Free boundary problems in partially saturated porous media

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joint project with Antonio Fasano (Firenze, Italy), Pavel Krejčí (WIAS, Germany), Mario Primicerio (Firenze, Italy)

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The starting point

M. ELEUTERI, P. KREJČÍ: *"Asymptotic behaviour of a Neumann parabolic problem with hysteresis"*, ZAMM - Z. Angew. Math. Mech., **87**, No. 4, 261-277 (2007).

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Statement of the problem

- Long time asymptotic stabilization of the solution of

$$\frac{\partial}{\partial t}(f(u)+w)-\triangle u=0 \qquad \text{in } \Omega\times(0,\infty),$$

  - $\Omega \subset \mathbb{R}^d$ with $d=2$ or $d=3$ of Lipschitz class
  - $f:\mathbb{R}\to\mathbb{R}$ given function
  - $w(x,t)=\mathscr{W}[\lambda,u](x,t)=\int_0^\infty g(r,\wp_r[\lambda(x,\cdot),u(x,\cdot)](t))\,dr$ output of a Preisach operator with initial memory configuration λ

- Homogeneous Neumann boundary conditions and initial conditions

$$\frac{\partial u}{\partial n}=0 \qquad \text{on } \partial\Omega\times(0,\infty) \qquad u(x,0)=u_0(x) \quad \text{and} \quad w(x,0)=w_0(x).$$

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Known results

- Existence results on bounded time intervals and asymptotic behaviour for Dirichlet boundary conditions:

N. Kenmochi, A. Visintin: Asymptotic stability for nonlinear PDEs with hysteresis. *Euro. Jnl. Appl. Math.* **5** (1994), 39–56.

A. Visintin: *Differential Models of Hysteresis.* Springer, Berlin - Heidelberg, 1994.

- Uniqueness:

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Preisach diagram

✔ $\mathscr{W}[\lambda,u](x,t) = \int_0^\infty g(r, \wp_r[\lambda(x,\cdot), u(x,\cdot)](t))\,dr$

output of a Preisach operator $\mathscr{W}$ with initial memory configuration λ and generating function g

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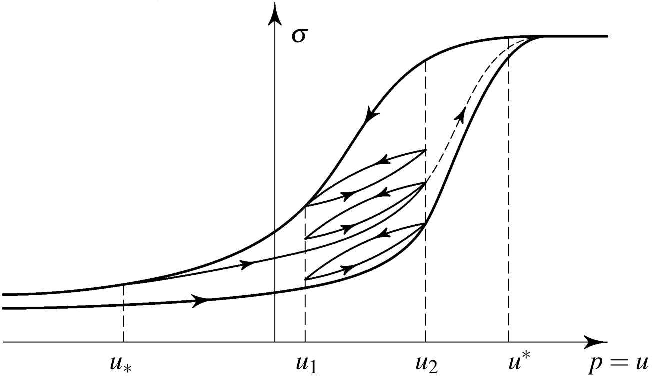

Mathematical difficulties

- We consider the initial condition for u within the invertibility domain of the constitutive operator
- By the parabolic maximum principle, the whole process takes place between these bounds
- The main result we prove is indeed in agreement with the case without hysteresis as well as with the 1D case, and therefore should not be surprising
- The way to prove it is, however, rather difficult

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Mathematical difficulties

- This is due to the fact that the state space of the process consists not only of all admissible space distributions of u, but also of all admissible space dependent memory configurations of the Preisach operator $\mathscr{W}$

- More specifically, besides the space variable $x \in \Omega$ and time $t > 0$, an additional memory variable $r > 0$ comes into play and makes the asymptotic analysis more complicated

- Note that, as it is often the case in the qualitative theory of PDEs with hysteresis, it is more convenient to use here the equivalent one-parametric representation of the Preisach operator, instead of the two-parametric one known from engineering literature

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Main result

Theorem

Under suitable assumptions on the data, the previous problem admits a unique continuous solution u such that

