} \text{Find } u_\varepsilon \in V \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in V . \end{cases} \quad (97)$$

*For each  $0 < \theta < 1$  consider the following conditions*

$$\text{C1} \quad \|u_\varepsilon\|_W = O(\varepsilon^{-(\theta+1)}) \quad \text{as } \varepsilon \rightarrow 0 ; \quad (98)$$

$$\text{C2} \quad \|u_\varepsilon\|_V = O(\varepsilon^{-(\theta+2)}) \quad \text{as } \varepsilon \rightarrow 0 ; \quad (99)$$

$$\text{C3} \quad f \in (W', V')_{\theta, \infty} . \quad (100)$$

*Then conditions C2 and C3 are equivalent. Moreover, C3 implies C1. Finally, C3 holds if and only if*

$$\varepsilon^{\theta+1/2} E(\varepsilon)^{1/2} \in L^\infty(0, 1) . \quad (101)$$

□

## 4.2 Problem order

We now use Corollary 4.1 to give a relationship between the problem order (cf. Definition 2.1) and the regularity of  $f$ . First, we have the following

**Theorem 4.2** *Fix  $f \in V'$  and consider the problem*

$$\begin{cases} \text{Find } u_\varepsilon \in V \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in V . \end{cases} \quad (102)$$

*Then  $E(\varepsilon, \beta) \in L^\infty(0, 1)$  if and only if  $f \in (W', V')_{\theta, \infty}$ , with  $\theta = (\beta - 1)/2$ .*

**Proof:** By Corollary 4.1 and recalling (cf. (21)) that  $E(\varepsilon, \beta) = \varepsilon^\beta E(\varepsilon)$ , we thus have that  $f \in (W', V')_{\theta, \infty}$  if and only if

$$\varepsilon^{(\theta+1/2-\beta/2)} E(\varepsilon, \beta)^{1/2} = \left( \varepsilon^{(2\theta+1-\beta)} E(\varepsilon, \beta) \right)^{1/2} \in L^\infty(0, 1) . \quad (103)$$

Since  $2\theta + 1 = \beta$ , we deduce that

$$E(\varepsilon, \beta) \in L^\infty(0, 1) \quad (104)$$

if and only if  $f \in (W', V')_{\theta, \infty}$ . □

We can finally prove the following theorem (cf. Definition 2.1).

**Theorem 4.3** *Fix  $f \in V'$  and consider the problem*

$$\begin{cases} \text{Find } u_\varepsilon \in V \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in V . \end{cases} \quad (105)$$

1. *If  $f \in (W', V')_{\theta, \infty}$  for some  $0 < \theta < 1$ , then problem (105) is of order  $\alpha$  given by*

$$\alpha = \inf \left\{ 2\theta + 1 : f \in (W', V')_{\theta, \infty}, 0 < \theta < 1 \right\} . \quad (106)$$

2. *If  $f \notin (W', V')_{\theta, \infty}$  for any  $0 < \theta < 1$ , then problem (105) is of order  $\alpha = 3$ .*

**Proof:** The proof is a direct consequence of Definition 2.1, after having noticed that, by Theorem 4.2, it holds

$$\inf \left\{ 2\theta + 1 : f \in (W', V')_{\theta, \infty}, 0 < \theta < 1 \right\} = \inf \left\{ \beta : E(\varepsilon, \beta) \in L^\infty(0, 1) \right\}. \quad (107)$$

□

Recalling that, independently of  $f \in \mathcal{U}'$ , one has

- if  $\beta < 1$ , then  $E(\varepsilon, \beta) \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$ ;
- $E(\varepsilon, \beta_1) = \varepsilon^{\beta_1 - \beta_2} E(\varepsilon, \beta_2)$  for every exponents  $\beta_1$  and  $\beta_2$ ,

a straightforward consequence of Theorems 4.2 and 4.3 is the following

**Corollary 4.2** *If the problem*

$$\begin{cases} \text{Find } u_\varepsilon \in V \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in V \end{cases} \quad (108)$$

