PDEs for multiphase ADvanced MATerials
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On a variational inequality of Bingham and Navier-Stokes type in three dimension

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1. Motivation

- Solid-Liquid phase transition

Well-posedness for the system between the Stefan and Navier-Stokes equations

- Interpretation of the liquid region

(i) Classical Stefan problem

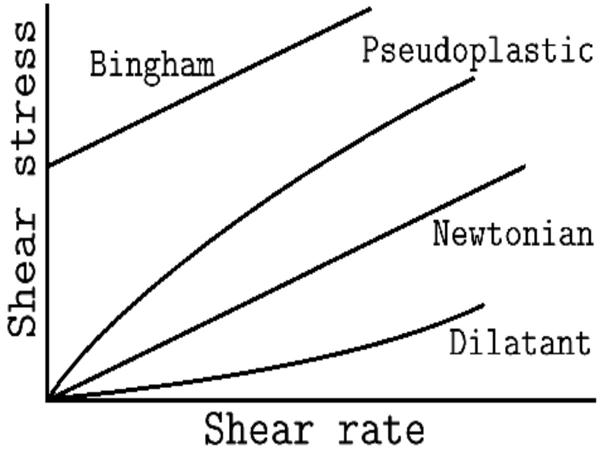
(ii) Enthalpy formulation (weak solution)

Variational formulation of Navier-Stokes equations in a material (solid-liquid) region Ω with any test function η (supp$\eta \subset \Omega_\ell(\theta)$),
(ex.) J. F. Rodrigues (2000). $\theta \in C(\overline{(0,T) \times \Omega})$.

$N = 2$, Navier-Stokes type.
$N = 3$, Non-Newtonian (dilatant fluid) type (mathematical reason)
It is not easy to formulate Navier-Stokes equations on the unknown liquid region $\Omega_\ell(\theta)$.

- Bingham fluid

Bingham fluid is characterized by a flow curve which is a straight line having an intercept on the shear stress axis.
(ex.) slurries, drilling muds, oil paints, toothpaste, etc.



If the yield limit is exceeded, then the structure completely disintegrates and the system behaves as a Newtonian fluid.

- Bingham fluid

$0 < T < +\infty$,

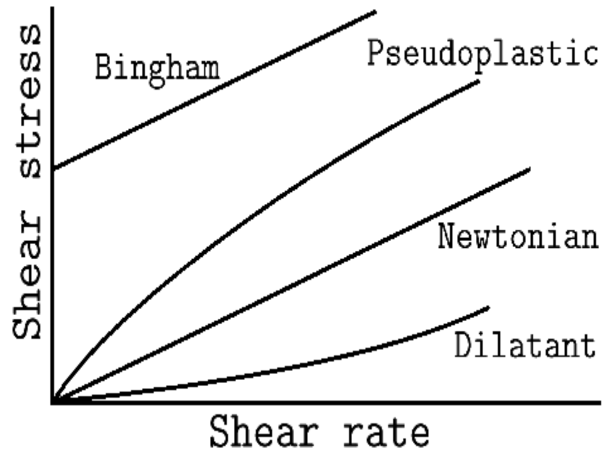
$\Omega \subset \mathbb{R}^3$: bounded domain with smooth boundary $\partial\Omega$,

$\boldsymbol{v} = (v_1, v_2, v_3)$: velocity,

p: pressure,

$\boldsymbol{f} = (f_1, f_2, f_3)$: body force,

$\boldsymbol{T} := \sigma_{ij}$: stress tensor,



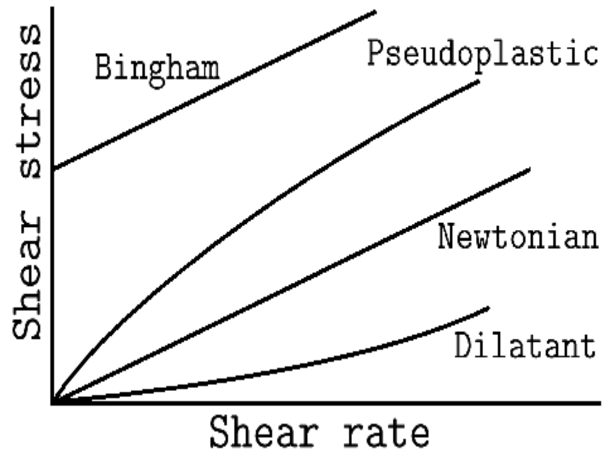
$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{f} + \mathrm{div}\boldsymbol{T} \quad \text{in } Q := (0, T) \times \Omega, \tag{1}$$

