Hysteresis Models for Piezoelectricity Based on Preisach Operators

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joint work with
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ADMAT2012, Cortona, September 17-21, 2012

Overview

- motivation
- Preisach operators
- a phenomenologlical model using Preisach operators
- a framework for thermodynamic consistency
- hysteresis potentials
- a wellposedness result in the elliptic case

Piezoelectric Transducers

Direct effect: apply mechanical force → measure electric voltage

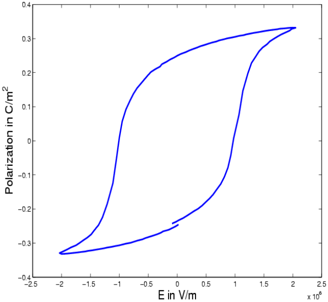
Indirect effect: impress electric voltage → observe mechanical displacement

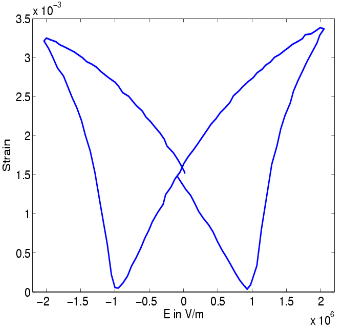
Application Areas:

- Ultrasound (medical imaging & therapy)
- Force- and acceleration Sensors
- Actor injection valves (common–rail Diesel engines)
- SAW (surface-acoustic-wave) sensors
- ...

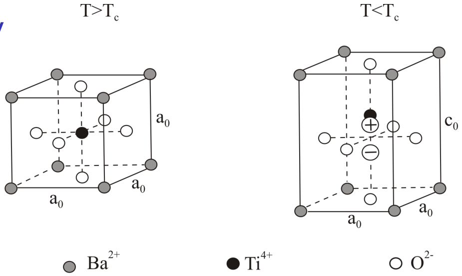
Hysteresis in Piezoelectricity

e.g. ferroelectric hysteresis:
dielectric displacement and mechanical strain
at high electric field intensities ($E \sim 2MV/m$):

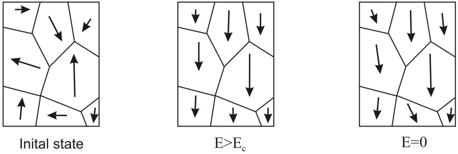




Piezoelectricity

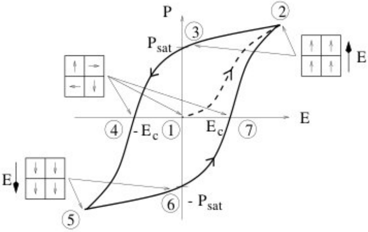


Unit cell of $BaTiO_3$ above (left) and below (right) the Curie temperature T_c

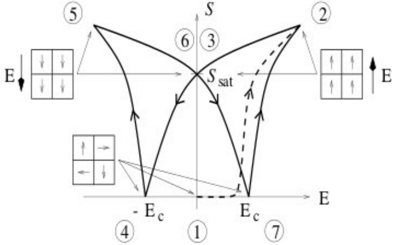


Orientation of the polarization of the grains at initial state (left), due to a strong external electric field (middle) and after switching it off (right)

Ferroelectricity



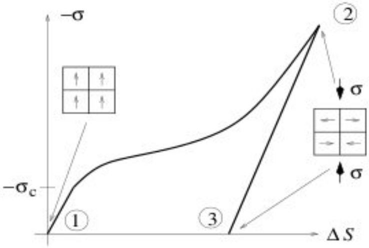
polarization hysteresis



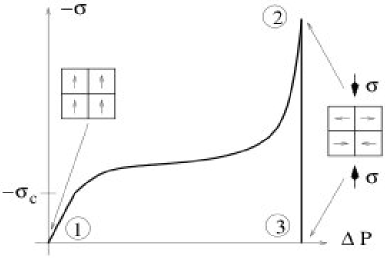
butterfly hysteresis

courtesy to M.Kamlah

Ferroelasticity



stress-strain relation



mechanical depolarization

courtesy to M.Kamlah

Models of Ferroelectricity/Ferroelasticity

(1) *Thermodynamically consistent models*
macroscopic view, 2nd law of thermodynamics
Bassiouny&Ghaleb'89, Kamlah&Böhle'01, Landis'04, Schröder&Romanowski'05, Su&Landis'07, Linnemann&Klinkel&Wagner'09

(2) *Micromechanical models*
consider material on level of single grains
Huber&Fleck'01, Fröhlich'01, Delibas&Arockiarajan&Seemann'05, Belov&Kreher'06, Huber'06, McMeeking&Landis&Jimeneza'07

(3) *Phase field models*
transition between phases (domain wall motion)
Wang&Kamlah&Zhang'10, Xu&Schrade&Müller&Gross&Granzow&Rödel'10,

(4) *Phenomenological models using hysteresis operators*
from input-output description for control purposes
Hughes&Wen'95, Kuhnen'01, Cimaa&Laboure&Muralt'02, Smith&Seelecke&Ounaies&Smith'03, Pasco&Berry04, Kuhnen&Krejčí'07, Ball&Smith&Kim&Seelecke'07, Hegewald&BK&MK&Lerch'08

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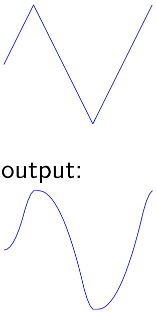
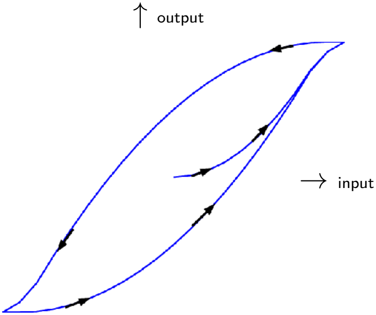
Hysteresis

input:

output:



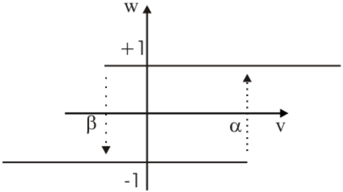


- magnetics
- piezoelectricity
- plasticity
- . . .