*is of order  $\alpha$ , then it holds the following.*

1. *If  $\beta > \alpha$  then  $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, \beta) = 0$ .*
2. *If  $1 \leq \beta < \alpha$  then  $\limsup_{\varepsilon \rightarrow 0} E(\varepsilon, \beta) = +\infty$ .*
3. *If  $\beta < 1$  then  $\lim_{\varepsilon \rightarrow 0} E(\varepsilon, \beta) = +\infty$ .*

□

**Remark 4.1** *We remark that we have*

$$\inf \left\{ 2\theta + 1 : f \in (W', V')_{\theta, \infty}, 0 < \theta < 1 \right\} = \inf \left\{ 2\theta + 1 : f \in (W', V')_{\theta, p}, 0 < \theta < 1 \right\}, \quad (109)$$

*for every  $p \in [1, +\infty]$ . The above equality is a consequence of the standard Interpolation inclusions (cf. [2] or [12])*

$$(W', V')_{\eta, p} \subset (W', V')_{\theta, q} \subset (W', V')_{\sigma, r} \quad (110)$$

*when  $0 < \eta < \theta < \sigma < 1$  and  $p, q, r \in [1, +\infty]$ .*

*This means that, in determining the problem order, one can decide to work with any  $p \in [1, +\infty]$ . Since the space of admissible displacements for a shell problem is a Hilbert space of Sobolev type, it turns out that  $p = 2$  is generally the easiest choice.*

## 5 On the asymptotic ratio between the bending and the total elastic energy

In this section we will investigate the asymptotic ratio between the bending energy and the total elastic energy. We begin with defining the function  $R(\varepsilon)$  as

$$R(\varepsilon) := \frac{\varepsilon^3 a^b(u_\varepsilon, u_\varepsilon)}{E(\varepsilon)}. \quad (111)$$

We will need the following lemma, whose proof follows from a direct computation.

**Lemma 5.1** *The real function defined by*

$$\varepsilon \longrightarrow E(\varepsilon) \quad (112)$$

*is differentiable in  $(0, 1]$ , and its derivative is given by*

$$E'(\varepsilon) = a^m(u_\varepsilon, u_\varepsilon) + 3\varepsilon^2 a^b(u_\varepsilon, u_\varepsilon) + 2\varepsilon a^m(u'_\varepsilon, u_\varepsilon) + 2\varepsilon^3 a^b(u'_\varepsilon, u_\varepsilon), \quad (113)$$

*where  $u'_\varepsilon \in \mathcal{V}$  is defined by*

$$u'_\varepsilon := \mathcal{V} - \lim_{h \rightarrow 0} \frac{u_{\varepsilon+h} - u_\varepsilon}{h} \quad (114)$$

*and the space  $\mathcal{V}$  is either  $\mathcal{U}$  (if  $f \notin \mathcal{U}_1^0$ ), or  $V$  (if  $f \in \mathcal{U}_1^0$ ).*

□

From Lemma 5.1, noting that

$$2\varepsilon a^m(u'_\varepsilon, u_\varepsilon) + 2\varepsilon^3 a^b(u'_\varepsilon, u_\varepsilon) = 2 \langle f, u'_\varepsilon \rangle = 2E'(\varepsilon), \quad (115)$$

and recalling that  $E(\varepsilon, \beta) = \varepsilon^\beta E(\varepsilon)$ , we easily get

$$\varepsilon E'(\varepsilon, \beta) = (\beta - 1)E(\varepsilon, \beta) - 2\varepsilon^{\beta+3} a^b(u_\varepsilon, u_\varepsilon). \quad (116)$$

