Alma Ticinensis Universitas · Università di Pavia
Dipartimento di Matematica "Felice Casorati"

On a phase field system interconnecting the Green and Naghdi types

Giacomo CANEVARI
Reporting on a joint work with **Pierluigi COLLI**

Multiphase for ADvanced MATerials
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Description of Problem $P_{\alpha,\beta}$

$$\partial_t^2 w - \alpha\Delta\partial_t w - \beta\Delta w + \partial_t u = f \qquad \text{on } [0,\, T] \times \Omega \tag{1}$$

$$\partial_t u - \Delta u + \gamma(u) + g(u) \ni \partial_t w \qquad \text{on } [0,\, T] \times \Omega \tag{2}$$

$$\partial_n w = 0\,, \qquad \partial_n u = 0 \qquad \text{on } [0,\, T] \times \partial\Omega \tag{3}$$

$$w(0,\,\cdot) = w_0\,, \quad \partial_t w(0,\,\cdot) = v_0\,, \quad u(0,\,\cdot) = u_0 \quad \text{on } \Omega\,. \tag{4}$$

- $\Omega \subseteq \mathbb{R}^3$ is a bounded smooth domain, $T > 0$ a finite time;
- u is the phase variable;
- w is the thermal displacement: if θ is the temperature, then by definition

$$w(t,\, x) = w_0(x) + \int_0^t \theta(s,\, x)\, ds \qquad \text{for } (t,\, x) \in [0,\, T] \times \Omega\,;$$

- $\alpha,\, \beta > 0$ are parameters, $\gamma \subseteq \mathbb{R}^2$ is a maximal monotone graph, g a Lipschitz–continuous function on $\mathbb{R}$, f a given source term.

- Let

$$\phi : \mathbb{R} \longrightarrow [0, +\infty] \quad \text{be a proper l.s.c. convex function,}$$

$$\phi(0) = 0\,, \qquad \partial\phi = \gamma\,,$$

and let G be a smooth function s.t. $G' = g$. If we define the **free energy**

$$\psi(\theta, u) = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + \phi(u) + G(u) - \frac{1}{2}\theta^2 - \theta u \right\}\,,$$

then the equation (2) follows from

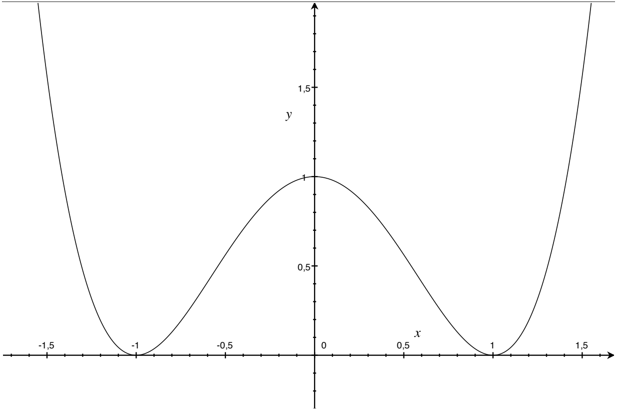
$$\partial_t u + d_u \psi(\theta, u) = 0\,.$$

- The equation (1) expresses the energetic balance

$$\partial_t (\theta + u) + \operatorname{div} \mathbf{q} = f\,,$$

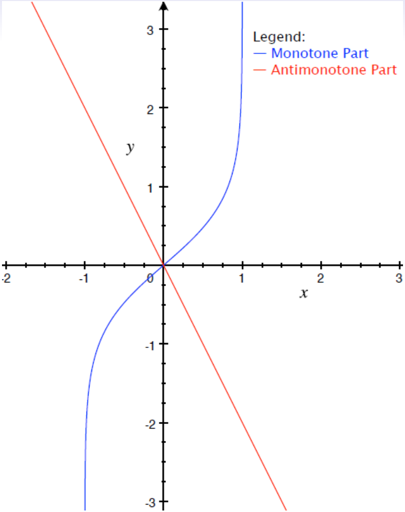
where $\theta + u = -d_\theta \psi(\theta, u)$ is the enthalpy and $\mathbf{q}$ the thermal flux.