* memory
* Volterra property
* rate independence

Krasnoselksii-Pokrovskii (1983), Mayergoyz (1991), Visintin (1994), Krejčí (1996), Brokate-Sprekels (1996)

A Simple Example I: The Relay

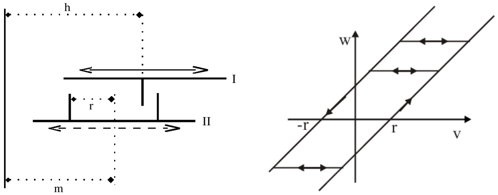


$$\mathcal{R}_{\beta,\alpha}[\mathrm{v}](t) = \mathrm{w}(t)$$

$$= \begin{cases} +1 & \text{if } \mathrm{v}(t) > \alpha \text{ or } (\mathrm{w}(t_i) = +1 \wedge \mathrm{v}(t) > \beta) \\ -1 & \text{if } \mathrm{v}(t) < \beta \text{ or } (\mathrm{w}(t_i) = -1 \wedge \mathrm{v}(t) < \alpha) \end{cases} \qquad t \in [t_i, t_{i+1}]$$

$t_0, t_1, t_2, \ldots$ sequence of local extrema of v,
i.e., v monotone on $[t_i, t_{i+1}]$.

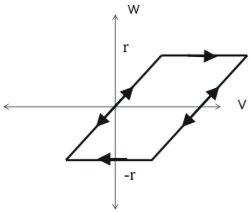
A Simple Example II: The Mechanical Play



$$\mathcal{F}_{\mathrm{r}}[\mathrm{v}](t) = \mathrm{w}(t) = \max\{\mathrm{v}(t) - \mathrm{r}, \min\{\mathrm{v}(t) + \mathrm{r}, \mathrm{w}(t_i)\}\} \quad t \in [t_i, t_{i+1}]$$

Relation to Relay operator: $\mathcal{F}_{\mathrm{r}}[\mathrm{v}](t) = \frac{1}{2} \int_{-\infty}^{\infty} \mathcal{R}_{\mathrm{s-r,s+r}}[\mathrm{v}](t)\, d\mathrm{s}$

A Simple Example III: The Elastic-Plastic Element



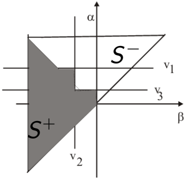
$\mathrm{v}\sim$ strain
$\mathrm{w}\sim$ stress
$\mathrm{r}\sim$ yield stress

$$\mathcal{S}_{\mathrm{r}}[\mathrm{v}](t) = \mathrm{w}(t) = \min\{\mathrm{r}, \max\{-\mathrm{r}, \mathrm{v}(t)\}\} \quad t \in [t_i, t_{i+1}]$$

Relation to mechanical Play: $\mathcal{S}_{\mathrm{r}}[\mathrm{v}](t) = \mathrm{v}(t) - \mathcal{F}_{\mathrm{r}}[\mathrm{v}](t)$

A General Hysteresis Model: the Preisach Operator

weighted superposition of elementary relays with Preisach weight function ω defined on Preisach plane $S = S^+ \cup S^-$:



$$\mathcal{P}^\omega[\mathrm{v}](t) = \iint_{\alpha,\beta \in S} \omega(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[\mathrm{v}](t)\, d(\alpha,\beta)$$

$$= \iint_{\alpha,\beta \in S^+(t)} \omega(\beta,\alpha)\, d(\alpha,\beta) - \iint_{\alpha,\beta \in S^-(t)} \omega(\beta,\alpha)\, d(\alpha,\beta)$$

$\pm$ high dimensionality

$+$ can model minor loops

$+$ can model saturation

(ω small for large values of β, α)

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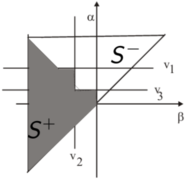

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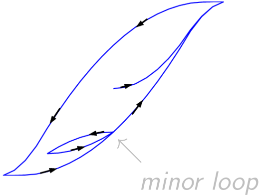

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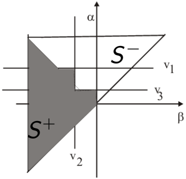

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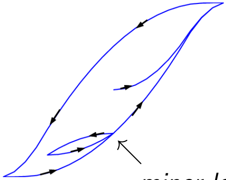


Memory Deletion and Everett Functions (I)

deletion rules:

- monotone deletion: only local extrema of the input are relevant for the subsequent output
- Madelung rule: inner *minor loops* are forgotten
- wipe out: previous local extrema are removed from memory by subsequent absolutely larger extrema

$$(\mathrm{v}(\tau))_{\tau\in[0,t]} \stackrel{\text{deletion}}{\longrightarrow} \mathrm{v}(t_i)_{i\in\{1,\ldots,N\}}$$

Memory Deletion and Everett Functions (II)

Everett (or shape-) **function:**

$$\begin{aligned} \mathcal{E}(v_*, v^*) &= 2\iint_{v_* \le \beta \le \alpha \le v^*} \omega(\beta,\alpha)\, d(\alpha,\beta) \text{ if } v_* < v^* \\ \mathcal{E}(-v^*, -v_*) &= \mathcal{E}(v_*, v^*) \end{aligned}$$

$$\mathcal{P}^\omega[\mathrm{v}](t) = \iint_{\alpha,\beta \in S} \omega(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[\mathrm{v}](t)\, d(\alpha,\beta) = \mathcal{P}^\omega[\mathrm{v}_0] + \sum_{i=1}^{N} \mathcal{E}(\mathrm{v}(t_i), \mathrm{v}(t_{i+1}))$$

$\rightsquigarrow$ crucial for efficient computation:

- avoid integral evaluation during the actual computations
- store only few time instances in the past (update)

Hysteresis in PDEs: each point in space has its own memory!