We are now in a position to prove the following theorem

**Theorem 5.1** *Consider the problem*

$$\begin{cases} \text{Find } u_\varepsilon \in \mathcal{V} \text{ such that} \\ \varepsilon a^m(u_\varepsilon, v) + \varepsilon^3 a^b(u_\varepsilon, v) = \langle f, v \rangle \quad \forall v \in \mathcal{V}. \end{cases} \quad (117)$$

*Suppose that there is an exponent  $\beta = \alpha$  such that there exist*

$$+\infty > \lim_{\varepsilon \rightarrow 0} E(\varepsilon, \alpha) > 0, \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{\alpha+3} a^b(u_\varepsilon, u_\varepsilon) \geq 0. \quad (118)$$

Then it holds

$$\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = \frac{\alpha - 1}{2}. \quad (119)$$

**Proof:** From (116) we get (cf. (111))

$$\left( R(\varepsilon) - \frac{\alpha - 1}{2} \right) \frac{1}{\varepsilon} = -\frac{1}{2} \frac{E'(\varepsilon, \alpha)}{E(\varepsilon, \alpha)}. \quad (120)$$

We now fix  $\varepsilon_0 < 1$  and consider  $\varepsilon$  with  $0 < \varepsilon < \varepsilon_0$ . By integration, we obtain from (120)

$$\int_{\varepsilon}^{\varepsilon_0} \left( R(\tau) - \frac{\alpha - 1}{2} \right) \frac{1}{\tau} d\tau = -\frac{1}{2} \int_{\varepsilon}^{\varepsilon_0} \frac{E'(\tau, \alpha)}{E(\tau, \alpha)} d\tau, \quad (121)$$

so that

$$\begin{aligned} \int_{\varepsilon}^{\varepsilon_0} \left( R(\tau) - \frac{\alpha - 1}{2} \right) \frac{1}{\tau} d\tau &= -\frac{1}{2} (\ln E(\varepsilon_0, \alpha) - \ln E(\varepsilon, \alpha)) \\ &= \frac{1}{2} \ln E(\varepsilon, \alpha) - \frac{1}{2} \ln E(\varepsilon_0, \alpha). \end{aligned} \quad (122)$$

By (118), it exists

$$-\infty < \lim_{\varepsilon \rightarrow 0} [\ln E(\varepsilon, \alpha)] < +\infty. \quad (123)$$

Since  $\varepsilon_0$  is fixed, it follows from (122) and (123) that

$$\left( R(\varepsilon) - \frac{\alpha - 1}{2} \right) \in L_*^1(0, 1). \quad (124)$$

Noting that it exists (cf. (118))

$$\lim_{\varepsilon \rightarrow 0} \left( R(\varepsilon) - \frac{\alpha - 1}{2} \right), \quad (125)$$

we conclude from (124) that

$$\lim_{\varepsilon \rightarrow 0} \left( R(\varepsilon) - \frac{\alpha - 1}{2} \right) = 0, \quad (126)$$

i.e.

$$\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = \frac{\alpha - 1}{2} . \quad (127)$$

The proof is complete. □

**Remark 5.1** *By analysing the work [6], E. Sanchez-Palencia conjectured that a relationship between  $\alpha$  and  $R(\varepsilon)$  might hold. Our Theorem 5.1 answers in a positive way this question.*

**Remark 5.2** *It has been proved (cf. [5] and [6]) that if there exists an  $\alpha$  satisfying (118), then such an exponent is unique and  $1 \leq \alpha \leq 3$ . We also remark that*

- *In the hypotheses of Theorem 3.1,  $\alpha$  does exist and  $\alpha = 3$ . Furthermore*

$$\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 1 ,$$

*i.e. asymptotically the energy goes entirely in bending.*

- *In the hypotheses of Theorem 3.2,  $\alpha$  does exist and  $\alpha = 1$ . Furthermore*

$$\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 0 ,$$

*i.e. asymptotically the energy goes entirely in membrane.*

- *In the hypotheses of Theorem 4.3, we are not able to prove that  $\alpha$  (satisfying (118)) exists. However, if it exists, then it holds the following*