- $u_* \leq u(x,t) \leq u^*$ *for all* $(x,t) \in \Omega \times \mathbb{R}_+$
- $\partial_t u, \triangle u \in L^2(\Omega \times \mathbb{R}_+) \cap L^\infty(\mathbb{R}_+; L^2(\Omega)),\ \partial_t \nabla u \in L^2(\Omega \times \mathbb{R}_+)$
- *there exists a constant* $u_\infty \in \mathbb{R}$ *such that*

$$\lim_{t \to \infty} \sup_{x \in \Omega} |u(x,t) - u_\infty| = 0$$

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Sketch of the proof of the main result

➜ **Tool:**

- time discrete approximation
- uniform estimates
- passage to the limit by compactness
- Estimate 1 ⇨ bounds for the solution

$$u_* \leq u(x,t) \leq u^* \text{ for all } (x,t) \in \Omega \times \mathbb{R}_+$$

- Estimate 2 and estimate 3: ⇨ regularity of the solution

$$\partial_t u, \triangle u \in L^2(\Omega \times \mathbb{R}_+) \cap L^\infty(\mathbb{R}_+; L^2(\Omega)), \partial_t \nabla u \in L^2(\Omega \times \mathbb{R}_+)$$

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Asymptotic behaviour

We distinguish two steps:

- Step 1: there exist two constants $A > 0$ and $c > 0$ such that

$$\left| u(x,t) - \frac{1}{|\Omega|} \int_\Omega u(x,t)\, dx \right| \le A\, e^{-ct}$$

- Step 2: there exists a constant u_∞ such that

$$\lim_{t \to \infty} U(t) := \frac{1}{|\Omega|} \int_\Omega u(x,t)\, dx = u_\infty$$

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The 3D-case

- In the case $d = 3$, we have assumed in addition the following restriction on the initial data

$$\gamma^2 \int_\Omega \left(|\nabla u_0|^2 + |\Delta u_0|^2 \right) dx \; < \; \frac{8 f_0^2}{81 \mu_4^4(\Omega)} \, ,$$

- If f is linear and $\mathscr{W}$ is the so-called *Prandtl-Ishlinskii operator*, then $\gamma = 0$
- In 3D, either the initial datum must be close to a constant, or the combined hysteresis nonlinearity $f + \mathscr{W}$ must be somehow close to a Prandtl-Ishlinskii operator within the range $[u_*, u^*]$
- Geometrically, the ascending/descending hysteresis branches have to be "almost convex/almost concave", respectively

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- The key point relies on the fact that, in order to obtain the asymptotic stabilization, it is not needed to show the regularity of the solution in the whole space-time cylinder $\Omega \times \mathbb{R}_+$, but only for sufficiently large times

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Physical interpretation of the model problem

➜ Soil hydrology

✎ Mass balance

$$\frac{\partial \sigma}{\partial t} + \mathrm{div}\, q = 0 \qquad \text{in } \Omega, \quad \langle q, n \rangle = 0 \text{ on } \partial\Omega$$

✎ Darcy's law

$$q = -k(p)\nabla(p)$$

✎ suitable constitutive relation between saturation σ and the pressure p

$$\sigma = f(p) + \mathscr{W}[\lambda, p]$$

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$$\frac{\partial \sigma}{\partial t} + \mathrm{div}\, q = 0 \qquad \text{in } \Omega, \quad \langle q, n \rangle = 0 \ \text{ on } \partial\Omega$$

✎ Darcy's law

$$q = -k(p)\nabla(p)$$

✎ suitable constitutive relation between saturation σ and the pressure p

$$\sigma = f(p) + \mathscr{W}[\lambda, p]$$

Physical interpretation of the model problem

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Interpretation of our results

- Deal with a column of (partially saturated) porous medium
- Homogeneous Neumann boundary conditions $\Leftrightarrow$ no flux (of water) from the top or the bottom of the column
- Asymptotic stabilization of the solution $\Leftrightarrow$ if there is no flux of water, after some time, the system will converge to an equilibrium
- From the *mathematical point of view*: results difficult to prove (due to the presence of hysteresis)
- From the *physical point of view*: expected
- The main question: what about considering more physically realistic boundary conditions?