A Phenomenological Model for Hysteresis in Piezoelectricity

$$\begin{aligned} \underline{S} &= \underline{S}^r + \underline{S}^i \\ \vec{D} &= \vec{D}^r + \vec{D}^i \end{aligned} \qquad \begin{aligned} \underline{S}^r &= \mathbf{s}^E \underline{\sigma} + \mathbf{d}_{\vec{P}}^T \vec{E} \\ \vec{D}^r &= \mathbf{d}_{\vec{P}} \underline{\sigma} + \varepsilon^\sigma \vec{E} \end{aligned} \qquad \begin{aligned} \vec{D}^i &= \vec{P} = \mathcal{P}^\omega[E] e_{\vec{P}} \\ \underline{S}^i &= f_S(\vec{P})(\tfrac{3}{2} e_{\vec{P}} e_{\vec{P}}^T - \tfrac{1}{2} I) \\ \mathbf{d}_{\vec{P}} &= f_{\mathbf{d}}(P) = \frac{P}{P_{sat}} \mathbf{d} \end{aligned}$$

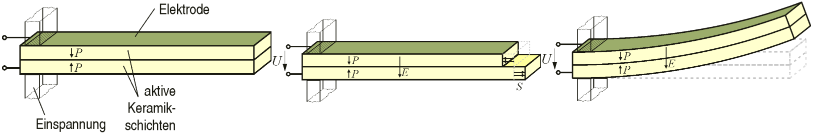
$\underline{S}^r$ ... reversible strain
$\underline{S}^i$ ... irreversible strain
$\vec{D}^r$ ... reversible polarization
$\vec{D}^i$ ... irreversible polarization

$\underline{\sigma}$ ... mech. stress
$\vec{E}$ ... electr. field
$\vec{P}$ ... polarization

$\mathbf{s}^E$ ... elast. coeff.
ε^σ ... dielectr. coeff.
$\mathbf{d}$ ... coupling coeff.

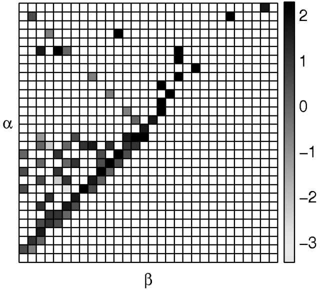
hysteresis identification: [Hegewald&B.K&M.K.&Lerch, J.Int.Mat.Sys.Struct.'08]
finite element formulation: [M.K&B.K.&Hegewald&Lerch J.Int.Mat.Sys.Struct.'09]

Numerical Results: Bending Actuator



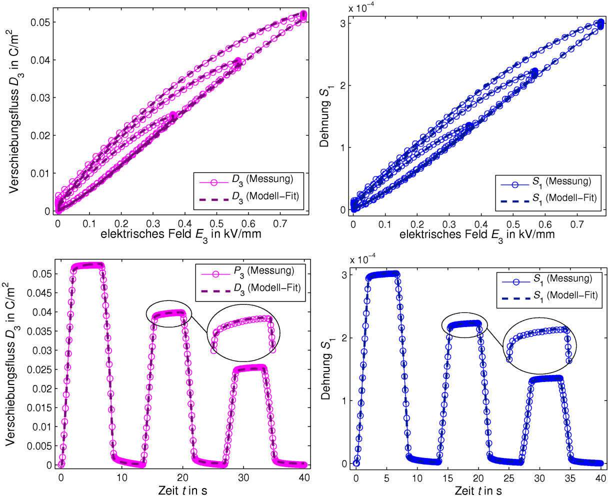
Numerical Results: Bending Actuator

Identified Preisach weight function:



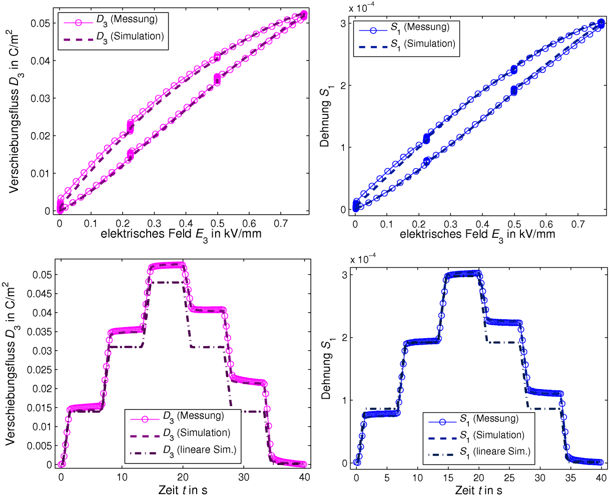
Numerical Results: Bending Actuator

comparison measurement – simulation with fitted Preisach operators:



Numerical Results: Bending Actuator

comparison measurement – simulation for alternative input signal:



A Framework for Thermodynamic Consistency (I)

(by P.Krejčí 2009, originally for magnetostriction; uniaxial loading)

$$\begin{aligned} S &= s^E \sigma - F_1[\sigma] U_2[E] \\ D &= \varepsilon^\sigma E - F_2[E] U_1[\sigma] \end{aligned}$$

with

$$\left\{ \begin{array}{rcll} F_1[\mathrm{v}]\dot{\mathrm{v}} - (U_1[\mathrm{v}])^{\cdot} & \geq & 0 & (*) \\ F_2[\mathrm{v}]\dot{\mathrm{v}} - (U_2[\mathrm{v}])^{\cdot} & \geq & 0 & (*) \\ U_1[\mathrm{v}] & \geq & 0 & \\ U_2[\mathrm{v}] & \geq & 0 \,. & \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{rcll} F_1[\mathrm{v}]\dot{\mathrm{v}} - (U_1[\mathrm{v}])^{\cdot} & = & 0 & \\ F_2[\mathrm{v}]\dot{\mathrm{v}} - (U_2[\mathrm{v}])^{\cdot} & \geq & 0 & (*) \\ U_1[\mathrm{v}] & \geq & 0 \,. & \end{array} \right.$$

$(*)\ldots U_i$ clockwise hysteresis potential for F_i,

A Framework for Thermodynamic Consistency (II)

Therewith for the free energy

$$W(\sigma, E) := -\frac{1}{2}(s^E\sigma^2 + \varepsilon^\sigma E^2) + S\sigma + DE + U_1[\sigma]U_2[E]$$

we get thermodynamic consistency

$$\dot{S}\sigma + \dot{D}E - \dot{W}$$

$$= (S\sigma + DE - W)^{\cdot} - S\dot{\sigma} - D\dot{E}$$

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$$\begin{aligned} & \dot{S}\sigma + \dot{D}E - \dot{W} \\ = \; & (S\sigma + DE - W)^{\cdot} - S\dot{\sigma} - D\dot{E} \\ = \; & s^E \sigma\dot{\sigma} + \varepsilon^\sigma E\dot{E} - (U_1[\sigma])^{\cdot} U_2[E] - U_1[\sigma](U_2[E])^{\cdot} \\ & -(s^E\sigma - F_1[\sigma]U_2[E])\dot{\sigma} - (\varepsilon^\sigma E - F_2[E]U_1[\sigma])\dot{E} \end{aligned}$$