- *If we are in the framework of part 1. of Theorem 4.3, then*

$$\alpha = \inf \left\{ 2\theta + 1 : f \in (W', V')_{\theta, \infty} , 0 < \theta < 1 \right\}$$

*and*

$$\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = \inf \left\{ \theta : f \in (W', V')_{\theta, \infty} , 0 < \theta < 1 \right\} .$$

- *If we are in the framework of part 2. of Theorem 4.3, then*

$$\alpha = 3$$

*and*

$$\lim_{\varepsilon \rightarrow 0} R(\varepsilon) = 1 .$$

## 6 An application to a cylindrical shell

The aim of this section is to analyse an instance of a shell of intermediate order, inspired by the considerations of [14]. We will consider a cylindrical shell of thickness  $\varepsilon$  whose midsurface is described by the region

$$\omega = \{(x, \rho, z) \mid -1 \leq x \leq 1, \quad 0 \leq \rho \leq 2\pi, \quad -\varepsilon/2 \leq z \leq \varepsilon/2\}. \quad (128)$$

Above,  $x$  is the axial coordinate,  $\rho$  the angular one and  $z$  the radial one. We will suppose that the shell is acted upon an axially constant normal pressure distribution which varies angularly as

$$f = f_0 \cos \rho, \quad (129)$$

where  $f_0$  is a constant. Due to the particular shape of the load, the problem can be reduced to a 1D (axially) problem (cf. [14]), whose unknown are

$$\begin{aligned} u &= u(x) && \text{the axial displacement} \\ v &= v(x) && \text{the angular displacement} \\ w &= w(x) && \text{the radial displacement.} \end{aligned} \quad (130)$$

Furthermore, the membrane strains are given by

$$\gamma_{11} = u', \quad \gamma_{12} = \frac{1}{2}(-u + v'), \quad \gamma_{22} = v + w, \quad (131)$$

while the bending strains are

$$\Upsilon_{11} = w'', \quad \Upsilon_{12} = \frac{1}{2}(-v' - w'), \quad \Upsilon_{22} = -v - w. \quad (132)$$

Introducing the notation  $\underline{u} = (u, v, w)$ , the membrane and bending energy forms are given by

$$\begin{aligned} \varepsilon a^m(\underline{u}, \underline{u}) &:= \frac{\varepsilon}{2} \int_{-1}^1 \left[ \nu(\gamma_{11}(\underline{u}) + \gamma_{22}(\underline{u}))^2 + (1 - \nu) \sum_{i,j=1}^2 \gamma_{ij}(\underline{u}) \right] dx \\ \varepsilon^3 a^b(\underline{u}, \underline{u}) &:= \frac{\varepsilon^3}{24} \int_{-1}^1 \left[ \nu(\Upsilon_{11}(\underline{u}) + \Upsilon_{22}(\underline{u}))^2 + (1 - \nu) \sum_{i,j=1}^2 \Upsilon_{ij}(\underline{u}) \right] dx, \end{aligned} \quad (133)$$

where  $\nu$  is the Poisson's ratio. Next, the external energy is given by

$$\langle f, \underline{u} \rangle = \int_{-1}^1 f_0 w \, dx . \quad (134)$$

We now fix the boundary conditions we are interested in: we suppose that  $w(-1) = w(1) = 0$  and the other displacement components are free. Hence the admissible displacement space is

$$\mathcal{U} = H^1(-1, 1) \times H^1(-1, 1) \times (H^2(-1, 1) \cap H_0^1(-1, 1)) = H^1 \times H^1 \times (H^2 \cap H_0^1) , \quad (135)$$

with the usual norm. Moreover, it is easily seen that (cf. (13))

$$\mathcal{U}_1 = \{ \underline{u} \in \mathcal{U}, a^m(\underline{u}, \underline{u}) = 0, \quad \forall \underline{u} \in \mathcal{U} \} = 0 . \quad (136)$$