$$\mathrm{div}\boldsymbol{v} = 0 \quad \text{in } Q, \tag{2}$$

$$\boldsymbol{v} = 0 \quad \text{on } \Sigma := (0, T) \times \partial\Omega, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \Omega. \tag{3}$$

$$\mathrm{div}\boldsymbol{T} := \left(\sum_{j=1}^{3} \frac{\partial \sigma_{1j}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial \sigma_{2j}}{\partial x_j}, \sum_{j=1}^{3} \frac{\partial \sigma_{3j}}{\partial x_j} \right).$$

- Bingham fluid



$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{f} + \mathrm{div}\boldsymbol{T} \quad \text{in } Q,$$

$$\mathrm{div}\boldsymbol{v} = 0 \quad \text{in } Q,$$

The stress tensor $\boldsymbol{T} := \sigma_{ij}$ is defined by $\boldsymbol{T} := -p\boldsymbol{I} + \boldsymbol{T}^D$ satisfying (*Deviation*)

$$\boldsymbol{T}^D = 2\mu\boldsymbol{D} + \sqrt{2}g\frac{\boldsymbol{D}}{|\boldsymbol{D}|} \quad \text{if } |\boldsymbol{D}| > 0, \tag{4}$$

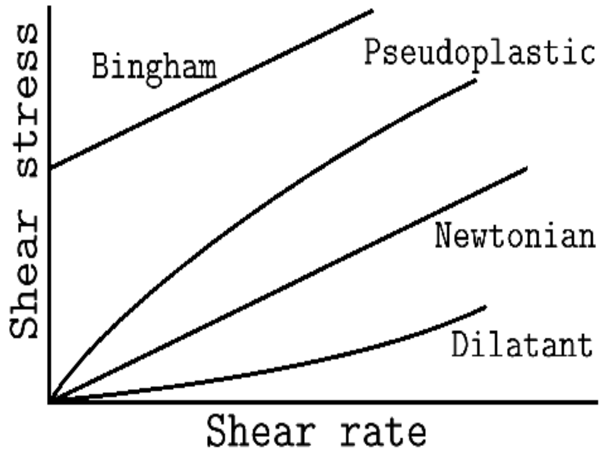
$$|\boldsymbol{T}^D| \leq \sqrt{2}g \quad \text{if and only if } |\boldsymbol{D}| = 0, \tag{5}$$

where, the shear rate (deformation rate) tensor is defined by

$$\boldsymbol{D} := \boldsymbol{D}(\boldsymbol{v}) := \epsilon_{ij} := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

$|\boldsymbol{D}| := (\sum_{i,j=1}^{3} |\epsilon_{ij}|^2)^{1/2}$, $\mu > 0$: viscosity coefficient.
$g : Q \to [0, +\infty)$ is a function, stands for the yield limit.

- Bingham fluid



$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{f} + \mathrm{div}\boldsymbol{T} \quad \text{in } Q,$$

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Remark.