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$$\begin{aligned}
& \dot{S}\sigma + \dot{D}E - \dot{W} \\
&= (S\sigma + DE - W)^{\cdot} - S\dot{\sigma} - D\dot{E} \\
&= s^E\sigma\dot{\sigma} + \varepsilon^\sigma E\dot{E} - (U_1[\sigma])^{\cdot}U_2[E] - U_1[\sigma](U_2[E])^{\cdot} \\
&\quad -(s^E\sigma - F_1[\sigma]U_2[E])\dot{\sigma} - (\varepsilon^\sigma E - F_2[E]U_1[\sigma])\dot{E} \\
&= (F_1[\sigma]\dot{\sigma} - (U_1[\sigma])^{\cdot})U_2[E] + (F_2[E]\dot{E} - (U_2[E])^{\cdot})U_1[\sigma] \\
&\geq 0
\end{aligned}$$

A Framework for Thermodynamic Consistency (II)

Therewith for the free energy

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&\geq 0
\end{aligned}$$

A Framework for Thermodynamic Consistency (III)

Thermodynamic consistency remains valid for

$$\begin{aligned} S &= s^E\sigma + dE - \sum_{i=1}^{n} F_1^{(i)}[\sigma]U_2^{(i)}[E] \\ D &= d\sigma + \varepsilon^\sigma E - \sum_{i=1}^{n} F_2^{(i)}[E]U_1^{(i)}[\sigma] \end{aligned}$$

with

$$F_j^{(i)}[\mathrm{v}]\dot{\mathrm{v}} - (U_j^{(i)}[\mathrm{v}])^{\cdot} \geq 0 \ \forall \mathrm{v} \text{ and } U_{j\pm1}[\mathrm{v}] \geq 0 \ \forall \mathrm{v}$$

or

$$F_j^{(i)}[\mathrm{v}]\dot{\mathrm{v}} - (U_j^{(i)}[\mathrm{v}])^{\cdot} = 0 \ \forall \mathrm{v}$$

A Framework for Thermodynamic Consistency (IV)

Thermodynamic consistency remains valid for

$$\begin{aligned} S &= s^E \sigma + dE + F_1[\sigma](EF_2[E] - U_2[E]) \\ D &= d\sigma + \varepsilon^\sigma E + F_2[E]U_1[\sigma] \end{aligned}$$

with $\left.\begin{aligned} F_1[\mathrm{v}]\dot{\mathrm{v}} - (U_1[\mathrm{v}])^{\cdot} &\geq 0 \\ \text{and } U_2[\mathrm{v}] - vF_2[v] &\geq 0 \end{aligned}\right\}$ or $F_1[\mathrm{v}]\dot{\mathrm{v}} - (U_1[\mathrm{v}])^{\cdot} = 0$

and $\left\{\begin{aligned} (F_2[\mathrm{v}])^{\cdot}\mathrm{v} - (U_2[\mathrm{v}])^{\cdot} &\geq 0 \quad (*) \\ \text{and } U_1[\mathrm{v}] &\geq 0\,. \end{aligned}\right.$

$(*)\ldots U_2$ counterclockwise hysteresis potential for F_2

Special Case: Pure Electric Loading

$$
\begin{aligned}
S &= S^{rev} + S^{ir} = s^E\sigma + dE + \tilde{\mathcal{Q}}[E] &&= s^E\sigma - F_1(\sigma)U_2[E] \\
D &= D^{rev} + D^{ir} = d\sigma + \varepsilon^\sigma E + \tilde{\mathcal{P}}[E] &&= \varepsilon^\sigma E - F_2[E]U_1(\sigma) \\
d &= \frac{d_0}{P_0}\tilde{\mathcal{P}}[E]
\end{aligned}
$$

$\tilde{\mathcal{P}}\dots$ polarization hysteresis
$\tilde{\mathcal{Q}}\dots$ butterfly hysteresis
σ small $\rightsquigarrow$ reversible (non)linearity wrt σ.

Special Case: Pure Electric Loading

$$S = S^{rev} + S^{ir} = s^E\sigma + dE + \tilde{\mathcal{Q}}[E] = s^E\sigma - F_1(\sigma)U_2[E]$$
$$D = D^{rev} + D^{ir} = d\sigma + \varepsilon^\sigma E + \tilde{\mathcal{P}}[E] = \varepsilon^\sigma E - F_2[E]U_1(\sigma)$$
$$d = \frac{d_0}{P_0}\tilde{\mathcal{P}}[E]$$

$\tilde{\mathcal{P}}$... polarization hysteresis

$\tilde{\mathcal{Q}}$... butterfly hysteresis

σ small $\rightsquigarrow$ reversible (non)linearity wrt σ.

$$F_1(\sigma) := \frac{d_{sat}}{P_{sat}}$$
$$U_1(\sigma) := \frac{d_{sat}}{P_{sat}}\sigma + 1 \geq 0 \text{ for } \sigma \text{ suff. small}$$
$$F_2[E] := -\tilde{\mathcal{P}}[E] \text{ (... clockwise pcw convex!)}$$
$$U_2[E] := \text{clockwise hysteresis potential for } F_2$$
$$\tilde{\mathcal{Q}}[E] := -\frac{d_0}{P_0}(U_2[E] + E\tilde{\mathcal{P}}[E])$$

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\end{aligned}
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$\tilde{\mathcal{P}}$... polarization hysteresis
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σ small $\rightsquigarrow$ reversible (non)linearity wrt σ.

$$
\begin{aligned}
F_1(\sigma) &:= \tfrac{d_{sat}}{P_{sat}} \\
U_1(\sigma) &:= \tfrac{d_{sat}}{P_{sat}}\sigma + 1 \ \geq 0 \text{ for } \sigma \text{ suff. small} \\
F_2[E] &:= -\tilde{\mathcal{P}}[E] \text{ (... clockwise pcw convex!)} \\
U_2[E] &:= \text{clockwise hysteresis potential for } F_2 \\
\tilde{\mathcal{Q}}[E] &:= -\tfrac{d_0}{P_0}(U_2[E] + E\tilde{\mathcal{P}}[E])
\end{aligned}
$$

Hysteresis Potentials

$$F[\mathrm{v}]\dot{\mathrm{v}} - (U[\mathrm{v}])^{\cdot} \geq 0 \quad \forall \mathrm{v} : [0, T] \to \mathbb{R}$$

relation to dissipated energy $\mathcal{D}[\mathrm{v}]$:

$$F[\mathrm{v}]\dot{\mathrm{v}} - (U[\mathrm{v}])^{\cdot} = \frac{d}{dt}\mathcal{D}[\mathrm{v}]$$

Hysteresis Potentials for Preisach Operators

Theorem (e.g., [Brokate&Sprekels, 1996], [Krejci, 1996])

$$\begin{aligned}\tilde{F}[\mathrm{v}](t) = \mathcal{P}^{\omega}[\mathrm{v}](t) &= \iint_{\alpha,\beta\in S} \omega(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[\mathrm{v}](t)\, d(\alpha,\beta) \\ \tilde{U}[\mathrm{v}](t) &= 2\iint_{\alpha,\beta\in S} (\alpha+\beta)\omega(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[\mathrm{v}](t)\, d(\alpha,\beta)\end{aligned}$$

with $\omega \geq 0$.
Then $\tilde{F}$ is a counterclockwise piecewise convex hysteresis operator and $\tilde{U}$ is a counterclockwise hysteresis potential for $\tilde{F}$.