Hence,  $\mathcal{U} = V$  (cf. (15) and (16)) and  $f \in \mathcal{U}_1^0 = V'$ . We notice that the membrane energy norm  $a^m(\underline{u}, \underline{u})^{1/2}$  is indeed equivalent to the norm (cf. (131) and the first equation of (133))

$$\|\underline{u}\|_W^2 = \|u, v, w\|_W^2 := \|u'\|_0^2 + \|-u + v'\|_0^2 + \|v + w\|_0^2 . \quad (137)$$

Our purpose is to establish for which  $\theta$  with  $0 < \theta < 1$  the linear functional

$$(u, v, w) \longrightarrow \int_{-1}^1 f_0 w \, dx \quad (138)$$

belongs to the interpolated space  $(W', V')_{\theta, 2}$  (cf. Remark 4.1). To this end, we first define the subspace  $N \subset H^1 \times H^1 \times H^2$  as

$$\begin{aligned} N &= \left\{ (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1) \in H^1 \times H^1 \times H^2 : \tilde{u}_1 = \frac{a-b}{2} , \right. \\ &\quad \left. \tilde{v}_1 = -\frac{a}{2}(1-x) - \frac{b}{2}(1+x) , \tilde{w}_1 = \frac{a}{2}(1-x) + \frac{b}{2}(1+x) \right\} , \end{aligned} \quad (139)$$

where  $a$  and  $b$  are arbitrary real constants. Clearly,  $N$  is a (closed) two dimensional subspace of  $H^1 \times H^1 \times H^2$ . We will need the next lemma.

**Lemma 6.1** *The linear map  $\varphi : H^1 \times H^1 \times H^2 \rightarrow V$  defined by  $\varphi(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w)$  where*

$$\begin{cases} u = \tilde{u} - \frac{\tilde{w}(-1) - \tilde{w}(1)}{2} \\ v = \tilde{v} + \frac{\tilde{w}(-1)}{2}(1-x) + \frac{\tilde{w}(1)}{2}(1+x) \\ w = \tilde{w} - \frac{\tilde{w}(-1)}{2}(1-x) - \frac{\tilde{w}(1)}{2}(1+x) \end{cases} , \quad (140)$$

is surjective, continuous and with kernel  $N$ . Moreover, let us introduce the space  $\mathcal{V} := (H^1 \times H^1 \times H^2)/N$  endowed with the usual quotient norm, and set  $[\tilde{u}, \tilde{v}, \tilde{w}]$  as the equivalence class of  $(\tilde{u}, \tilde{v}, \tilde{w}) \in H^1 \times H^1 \times H^2$ . Then the linear map  $\psi : \mathcal{V} \rightarrow V$  (cf. (140)) defined by

$$\psi([\tilde{u}, \tilde{v}, \tilde{w}]) := \varphi(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w) \quad (141)$$

is an isomorphism.

**Proof:** We first notice that from (140) we have

$$\begin{cases} w(-1) = \tilde{w}(-1) - \tilde{w}(1) = 0 \\ w(1) = \tilde{w}(1) - \tilde{w}(-1) = 0 \end{cases}, \quad (142)$$

so that  $\varphi(\tilde{u}, \tilde{v}, \tilde{w}) \in V$ , for each  $(\tilde{u}, \tilde{v}, \tilde{w}) \in H^1 \times H^1 \times H^2$ .