Some examples for the **bulk potential** $\phi + G$:



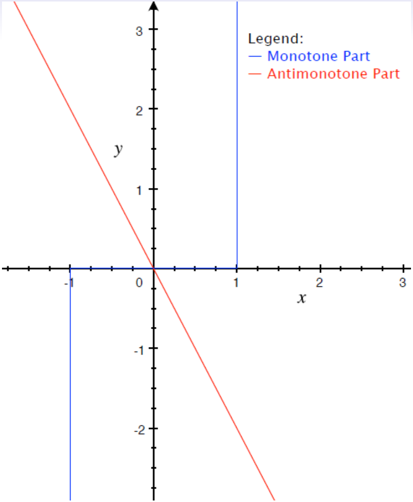
- the Caginalp "double well" potential

$$\phi(u) + G(u) = (1 - u^2)^2\,, \qquad u \in \mathbb{R}\,;$$



- the logaritmic potential, defined on $(-1, 1)$ by

$$\gamma(u) = \log(1+u) - \log(1-u)\,, \quad g(u) = -2u\,;$$



- the "double obstacle" potential, s.t.

$$\gamma = \partial \mathcal{I}_{[-1,1]}\,, \quad g(u) = -2u\,.$$

Some **constitutive assumptions** for the thermal flux $\mathbf{q}$, according to the Green and Naghdi's theory, in the linearized versions:

- Type I (Fourier)

$$\mathbf{q} = -\alpha \nabla \partial_t w \,, \qquad \alpha > 0$$

- Type II (Gurtin–Pipkin)

$$\mathbf{q} = -\beta \nabla w \,, \qquad \beta > 0$$

- Type III

$$\mathbf{q} = -\alpha \nabla \partial_t w - \beta \nabla w \,, \qquad \alpha \,, \beta > 0 \,.$$

In this work, we consider:

Problem $\mathbf{P}_{\alpha,\,\beta}$	Type III
Limit as $\beta \searrow 0$	Type I
Limit as $\alpha \searrow 0$	Type II

References

- For the original Caginalp model, see **Caginalp**, 1986, and **Caginalp, Nishiura**, 1991.
- For the Green and Naghdi's theory, see **Green and Naghdi**, 1991, 1992, 1993, and 1995.
- For the use of non–smooth potentials in thermodynamics, see **Frémond**, 2002.
- For some generalizations of the Caginalp model, involving type II and III laws and logarithmic potentials, see **Miranville, Quintanilla**, 2009 (two papers), 2010, and 2011.
- For the study of Problem $\mathbf{P}_\beta$, see **Colli, Gilardi, Grasselli**, 1997 (two papers).
- For the asymptotics on phase field problem, see, e.g., **Bonetti**, 1999.
- The subject of this talk is studied in **G.C., Colli**, 2012 and t.a.

Study of Problem $\mathbf{P}_{\alpha,\,\beta}$

Set

$$V = H^1(\Omega)\,, \qquad H = L^2(\Omega)\,,$$

$$W = \left\{ v \in H^2(\Omega) : \ \partial_n v = 0 \text{ on } \partial\Omega \right\} \,.$$

Problem $\mathbf{P}_{\alpha,\,\beta}$. Finding $(w,\, u,\, \xi)$ which satisfies, for all $v \in V$ and a.a. $t \in [0,\, T]$,

$$w \in W^{2,\,1}\left(0,\, T;\, V'\right) \cap W^{1,\,\infty}\left(0,\, T;\, H\right) \cap H^1\left(0,\, T;\, V\right)$$

$$u \in H^1\left(0,\, T;\, V'\right) \cap C^0\left([0,\, T];\, H\right) \cap L^2\left(0,\, T;\, V\right)$$