Hysteresis Potentials for Preisach Operators

Corollary

$$\begin{aligned}\tilde{F}[\mathrm{v}](t) = \mathcal{P}^{\omega}[\mathrm{v}](t) &= \iint_{\alpha,\beta\in S} \omega(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[\mathrm{v}](t)\, d(\alpha,\beta) \\ \tilde{U}[\mathrm{v}](t) &= 2\iint_{\alpha,\beta\in S} (\alpha+\beta)\omega(\beta,\alpha)\mathcal{R}_{\beta,\alpha}[\mathrm{v}](t)\, d(\alpha,\beta)\end{aligned}$$

with $\omega \geq 0$ and $\tilde{F}$ invertible.
Then $F = \tilde{F}^{-1}$ is a clockwise piecewise convex hysteresis operator and $U = \tilde{U} \circ \tilde{F}^{-1}$ is a clockwise hysteresis potential for F.

Efficient evaluation of $\tilde{F}^{-1}$: see, e.g. [Davino&Giustiani&Visone,2007]

Insert Into Thermodynamically Consistent Model

$$
\begin{aligned}
S &= s^E \sigma - F_1(\sigma) U_2[E] \\
D &= \varepsilon^\sigma E - F_2[E] U_1(\sigma)
\end{aligned}
$$

$$
\begin{aligned}
S &= s^E \sigma - F_1(\sigma) (\tilde{U}_2 \circ \tilde{F}_2^{-1})[E] \\
D &= \varepsilon^\sigma E - \underbrace{\tilde{F}_2^{-1}[E]}_{=-P} U_1(\sigma)
\end{aligned}
$$

Insert Into Thermodynamically Consistent Model

$$
\begin{aligned}
S &= s^E \sigma - F_1(\sigma) U_2[E] \\
D &= \varepsilon^\sigma E - F_2[E] U_1(\sigma)
\end{aligned}
$$

$$
\begin{aligned}
S &= s^E \sigma - F_1(\sigma)(\tilde{U}_2 \circ \tilde{F}_2^{-1})[E] \\
D &= \varepsilon^\sigma E - \underbrace{\tilde{F}_2^{-1}[E]}_{=-P} U_1(\sigma)
\end{aligned}
$$

$$
\begin{aligned}
S &= s^E \sigma - F_1(\sigma) \tilde{U}_2[-P] \\
D &= \varepsilon^\sigma \tilde{F}_2[-P] + P U_1(\sigma)
\end{aligned}
$$

Insert Into Thermodynamically Consistent Model

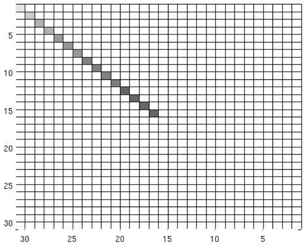
$$
\begin{aligned}
S &= s^E\sigma - F_1(\sigma)U_2[E] \\
D &= \varepsilon^\sigma E - F_2[E]U_1(\sigma)
\end{aligned}
$$

$$
\begin{aligned}
S &= s^E\sigma - F_1(\sigma)(\tilde{U}_2 \circ \tilde{F}_2^{-1})[E] \\
D &= \varepsilon^\sigma E - \underbrace{\tilde{F}_2^{-1}[E]}_{=-P} U_1(\sigma)
\end{aligned}
$$

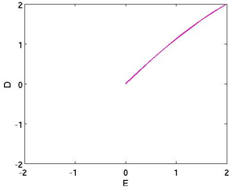
$$
\begin{aligned}
S &= s^E\sigma - F_1(\sigma)\tilde{U}_2[-P] \\
D &= \varepsilon^\sigma \tilde{F}_2[-P] + PU_1(\sigma)
\end{aligned}
$$

A Simple Test: Ferroelectricity

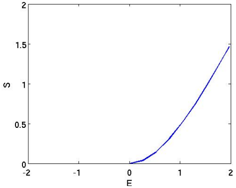
Preisach weight function:



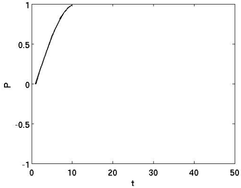
A Simple Test: Ferroelectricity



$P(E)$

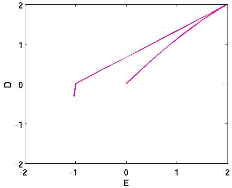


$S(E)$

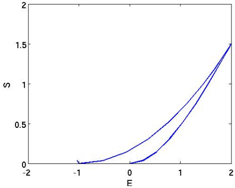


$P(t)$

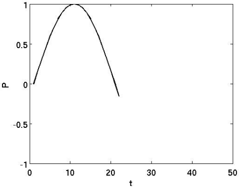
A Simple Test: Ferroelectricity



$P(E)$

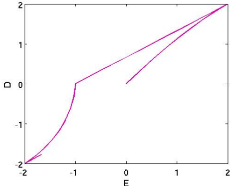


$S(E)$

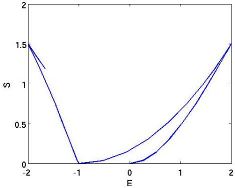


$P(t)$

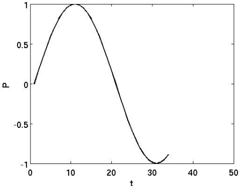
A Simple Test: Ferroelectricity



$P(E)$

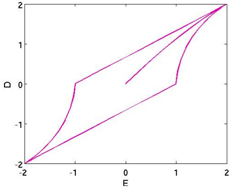


$S(E)$

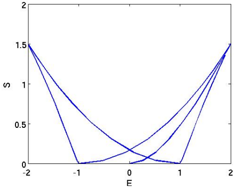


$P(t)$

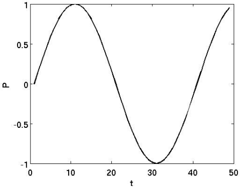
A Simple Test: Ferroelectricity



$P(E)$



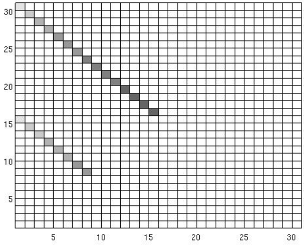
$S(E)$



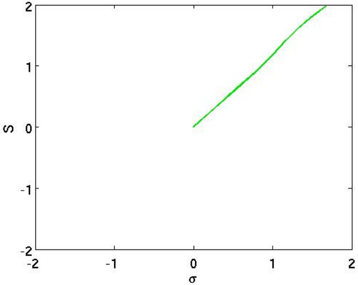
$P(t)$

A Simple Test: Ferroelasticity

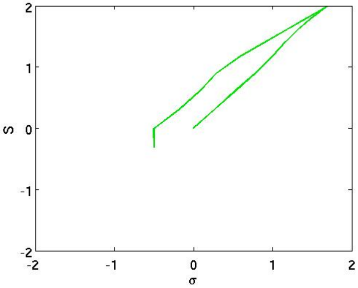
Preisach weight function:



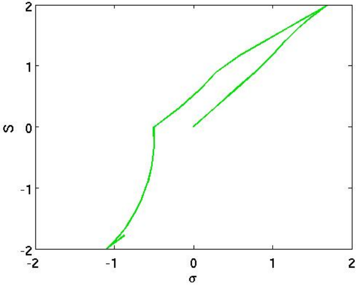
A Simple Test: Ferroelasticity



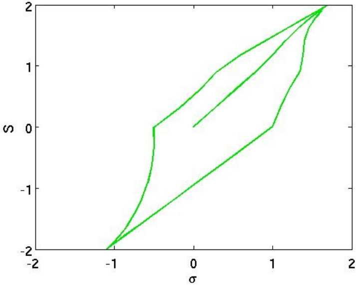
A Simple Test: Ferroelasticity



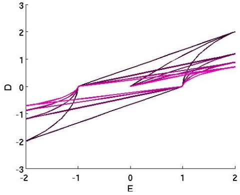
A Simple Test: Ferroelasticity

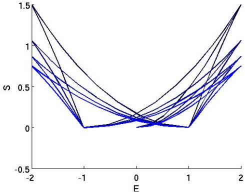


A Simple Test: Ferroelasticity



A Simple Test: Ferroelectricity and Ferroelasticity





Identification from Measurements

ferroelectricity: excitation via E; $\sigma = const.$

$$\begin{array}{rcl} S &=& s^E\sigma - F_1(\sigma)\tilde{U}_2[-P] \\ D &=& \varepsilon^\sigma \underbrace{\tilde{F}_2[-P]}_{=E} + PU_1(\sigma) \end{array}$$

Algorithm

SET $U_1 P = D - \varepsilon^\sigma E$

NORMALIZE $\mathrm{v} = -\frac{U_1 P}{\max |U_1 P|}$

$k = 0$: CHOOSE ω^0;

UNTIL INCREMENTS ARE SUFFICIENTLY SMALL DO

SET $F_1^{k+1} = \mathrm{ARGMIN}_{F_1} \| S - s^E\sigma + F_1 \tilde{U}^{\omega^k} \|^2$

SET $\omega^{k+1} = \mathrm{ARGMIN}_\omega \left\| \begin{pmatrix} E^k - \mathcal{P}^\omega[\mathrm{v}] \\ S - s^E\sigma + F_1^{k+1}\tilde{U}^\omega[\mathrm{v}] \end{pmatrix} \right\|$

END DO

$S^{fit} := s^E\sigma - F_1^\infty \tilde{U}^{\omega^\infty}$

$D := \varepsilon^\sigma \mathcal{P}^{\omega^\infty}[\mathrm{v}] + U_1 P$

Identification from Measurements

ferroelectricity: excitation via E; $\sigma = const.$

$$\begin{aligned} S &= s^E\sigma - F_1(\sigma)\tilde{U}_2[-P] \\ D &= \varepsilon^\sigma \underbrace{\tilde{F}_2[-P]}_{=E} + PU_1(\sigma) \end{aligned}$$

Algorithm

SET $U_1P = D - \varepsilon^\sigma E$

NORMALIZE $\mathrm{v} = -\frac{U_1P}{\max|U_1P|}$

$k = 0$: CHOOSE ω^0;

UNTIL INCREMENTS ARE SUFFICIENTLY SMALL DO

SET $F_1^{k+1} = \mathrm{ARGMIN}_{F_1}\|S - s^E\sigma + F_1\tilde{U}^{\omega^k}\|^2$

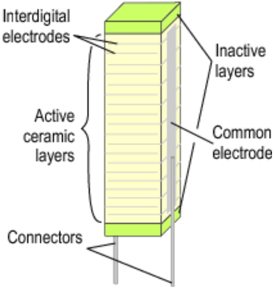
SET $\omega^{k+1} = \mathrm{ARGMIN}_\omega \left\| \begin{pmatrix} E^k - \mathcal{P}^\omega[\mathrm{v}] \\ S - s^E\sigma + F_1^{k+1}\tilde{U}^\omega[\mathrm{v}] \end{pmatrix} \right\|$

END DO

$S^{fit} := s^E\sigma - F_1^\infty \tilde{U}^{\omega^\infty}$

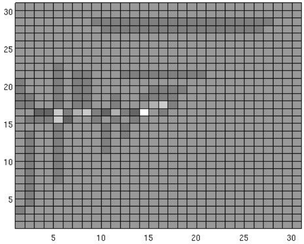
$D := \varepsilon^\sigma \mathcal{P}^{\omega^\infty}[\mathrm{v}] + U_1P$