✍ Joint project with A. Fasano and M. Primicerio

OCCURRENCE OF FREE BOUNDARY PROBLEMS

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Picture of the front initially prescribed

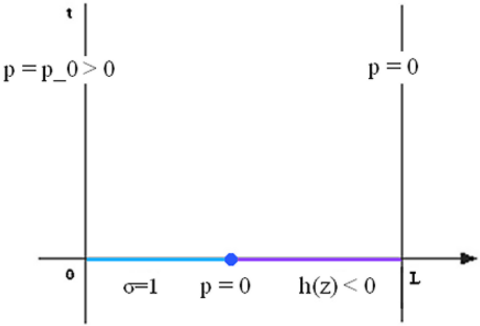

Figure 2: Picture of the front initially prescribed.

Problem (unsaturated region)

To find a triple (T, s, p) such that
(a) $0 < T \leq T_0 \qquad s \in \mathscr{C}^0([0,T]), \qquad s \in (0,L)$;
(b) p and p_z are continuous in $\overline{D}_T$, p_t and p_{zz} are continuous in D_T, where

$$D_T = \{(x,t) : s(t) < z < L,\ 0 < t < T\};$$

(c) the following equations are satisfied:

$$\begin{cases} a(p)p_t - p_{zz} = 0 & \text{in } D_T \\ s(0) = b & \\ p(x,0) = h(x) \leq 0 & b < x < L \\ p(s(t),t) = 0 & 0 < t < T \\ p_z(s(t),t) = -\frac{p_0(t)}{s(t)}, & 0 < t < T \\ p(L,t) = 0 & 0 < t < T \end{cases}$$

A. Fasano, M. Primicerio: *Liquid flow in partially saturated media*, J. Inst. Maths. Applics, (1979), **23**, 503-517.

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Analysis of occurrence of further free boundary problems

- Is there contradiction to have the presence of further additional free boundaries?

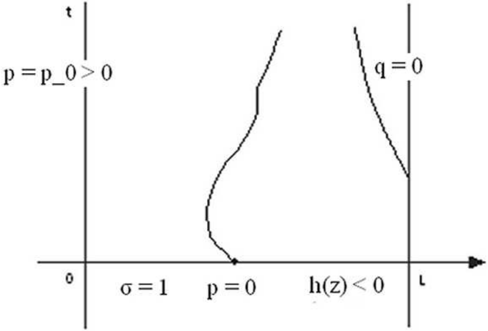

Figure 3: May a second front arise?.

May a new front arise?

- We have the creation of another region completely saturated
- For $z = L$ we have the condition of zero flux (i.e. $q(L,t) = 0$)
- As long as the new region is fully saturated, there is no flux of water through the region
- Therefore the same condition $q = 0$ remains constantly on the second front also
- Thus the conditions on the new front are

$$\begin{cases} p(s(t),t) = 0 \\ p_z(s(t),t) = 0 \end{cases}$$

which is a contradiction (Hopf lemma)

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- Thus the conditions on the new front are

$$\begin{cases} p(s(t),t) = 0 \\ p_z(s(t),t) = \rho\, g > 0 \end{cases}$$

which produces no contradiction

May a third front arise?

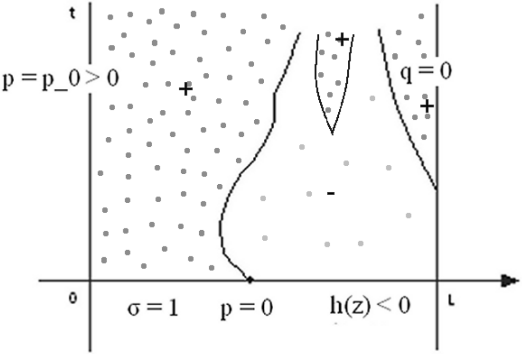

Figure 4: May a third front arise?.