BV functions and variational models in plasticity

(Maria Giovanna Mora, Enrico Vitali – Pavia)

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- Basic properties of the space BV of *functions with bounded variation* and of the space BD of *functions with bounded deformation*.

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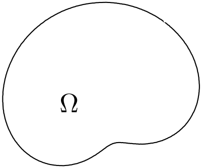
- Basic properties of the space *BV* of *functions with bounded variation* and of the space *BD* of *functions with bounded deformation*.
- Analysis of a variational model in plasticity (in the functional framework introduced in the first part).

A sketch of the main motivating problem



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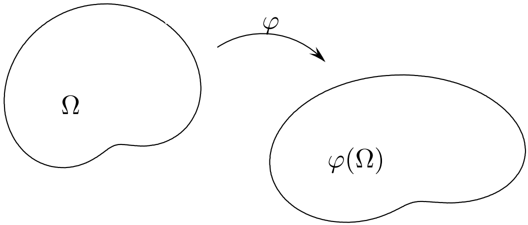

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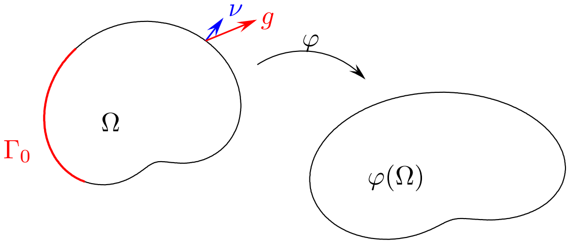

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$+$ B.C. (u prescribed on Γ_0 and external force prescribed on the complement)

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Assumption: $\mathbb{K} = K + \mathbb{R}I$, with K convex, compact neighbourhood of 0 in $M^{n\times n}_{D}$.

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The variational approach involves the energy functional

$$F(u) = \frac{1}{2}\int_\Omega Q(e)\,\mathrm{d}x + \int_\Omega H(p)\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_0} g(x)u(x)\mathrm{d}\mathscr{H}^{n-1}$$

where:

Q : positive definite quadratic form (elastic energy $\mathbb{C}e : e$)

H : positively 1-homogeneous convex function. (H is the support function of K).

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Key fact: since H has linear growth, the minimization problem for F has, in general, no solution in Sobolev spaces; in the natural weak formulation, plastic deformations are allowed to take measure values. This agrees with the points of view of mechanics: shear deformations concentrates, and shear bands can be thought of as sharp discontinuities of the displacement.

This naturally leads to the space of functions with bounded deformation

$$BD(\Omega) = \{u \in L^1(\Omega;\mathbb{R}^n) : \ Eu \text{ bounded} \\ \text{(matrix-valued) Radon measure}\}$$

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Thus, it looks quite natural the ‘preliminary’ study of the space of functions with bounded variation:

$$BV(\Omega) = \{u \in L^1(\Omega;\mathbb{R}^n) : \ \nabla u \text{ bounded} \\ \text{(matrix-valued) Radon measure}\}$$

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On the other hand, we point out the a wide classical literature makes the space BV a relevant functional space in modern variational analysis.

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$$0 = t_k^k < t_1^k < \ldots < t_k^k = T \qquad (\max |t_i^k - t_{i-1}^k| \xrightarrow{k} 0)$$

we define a piecewise-constant evolution by minimizing iteratively

$$\frac{1}{2}\int_\Omega Q(e)\,\mathrm{d}x + \int_\Omega H(p - p_{i-1}^k)\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_0} g(t_i^k, x)u(x)\mathrm{d}\mathscr{H}^{n-1}$$

(with respect to the triple (u, e, p)).

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The relevant result is now passing to the limit (as $k \to \infty$) in order to get a time-continuous evolution.

Background: standard measure theory and functional analysis; basic results on Sobolev spaces.

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Sede:	Pavia
Orario:	28-32 ore, 4 ore/settimana (eventualmente 2+2 matt.+pom.)
	15 aprile – 15 giugno (approssimativamente)