Numerical Results: Stack Actuator



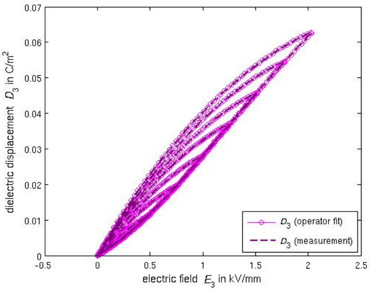
Numerical Results: Stack Actuator

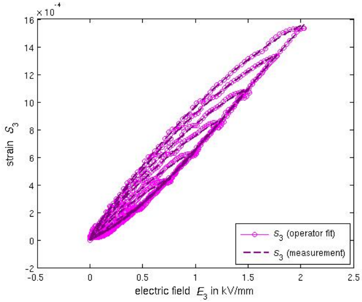
Identified Preisach weight function:



Numerical Results: Stack Actuator

comparison measurement – simulation with fitted Preisach operators:





Wellposedness in the Elliptic Case: Piezo Model

$$
\begin{aligned}
\rho u_{tt} - \mathrm{DIV}\underline{\boldsymbol{\sigma}}(t) &= \vec{\mathbf{f}}^m(t) \text{ in } \Omega \\
-\mathrm{div}\vec{\mathbf{D}}(t) &= \mathbf{f}^e(t) \text{ in } \Omega \\
\underline{\boldsymbol{\sigma}}(t) \cdot n &= 0 \text{ on } \Omega \setminus \Gamma^m, \quad \vec{\mathbf{u}}(t) = \vec{\mathbf{u}}^m(t) \text{ on } \Gamma^m \\
\vec{\mathbf{D}}(t) \cdot n &= 0 \text{ on } \Omega \setminus \Gamma^e, \quad \boldsymbol{\phi}(t) = \boldsymbol{\phi}^e(t) \text{ on } \Gamma^e \\
\underline{\mathbf{S}}(t) &= \mathbf{s}^E \underline{\boldsymbol{\sigma}}(t) + \mathbf{d}^T \vec{\mathbf{E}}(t) - F_1[\underline{\boldsymbol{\sigma}}](t) U_2[\vec{\mathbf{E}}](t) \\
\vec{\mathbf{D}}(t) &= \mathbf{d}\underline{\boldsymbol{\sigma}}(t) + \varepsilon^\sigma \vec{\mathbf{E}}(t) - F_2[\vec{\mathbf{E}}](t) U_1[\underline{\boldsymbol{\sigma}}](t) \\
\underline{\mathbf{S}}(t) &= \mathrm{DIV}^T \vec{\mathbf{u}}(t) \\
\vec{\mathbf{E}}(t) &= -\mathrm{grad}\boldsymbol{\phi}(t) \\
\vec{\mathbf{u}}(0) &= u_0 \\
\boldsymbol{\phi}(0) &= \phi_0
\end{aligned}
$$

Wellposedness in the Elliptic Case: Piezo Model

$$
\begin{aligned}
\cancel{\rho u_{tt}} - \mathrm{DIV}\underline{\boldsymbol{\sigma}}(t) &= \vec{\mathbf{f}}^m(t) \text{ in } \Omega \\
-\mathrm{div}\vec{\mathbf{D}}(t) &= \mathbf{f}^e(t) \text{ in } \Omega \\
\underline{\boldsymbol{\sigma}}(t) \cdot n &= 0 \text{ on } \Omega \setminus \Gamma^m, \quad \vec{\mathbf{u}}(t) = \vec{\mathbf{u}}^m(t) \text{ on } \Gamma^m \\
\vec{\mathbf{D}}(t) \cdot n &= 0 \text{ on } \Omega \setminus \Gamma^e, \quad \boldsymbol{\phi}(t) = \boldsymbol{\phi}^e(t) \text{ on } \Gamma^e \\
\underline{\mathbf{S}}(t) &= \mathbf{s}^E \underline{\boldsymbol{\sigma}}(t) + \mathbf{d}^T \vec{\mathbf{E}}(t) - F_1[\underline{\boldsymbol{\sigma}}](t) U_2[\vec{\mathbf{E}}](t) \\
\vec{\mathbf{D}}(t) &= \mathbf{d}\underline{\boldsymbol{\sigma}}(t) + \varepsilon^\sigma \vec{\mathbf{E}}(t) - F_2[\vec{\mathbf{E}}](t) U_1[\underline{\boldsymbol{\sigma}}](t) \\
\underline{\mathbf{S}}(t) &= \mathrm{DIV}^T \vec{\mathbf{u}}(t) \\
\vec{\mathbf{E}}(t) &= -\mathrm{grad}\boldsymbol{\phi}(t) \\
\vec{\mathbf{u}}(0) &= u_0 \\
\boldsymbol{\phi}(0) &= \phi_0
\end{aligned}
$$

Wellposedness in the Elliptic Case: Piezo Model

$$
\begin{aligned}
-\mathrm{DIV}\underline{\boldsymbol{\sigma}}(t) &= \vec{\mathbf{f}}^m(t) \text{ in } \Omega \\
-\mathrm{div}\vec{\mathbf{D}}(t) &= \mathbf{f}^e(t) \text{ in } \Omega \\
\underline{\boldsymbol{\sigma}}(t)\cdot n &= 0 \text{ on } \Omega\setminus\Gamma^m, \quad \vec{\mathbf{u}}(t) = \vec{\mathbf{u}}^m(t) \text{ on } \Gamma^m \\
\vec{\mathbf{D}}(t)\cdot n &= 0 \text{ on } \Omega\setminus\Gamma^e, \quad \boldsymbol{\phi}(t) = \boldsymbol{\phi}^e(t) \text{ on } \Gamma^e \\
\underline{\mathbf{S}}(t) &= \mathbf{s}^E\underline{\boldsymbol{\sigma}}(t) + \mathbf{d}^T\vec{\mathbf{E}}(t) - F_1[\underline{\boldsymbol{\sigma}}](t)U_2[\vec{\mathbf{E}}](t) \\
\vec{\mathbf{D}}(t) &= \mathbf{d}\underline{\boldsymbol{\sigma}}(t) + \varepsilon^\sigma\vec{\mathbf{E}}(t) - F_2[\vec{\mathbf{E}}](t)U_1[\underline{\boldsymbol{\sigma}}](t) \\
\underline{\mathbf{S}}(t) &= \mathrm{DIV}^T\vec{\mathbf{u}}(t) \\
\vec{\mathbf{E}}(t) &= -\mathrm{grad}\boldsymbol{\phi}(t) \\
\vec{\mathbf{u}}(0) &= u_0 \\
\boldsymbol{\phi}(0) &= \phi_0
\end{aligned}
$$