Moreover, it is trivially seen that  $\varphi$  is surjective: take  $(u, v, w) \in V$  and define  $\tilde{u} = u$ ,  $\tilde{v} = v$ ,  $\tilde{w} = w$ ; then  $\varphi(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w)$ , since  $\tilde{w}(-1) = \tilde{w}(1) = 0$ . To get continuity, we only notice that if  $\varphi(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w)$ , then (cf. (140))

$$\begin{cases} \|u\|_1 \leq \|\tilde{u}\|_1 + \frac{|\tilde{w}(-1) - \tilde{w}(1)|}{2} \leq C(\|\tilde{u}\|_1 + \|\tilde{w}\|_2) \\ \|v\|_1 \leq \|\tilde{v}\|_1 + \frac{|\tilde{w}(-1)|}{2}\|(1-x)\|_1 + \frac{|\tilde{w}(1)|}{2}\|(1+x)\|_1 \leq C(\|\tilde{v}\|_1 + \|\tilde{w}\|_2) , \\ \|w\|_2 \leq \|\tilde{w}\|_2 + \frac{|\tilde{w}(-1)|}{2}\|(1-x)\|_2 + \frac{|\tilde{w}(1)|}{2}\|(1+x)\|_2 \leq C\|\tilde{w}\|_2 \end{cases} \quad (143)$$

where we have used the continuity of the trace operator. Hence

$$\|(u, v, w)\|_V \leq C\|(\tilde{u}, \tilde{v}, \tilde{w})\|_{H^1 \times H^1 \times H^2}. \quad (144)$$

To continue, if  $\varphi(\tilde{u}, \tilde{v}, \tilde{w}) = (0, 0, 0)$ , then (cf. (140))

$$\begin{cases} \tilde{u} = \frac{\tilde{w}(-1) - \tilde{w}(1)}{2} \\ \tilde{v} = -\frac{\tilde{w}(-1)}{2}(1-x) - \frac{\tilde{w}(1)}{2}(1+x) , \\ \tilde{w} = \frac{\tilde{w}(-1)}{2}(1-x) + \frac{\tilde{w}(1)}{2}(1+x) \end{cases} \quad (145)$$

so that, setting  $a = \tilde{w}(-1)$  and  $b = \tilde{w}(1)$  we see that  $(\tilde{u}, \tilde{v}, \tilde{w}) \in N$  (cf. (139)). Hence  $\text{Ker}\varphi \subset N$ . A direct computation gives  $N \subset \text{Ker}\varphi$ , so that  $\text{Ker}\varphi = N$ . Therefore, standard arguments show that the map  $\psi : \mathcal{V} \rightarrow V$  defined by (141) is a linear continuous bijection, and the Open Mapping Theorem allows to conclude that  $\psi$  is an isomorphism.  $\square$

Let us now define the following norm on  $\mathcal{V}$  (cf. (137)):

$$\|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{\mathcal{W}}^2 := \|\tilde{u}'\|_0^2 + \|-\tilde{u} + \tilde{v}'\|_0^2 + \|\tilde{v} + \tilde{w}\|_0^2 . \quad (146)$$

Notice that if  $(\tilde{u}_1, \tilde{v}_1, \tilde{w}_1) \in N$  (cf. (139)), then  $\tilde{u}'_1 = -\tilde{u}_1 + \tilde{v}'_1 = \tilde{v}_1 + \tilde{w}_1 = 0$ . It follows that the above position is well-defined and it is nothing but the pull-back of the  $W$ -norm on  $V$  through the isomorphism  $\psi$ . We will thus denote with  $\mathcal{W}$  the completion of  $\mathcal{V}$  with respect to the norm in (146). Notice also that  $\psi$  can be extended to an isomorphism (still denoted with  $\psi$ ) from  $\mathcal{W}$  to  $W$ . We have the following

**Lemma 6.2** *The norm defined by (146) is equivalent to the standard quotient norm in  $(H^1 \times H^1 \times L^2)/N$ .*

**Proof:** Since for every  $(\tilde{u}_1, \tilde{v}_1, \tilde{w}_1) \in N$  we have (cf. (146))

$$\|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{\mathcal{W}}^2 = \|[\tilde{u} + \tilde{u}_1, \tilde{v} + \tilde{v}_1, \tilde{w} + \tilde{w}_1]\|_{\mathcal{W}}^2 \leq C \left( \|\tilde{u} + \tilde{u}_1\|_1^2 + \|\tilde{v} + \tilde{v}_1\|_1^2 + \|\tilde{w} + \tilde{w}_1\|_0^2 \right) , \quad (147)$$

it holds

$$\|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{\mathcal{W}} \leq C \|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{(H^1 \times H^1 \times L^2)/N} . \quad (148)$$