$$\xi \in L^2([0,\, T] \times \Omega)\,, \qquad u \in D(\gamma) \text{ and } \xi \in \gamma(u) \text{ a.e.}$$

$$\left\langle \partial_t^2 w(t),\, v \right\rangle + \alpha \left(\nabla \partial_t w(t) + \beta \nabla w(t),\, \nabla v \right)_H + \left\langle \partial_t u(t),\, v \right\rangle = \left\langle f(t),\, v \right\rangle$$

$$\left\langle \partial_t u(t),\, v \right\rangle + \left(\nabla u(t),\, \nabla v \right)_H + \left(\xi(t) + g(u)(t),\, v \right)_H = \left(\partial_t w(t),\, v \right)_H .$$

We assume

$$f \in L^2(0,\,T;\,V') + L^1(0,\,T;\,H) \tag{5}$$

$$w_0 \in V\,, \qquad v_0 \in H\,, \qquad u_0 \in H\,, \qquad \phi(u_0) \in L^1(\Omega)\,. \tag{6}$$

Theorem (Existence and uniqueness for Problem $\mathbf{P}_{\alpha,\,\beta}$)

Under the assumptions (5)–(6), *Problem* $\mathbf{P}_{\alpha,\,\beta}$ *admits a unique solution.*

The proof is based on

- the Faedo–Galerkin approximation scheme;
- the Yosida regularization of γ:

$$\gamma_\varepsilon = \frac{1}{\varepsilon}\left\{\mathrm{Id} - (\mathrm{Id} + \varepsilon\gamma)^{-1}\right\}\,, \qquad 0 < \varepsilon \leq 1\,.$$

Theorem (Regularity and strong solution)

If the hypotheses

$$f \in L^2(0,\,T;\,H) + L^1(0,\,T;\,V) \tag{7}$$

$$w_0 \in W\,, \qquad v_0 \in V\,, \qquad u_0 \in V\,, \qquad \phi(u_0) \in L^1(\Omega)\,, \tag{8}$$

hold, then the solution $(w,\,u,\,\xi)$ *of Problem* $\mathbf{P}_{\alpha,\,\beta}$ *fulfills*

$$w \in W^{2,1}(0,\,T;\,H) \cap W^{1,\infty}(0,\,T;\,V) \cap H^1(0,\,T;\,W)$$

$$u \in H^1(0,\,T;\,H) \cap C^0([0,\,T];\,V) \cap L^2(0,\,T;\,W)$$

and, in particular, it is a strong solution, i.e., it satisfies the equations (1)–(4) *pointwise a.e.*

The two results above hold true when $\Omega \subseteq \mathbb{R}^N$, for all $N \geq 1$.
On the other hand, the assumption $N \leq 3$ will be exploited in the sequel.

Theorem (L^∞ estimates)

Let $\gamma^0(s)$ denote the unique element of $\gamma(s)$ having minimal modulus, for all $s \in \mathbb{R}$. In addition to (7)–(8)*, we assume*

$$u_0 \in W\,, \qquad u_0 \in D(\gamma) \quad q.o., \qquad \gamma^0(u_0) \in H\,; \tag{9}$$

then, we have

$$u \in W^{1,\infty}(0,\,T;\,H) \cap H^1(0,\,T;\,V) \cap L^\infty(0,\,T;\,W)$$

and, in particular, $u \in C^0\left([0,\,T] \times \overline{\Omega}\right)$. Furthermore, if the assumptions

$$f \in L^\infty(0,\,T;\,H) + L^r(0,\,T;\,V) \qquad \text{for some } r > 4/3 \tag{10}$$

$$\gamma^0(u_0) \in L^\infty(\Omega) \tag{11}$$

hold, then

$$\partial_t w \in L^\infty((0,\,T) \times \Omega)\,, \qquad \xi \in L^\infty((0,\,T) \times \Omega)\,.$$

Limit as $\beta \searrow 0$

In this section $\alpha > 0$ is fixed. We denote by $(w_\beta,\, u_\beta,\, \xi_\beta)$ the solution of Problem $\mathbf{P}_{\alpha,\,\beta}$.