Newton fluid $\quad \boldsymbol{T}^D = 2\mu\boldsymbol{D} \quad (g \equiv 0),$

Dilatant fluid $\quad \boldsymbol{T}^D = 2\mu|\boldsymbol{D}|^{p-2}\boldsymbol{D} \quad (\text{for } p > 2).$

2. Well-posedness of variational inequality

- Notation

$H := L^2(\Omega)$, $V := W_0^{1,2}(\Omega)$, V^*: dual space of V;
$V \hookrightarrow H \hookrightarrow V^*$ (dense and compact imbeddings);
$\boldsymbol{\mathcal{D}}_\sigma(\Omega) := \{\boldsymbol{u} \in \boldsymbol{C}_0^\infty(\Omega) := (C_0^\infty(\Omega))^3; \mathrm{div}\boldsymbol{u} = 0 \text{ in } \Omega\}$;
$\boldsymbol{H} := \boldsymbol{L}_\sigma^2(\Omega)$, $\boldsymbol{V} := \boldsymbol{W}_{0,\sigma}^{1,2}(\Omega)$; $\boldsymbol{V} \hookrightarrow \boldsymbol{H} \hookrightarrow \boldsymbol{V}^*$;
Bilinear functional $a(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and trilinear functional $b(\cdot,\cdot,\cdot)$: $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$a(\boldsymbol{u},\boldsymbol{w}) := 2\mu \int_\Omega \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{w}) dx := 2\mu \sum_{i,j=1}^{3} \int_\Omega \epsilon_{ij}(\boldsymbol{u})(x)\epsilon_{ij}(\boldsymbol{w})(x) dx,$$

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \sum_{i,j=1}^{3} \int_\Omega u_i(x) \frac{\partial v_j}{\partial x_i}(x) w_j(x) dx \quad ({}^\forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}),$$

noting that $b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = -b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})$, $b(\boldsymbol{u},\boldsymbol{w},\boldsymbol{w}) = 0$ for all $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}$;
$\|\boldsymbol{u}\| := a(\boldsymbol{u},\boldsymbol{u})^{\frac{1}{2}}$, the equivalent norm of $|\boldsymbol{u}|_{\boldsymbol{V}}$.

The following convex constraint set plays the important role,

$$K := \left\{ \boldsymbol{z} \in \boldsymbol{V}; |\boldsymbol{z}(x)|_{\mathbb{R}^3} \le c \quad \text{for a.a. } x \in \Omega \right\}, \tag{6}$$

$c > 0$: constant. See, G. Prouse (1979), F - N. Kenmochi (2010).

Theorem. *Assume* $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H})$, $\boldsymbol{v}_0 \in K$ *and* $g \in W^{1,1}(0,T;H)$ *with* $g \ge 0$. *Then,* $^{\exists_1}\boldsymbol{v} \in W^{1,2}(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V})$ *s.t.*

$$\boldsymbol{v}(t) \in K \quad {}^\forall t \in [0,T], \tag{7}$$

$$\begin{aligned} &(\boldsymbol{v}'(t), \boldsymbol{v}(t) - \boldsymbol{z})_{\boldsymbol{H}} + a(\boldsymbol{v}(t), \boldsymbol{v}(t) - \boldsymbol{z}) \\ &\qquad + b(\boldsymbol{v}(t), \boldsymbol{v}(t), \boldsymbol{v}(t) - \boldsymbol{z}) + \sqrt{2}\int_\Omega g(t,x)|\boldsymbol{D}(\boldsymbol{v})(t,x)|dx \\ &\le \sqrt{2}\int_\Omega g(t,x)|\boldsymbol{D}(\boldsymbol{z})(x)|dx + (\boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{z})_{\boldsymbol{H}} \\ &\qquad\qquad {}^\forall \boldsymbol{z} \in K \text{ and for a.a. } t \in (0,T), \end{aligned} \tag{8}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in } \boldsymbol{H}. \tag{9}$$

- Known result

$\diamond$ G. Duvaut - J. L. Lions, *Inequalities in mechanics and physics*, Springer Verlag, Berlin-Heidelberg-New York, 1976.

$N = 2$, abstract form, constant $g > 0$

$\diamond$ H. Brézis, J. Math. Anal. Appl., **39**(1972), 159–165.

$N = 3$, abstract form, constant $g > 0$

$\diamond$ J. Naumann, Ann. Mat. Pure Appl.(4), **124**(1980), 107–125.

$$\boldsymbol{v}'(t) + A\boldsymbol{v}(t) + B\boldsymbol{v}(t) + \partial\varphi(\boldsymbol{v}(t)) \ni \boldsymbol{f}(t) \quad \text{in } \boldsymbol{W}^*.$$

$N = 3$, related abstract form.