To obtain the converse inequality, fix  $[\tilde{u}, \tilde{v}, \tilde{w}] \in (H^1 \times H^1 \times L^2)/N$  and take a representative  $(\tilde{u}, \tilde{v}, \tilde{w}) \in H^1 \times H^1 \times L^2$  such that  $\tilde{u}$  and  $\tilde{v}$  have both zero mean value (notice that, due to (139), this is always possible). Then we get

$$\begin{aligned} \|\tilde{u}\|_0 + \|\tilde{u}'\|_0 + \|\tilde{v}\|_0 + \|\tilde{v}'\|_0 + \|\tilde{w}\|_0 \\ \leq \|\tilde{u}\|_0 + \|\tilde{u}'\|_0 + \|\tilde{v}\|_0 + \|-\tilde{u} + \tilde{v}'\|_0 + \|\tilde{u}\|_0 + \|\tilde{v} + \tilde{w}\|_0 + \|\tilde{v}\|_0 . \end{aligned} \quad (149)$$

Hence

$$\|\tilde{u}\|_1 + \|\tilde{v}\|_1 + \|\tilde{w}\|_0 \leq C (\|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{\mathcal{W}} + \|\tilde{u}\|_0 + \|\tilde{v}\|_0) . \quad (150)$$

But since  $\tilde{u}$  and  $\tilde{v}$  have both zero mean value, we have

$$\|\tilde{u}\|_0 + \|\tilde{v}\|_0 \leq C (\|\tilde{u}'\|_0 + \|\tilde{v}'\|_0) \leq C \|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{\mathcal{W}} . \quad (151)$$

It follows from (150) and (151) that

$$\|\tilde{u}\|_1 + \|\tilde{v}\|_1 + \|\tilde{w}\|_0 \leq C \|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{\mathcal{W}} . \quad (152)$$

We finally get from (152)

$$\|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{(H^1 \times H^1 \times L^2)/N} \leq C \|[\tilde{u}, \tilde{v}, \tilde{w}]\|_{\mathcal{W}} . \quad (153)$$

The proof is thus complete. □

By Lemma 6.2 we see that for  $0 < \sigma < 1$

$$(\mathcal{V}, \mathcal{W})_{\sigma,2} = (H^1 \times H^1 \times H^{2(1-\sigma)})/N , \quad (154)$$

and this latter space is  $\psi$ -isomorphic to  $(V, W)_{\sigma,2}$ .

We now turn back to our functional on  $V$  (cf. (138))

$$(u, v, w) \longrightarrow \int_{-1}^1 f_0 w \, dx . \quad (155)$$

We have the following

**Proposition 6.1** *The functional  $f \in V'$  given by (155) is in the space  $(W', V')_{\theta,2}$  if and only if  $1/4 < \theta < 1$ . Hence the order of the shell problem detailed in this section is  $\alpha = 3/2$ .*

**Proof:** We consider the pull-back through  $\psi$  of  $f$ : it is given by (cf. (140))

$$\mathcal{F} : [\tilde{u}, \tilde{v}, \tilde{w}] \longrightarrow \int_{-1}^1 f_0 \left( \tilde{w} - \frac{\tilde{w}(-1)}{2}(1-x) - \frac{\tilde{w}(1)}{2}(1+x) \right) dx , \quad (156)$$

if  $\psi([\tilde{u}, \tilde{v}, \tilde{w}]) = (u, v, w)$ . Hence  $\mathcal{F} \in (\mathcal{V}, \mathcal{W})'_{\sigma,2} = (\mathcal{W}', \mathcal{V}')_{1-\sigma,2}$  as long as the trace for  $\tilde{w}$  makes sense, i.e. for  $\sigma$  such that  $0 < \sigma < 3/4$  (cf. (154)). As a consequence, we get that  $f \in V'$  belongs to  $(W', V')_{\theta,2}$  if and only if  $1/4 < \theta < 1$ . The order of our problem is thus (cf. Theorem 4.3 and Remark 4.1)  $\alpha = 3/2$ . □