Question. We ask whether, as $\beta \searrow 0$, there is any convergence

$$(w_\beta,\, u_\beta,\, \xi_\beta) \longrightarrow (w,\, u,\, \xi)\,,$$

where $(w,\, u,\, \xi)$ is a solution of Problem $\mathbf{P}_\alpha$:

$$\partial_t^2 w - \alpha\Delta\partial_t w + \partial_t u = f \qquad \text{on } [0,\, T] \times \Omega$$

$$\partial_t u - \Delta u + \xi + g(u) = \partial_t w \qquad \text{on } [0,\, T] \times \Omega$$

$$u \in D(\gamma)\,, \qquad \xi \in \gamma(u) \quad \text{a.e. on } [0,\, T] \times \Omega$$

$$\partial_n w = 0\,, \qquad \partial_n u = 0 \qquad \text{on } [0,\, T] \times \partial\Omega$$

$$w(\cdot,\, 0) = w_0\,, \qquad \partial_t w(\cdot,\, 0) = v_0\,, \qquad u(\cdot,\, 0) = u_0 \qquad \text{on } \Omega\,.$$

Theorem (First error estimate as $\beta \searrow 0$)

Under the assumptions (5)–(6), *Problem* $\mathbf{P}_\alpha$ *has a unique solution; that is, there exists a unique triplet* $(w,\, u,\, \xi)$ *fulfilling*

$$w \in W^{2,1}(0,\, T;\, V') \cap W^{1,\infty}(0,\, T;\, H) \cap H^1(0,\, T;\, V)$$

$$u \in H^1(0,\, T;\, V') \cap C^0([0,\, T];\, H) \cap L^2(0,\, T;\, V)$$

$$\xi \in L^2([0,\, T] \times \Omega)\,, \qquad u \in D(\gamma) \text{ and } \xi \in \gamma(u) \text{ a.e.}$$

$$\left\langle \partial_t^2 w(t),\, v \right\rangle + \alpha\, (\nabla \partial_t w(t),\, \nabla v)_H + \langle \partial_t u(t),\, v\rangle = \langle f(t),\, v\rangle$$

$$\langle \partial_t u(t),\, v\rangle + (\nabla u(t),\, \nabla v)_H + (\xi(t) + g(u)(t),\, v)_H = (\partial_t w(t),\, v)_H$$

for all $v \in V$ *and a.a.* $t \in [0,\, T]$. *Furthermore, there is a constant* c *independent of* β *s.t.*

$$\|w_\beta - w\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|u_\beta - u\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\beta\,.$$

We have better estimates on the convergence error when γ is a single–valued, smooth function (e.g.: "double–well" or logarithmic potential).

Theorem (Second error estimate as $\beta \searrow 0$)

Suppose that $\gamma : D(\gamma) \longrightarrow \mathbb{R}$ is a single–valued, locally Lipschitz–continuous function, and that f and the initial data fulfill the strongest hypotheses. Then, the estimate

$$\|w_\beta - w\|_{W^{1,\infty}(0,T;V)\cap H^1(0,T;W)} + \|u_\beta - u\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c\beta\,,$$

holds true, for some constant c independent of β.

Limit as $\alpha \searrow 0$

Let us fix $\beta > 0$ once and for all. We denote by $(w_\alpha,\, u_\alpha,\, \xi_\alpha)$ the solution of Problem $\mathbf{P}_{\alpha,\,\beta}$.