$\diamond$ M. Ôtani, J. Differential Equations, **46**(1982), 268–299.

$$\boldsymbol{v}'(t) + \partial\varphi^t(\boldsymbol{v}(t)) + B^t\boldsymbol{v}(t) \ni \boldsymbol{f}(t) \quad \text{in } \boldsymbol{H}.$$

$N = 3$, constant $g \geq 0$

$\diamond$ Y. Kato, Nagoya Math. J., **129**(1993), 53–95.

- Known result for Boussinesq-Stefan

$N = 3$, Dilatant fluid $p > 3$, in order to obtain $\theta \in C(\overline{Q})$.
◇ J. F. Rodrigues, On the evolution Boussinesq-Stefan problem for non-Newtonian fluids, pp.390–397 in **Vol.14**, GAKUTO Internat. Ser. Math. Sci. Appl., 2000.

Moreover, there are many results by his team for the Boussinesq-Stefan problem of "stationary Boussinesq-Stefan problem with the Bingham fluid on the unknown liquid region with a non-negative constant $g \geq 0$", "Boussinesq-Stefan of different Non-Newtonian type", etc.

Remark. In the above paper, he gave us the conjecture of the existence with respect to the special case of "solidification of Boussinesq-Stefan problem of Bingham type with a constant $g \geq 0$" under the assumption that the liquid zone decreases.

- Key of the proof R. Kano - N. Kenmochi -Y. Murase (2009):

$$\boldsymbol{v}'(t) + \partial\varphi^t(\boldsymbol{v};\boldsymbol{v}(t)) \ni \boldsymbol{g}(t) \quad \text{in } \boldsymbol{H}.$$

$\mathcal{V}(-\delta_0, t) :=$

$$\left\{\boldsymbol{u} \in W^{1,2}(-\delta_0, t; \boldsymbol{H}) \cap L^\infty(-\delta_0, t; \boldsymbol{V}); \boldsymbol{u}(t) \in K,\ {}^\forall t \in [-\delta_0, t]\right\}.$$

Moreover, the functional $\varphi^t : \mathcal{V}(-\delta_0, t) \times \boldsymbol{H} \to \mathbb{R} \cup \{+\infty\}$ by

$$\varphi^t(\bar{\boldsymbol{u}};\boldsymbol{z}) := \begin{cases} \dfrac{\mu}{2}\|\boldsymbol{z}\|^2 + \sqrt{2}\displaystyle\int_\Omega g(t,x)|\boldsymbol{D}(\boldsymbol{z})(x)|dx \\ \qquad + b(\bar{\boldsymbol{u}}(t), \bar{\boldsymbol{u}}(t), \boldsymbol{z}) + c' & \text{if } \boldsymbol{z} \in K, \\ +\infty & \text{otherwise,} \end{cases} \quad {}^\forall \bar{\boldsymbol{u}} \in \mathcal{V}(-\delta_0, t).$$

Lemma. [A. Ito - N. Yamazaki - N. Kenmochi (2008)] ${}^\exists c_1 > 0$: *constant s.t.*

$$\begin{aligned} &|\varphi^t(\bar{\boldsymbol{u}};\boldsymbol{z}) - \varphi^s(\bar{\boldsymbol{u}};\boldsymbol{z})| \\ &\quad \le c_1\{|g(t) - g(s)|_H + |\bar{\boldsymbol{u}}(t) - \bar{\boldsymbol{u}}(s)|_{\boldsymbol{H}}\}(1 + \varphi^s(\bar{\boldsymbol{u}};\boldsymbol{z})), \end{aligned}$$

${}^\forall \boldsymbol{z} \in K$, ${}^\forall \bar{\boldsymbol{u}} \in \mathcal{V}(-\delta_0, T)$, ${}^\forall s, t \in [0, T]$.

We need $g \in W^{1,1}(0,T;H)$, namely we can treat the dependence of g with respect to the temperature $\theta \in W^{1,1}(0,T;H)$, cf. J. F. Rodrigues.