**Acknowledgements:** This work has been partly supported by I.A.N. of C.N.R. in Pavia. The authors are grateful to Prof. F. Brezzi and Prof. G. Savaré for the very useful discussions about the shell problem.

## References

- [1] Baiocchi C., Savaré G., *Singular Perturbation and Interpolation*, Math. Models and Methods Appl. Sci., 1994, 4, 557–570.
- [2] Bergh J., Löfstrom J., **Interpolation Spaces: an introduction**, Springer–Verlag, 1976.

- [3] Bernadou M., Ciarlet P.G., *Sur l'ellipticité du modèle linéaire de W. T. Koiter*, in Computing Methods and Engineering, eds. R. Glowinski and J.-L. Lions, Springer, 1976.
- [4] Blouza A., Le Dret H., *Existence and uniqueness for the linear Koiter model for shells with little regularity*, to appear in Quarterly of Applied Mathematics.
- [5] Blouza A., Brezzi F., Lovadina C., *A New Classification for Shell Problems*, Pubblicazioni IAN-CNR n. 1128, 1999.
- [6] Blouza A., Brezzi F., Lovadina C., *Sur la classification des coques linéairement élastiques*, to appear in C.R. Acad. Sci. Paris, Serie I, Tome 328, 1999.
- [7] Caillerie D., *Étude générale d'un type de problèmes raides et de perturbation singulière*, C. R. Acad. Sci. Paris, 1996, 323, série I, 835-840.
- [8] Chapelle D., Bathe K.J., *Fundamental considerations for the finite element analysis of shell structures*, Computer and Structures, 1998, 66(1), 19-36.
- [9] D. Chenais and J.C. Paumier, *On the locking phenomenon for a class of elliptic problems*, Numer. Math., 1994, 67, 427-440.
- [10] Ciarlet P.G., **Introduction to Linear Shell Theory**, Series in Applied Mathematics, Gauthier-Villars, 1998.
- [11] Ciarlet P.G., Lods V., *Asymptotic Analysis of linear Elastic Shells. III. Justification of Koiter's Shell Equations*. Arch. Rational Mech. Anal. 136, 191-200, Springer-Verlag, 1996.
- [12] Lions J.-L., Peetre J., *Sur une classe d'espaces d'interpolation*, Pubbl. I.H.E.S., 1964, 19, 5-68.
- [13] Lions J.-L., Sanchez-Palencia E., *Problèmes sensitifs et coques élastiques minces*, in Partial Differential Equations and Functional Analysis, eds. J. Cea et al., Birkhäuser, 1996.
- [14] Piila J., Leino Y., Ovaskainen O., Pitkaranta J., *Shell deformation states and the finite element method: a benchmark study of cylindrical shells*, Comput. Methods Appl. Mech. Engrg, 1995, 128, 81-121.
- [15] Sanchez-Palencia E., *Statique et dynamique des coques minces. I. Cas de flexion pure non inhibée*, C. R. Acad. Sci. Paris, 1989, 309 série I, 411-417.
- [16] Sanchez-Palencia E., *Statique et dynamique des coques minces. II. Cas de flexion pure inhibée - Approximation membranaire*, C. R. Acad. Sci. Paris, 1989, 309 série I, 531-537.
- [17] Sanchez-Palencia E., *Asymptotic and spectral properties of a class of singular-stiff problems*, J. Math. Pures Appl., 1992, 71, 379-406.