Question. As $\alpha \searrow 0$, we ask whether the convergence

$$(w_\alpha,\, u_\alpha,\, \xi_\alpha) \longrightarrow (w,\, u,\, \xi)\,, \tag{12}$$

holds in some sense, where $(w,\, u,\, \xi)$ is a solution of Problem $\mathbf{P}_\beta$:

$$\partial_t^2 w - \beta\Delta w + \partial_t u = f \qquad \text{on } [0,\, T] \times \Omega$$

$$\partial_t u - \Delta u + \xi + g(u) = \partial_t w \qquad \text{on } [0,\, T] \times \Omega$$

$$u \in D(\gamma)\,, \qquad \xi \in \gamma(u) \quad \text{a.e. on } [0,\, T] \times \Omega$$

$$\partial_n w = 0\,, \qquad \partial_n u = 0 \qquad \text{on } [0,\, T] \times \partial\Omega$$

$$w(\cdot,\, 0) = w_0\,, \qquad \partial_t w(\cdot,\, 0) = v_0\,, \qquad u(\cdot,\, 0) = u_0 \qquad \text{on } \Omega\,.$$

The well–posedness of Problem $\mathbf{P}_\beta$ is already proved in literature.

- Assuming (5)–(6), we can prove the *weak* convergence in (12).
- Under stronger hypotheses, we infer the strong convergence, with an estimate on the convergence error.

Theorem (First error estimate as $\alpha \searrow 0$)

If we assume (5)–(6), *as well as*

$$f \in L^2(0, T; H) + L^1(0, T; V)$$

$$w_0 \in W, \qquad v_0 \in V, \qquad u_0 \in V, \qquad \phi(u_0) \in L^1(\Omega),$$

then there exists a constant c, independent of α, s.t.

$$\|w_\alpha - w\|_{W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)} + \|u_\alpha - u\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\alpha^{1/2}.$$

- For the proof, it is convenient to introduce the new variable

$$y(t,\,x) := \int_0^t u(s,\,x)\,ds + w(t,\,x) \qquad \text{for} \quad (t,\,x) \in [0,\,T] \times \Omega\,.$$

- We obtain better estimates on the convergence error when

$$\begin{array}{c}\gamma : \mathbb{R} \longrightarrow \mathbb{R} \text{ is a single–valued,} \\ \text{locally Lipschitz–continuous function}\end{array} \tag{13}$$

  (e.g.: the Caginalp “double well” potential).
- If (13) holds, we can prove regularity results for Problems $\mathbf{P}_{\alpha,\,\beta}$ and $\mathbf{P}_{\beta}$, with α–independent estimates on the solution.

Theorem (Second error estimate as $\alpha \searrow 0$)

In addition to (13), *we require*

$$f \in W^{2,\,1}(0,\,T;\,H) + W^{1,\,1}(0,\,T;\,V)$$

$$w_0 \in W\,, \quad v_0 \in W\,, \quad u_0 \in H^3(\Omega) \cap W\,, \quad \alpha\Delta v_0 + \beta\Delta v_0 + f(0) \in V\,.$$

Then, there exists a constant c, independent of α, which fulfills

$$\|w_\alpha - w\|_{W^{1,\,\infty}(0,\,T;\,H)\cap L^\infty(0,\,T;\,V)} + \|u_\alpha - u\|_{H^1(0,\,T;\,H)\cap L^\infty(0,\,T;\,V)\cap L^2(0,\,T;\,W)} \le c\alpha$$

$$\|w_\alpha - w\|_{W^{1,\,\infty}(0,\,T;\,V)\cap L^\infty(0,\,T;\,W)} \le c\alpha^{1/2}\,.$$

Final remarks

- All the results hold true when $\Omega \subseteq \mathbb{R}^3$ is, for instance, a convex polyhedron.
- Problem $\mathbf{P}_{\alpha,\,\beta}$ can be exploited to approximate Problem $\mathbf{P}_{\beta}$, via an **artificial viscosity method**. The convergence results as $\alpha \searrow 0$ can therefore be interpreted as estimates on the consistency error.
- Possible generalizations of this model:
  - Non linear coupling of θ and u in the free energy functional

$$\overline{\psi}(\theta,\,u) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla u|^2 + \phi(u) + G(u) - \frac{1}{2}\theta^2 - h(u)\theta \right\};$$

  - Non linearized versions of the Green and Nagdhi's constitutive hypotheses.