Wellposedness in the Elliptic Case: Piezo Model (weak form)

$$\int_\Omega \underline{\boldsymbol{\sigma}}(t) : \mathrm{DIV}^T \vec{v}\, dx = \int_\Omega \vec{\mathbf{f}}^m(t)\vec{v}\, dx \quad \forall \vec{v} \in H^1_0(\Omega, \Gamma_m)$$
$$\int_\Omega \vec{\mathbf{D}}(t) \cdot \mathrm{grad} v\, dx = \int_\Omega \mathbf{f}^e(t) v\, dx \quad \forall v \in H^1_0(\Omega, \Gamma_e)$$

$$\vec{\mathbf{u}}(t) \in H^1_{\vec{\mathbf{u}}^m(t)}(\Omega, \Gamma_m) = \{\vec{v} \in H^1(\Omega)^3 \,:\, \vec{v}_{\Gamma^m} = \vec{\mathbf{u}}^m(t)\}$$
$$\boldsymbol{\phi}(t) \in H^1_{\boldsymbol{\phi}^e(t)}(\Omega, \Gamma_e) = \{v \in H^1(\Omega) \,:\, v_{\Gamma^e} = \boldsymbol{\phi}^e(t)\}$$

$$\mathrm{DIV}^T \vec{\mathbf{u}}(t) = \mathbf{s}^E \underline{\boldsymbol{\sigma}}(t) + \mathbf{d}^T(-\mathrm{grad}\boldsymbol{\phi}(t)) - F_1[\underline{\boldsymbol{\sigma}}](t) U_2[-\mathrm{grad}\boldsymbol{\phi}(t)](t)$$
$$\vec{\mathbf{D}}(t) = \mathbf{d}\underline{\boldsymbol{\sigma}}(t) + \varepsilon^\sigma(-\mathrm{grad}\boldsymbol{\phi}(t)) - F_2[-\mathrm{grad}\boldsymbol{\phi}(t)](t) U_1[\underline{\boldsymbol{\sigma}}](t)$$

$$\vec{\mathbf{u}}(0) = u_0$$

$$\boldsymbol{\phi}(0) = \phi_0$$

Wellposedness in the Elliptic Case: Piezo Model (weak form)

$$\int_\Omega \underline{\boldsymbol{\sigma}}(t) : \mathrm{DIV}^T \vec{v}\, dx = \int_\Omega \vec{\mathbf{f}}^m(t)\vec{v}\, dx \quad \forall \vec{v} \in H_0^1(\Omega, \Gamma_m)$$

$$\int_\Omega \vec{\mathbf{D}}(t) \cdot \mathrm{grad} v\, dx = \int_\Omega \mathbf{f}^e(t) v\, dx \quad \forall v \in H_0^1(\Omega, \Gamma_e)$$

$$\vec{\mathbf{u}}(t) \in H^1_{\vec{\mathbf{u}}^m(t)}(\Omega, \Gamma_m) = \{\vec{v} \in H^1(\Omega)^3 \,:\, \vec{v}_{\Gamma^m} = \vec{\mathbf{u}}^m(t)\}$$

$$\boldsymbol{\phi}(t) \in H^1_{\boldsymbol{\phi}^e(t)}(\Omega, \Gamma_e) = \{v \in H^1(\Omega) \,:\, v_{\Gamma^e} = \boldsymbol{\phi}^e(t)\}$$

$$\mathrm{DIV}^T \vec{\mathbf{u}}(t) = \mathbf{s}^E \underline{\boldsymbol{\sigma}}(t) + \mathbf{d}^T(-\mathrm{grad}\boldsymbol{\phi}(t)) - F_1[\underline{\boldsymbol{\sigma}}](t) U_2[-\mathrm{grad}\boldsymbol{\phi}(t)](t)$$

$$\vec{\mathbf{D}}(t) = \mathbf{d}\underline{\boldsymbol{\sigma}}(t) + \varepsilon^\sigma(-\mathrm{grad}\boldsymbol{\phi}(t)) - F_2[-\mathrm{grad}\boldsymbol{\phi}(t)](t) U_1[\underline{\boldsymbol{\sigma}}](t)$$

$$\vec{\mathbf{u}}(0) = u_0$$

$$\boldsymbol{\phi}(0) = \phi_0$$

Wellposedness in the Elliptic Case: Piezo Model (weak form)

$$\int_\Omega \underline{\boldsymbol{\sigma}}(t) : \mathrm{DIV}^T \vec{v}\, dx = \int_\Omega \vec{\mathbf{f}}^m(t)\vec{v}\, dx \quad \forall \vec{v} \in H_0^1(\Omega, \Gamma_m)$$

$$\int_\Omega \vec{\mathbf{D}}(t) \cdot \mathrm{grad} v\, dx = \int_\Omega \mathbf{f}^e(t) v\, dx \quad \forall v \in H_0^1(\Omega, \Gamma_e)$$

$$\vec{\mathbf{u}}(t) \in H^1_{\vec{\mathbf{u}}^m(t)}(\Omega, \Gamma_m) = \{\vec{v} \in H^1(\Omega)^3 \ : \ \vec{v}_{\Gamma^m} = \vec{\mathbf{u}}^m(t)\}$$

$$\boldsymbol{\phi}(t) \in H^1_{\boldsymbol{\phi}^e(t)}(\Omega, \Gamma_e) = \{v \in H^1(\Omega) \ : \ v_{\Gamma^e} = \boldsymbol{\phi}^e(t)\}$$

$$\underline{\boldsymbol{\sigma}}(t) = \Big[\mathbf{s}^E \mathrm{id} - F_1[\cdot](t) U_2[-\mathrm{grad}\boldsymbol{\phi}(t)]\Big]^{-1} \Big(\mathrm{DIV}^T \vec{\mathbf{u}}(t) - \mathbf{d}^T(-\mathrm{grad}\boldsymbol{\phi}(t))\Big)$$

$$\vec{\mathbf{D}}(t) = \mathbf{d}\underline{\boldsymbol{\sigma}}(t) + \varepsilon^\sigma(-\mathrm{grad}\boldsymbol{\phi}(t)) - F_2[-\mathrm{grad}\boldsymbol{\phi}(t)](t) U_1[\underline{\boldsymbol{\sigma}}](t)$$

$$\vec{\mathbf{u}}(0) = u_0$$

$$\boldsymbol{\phi}(0) = \phi_0$$

Wellposedness in the Elliptic Case: Piezo Model (weak form)

$$\int_\Omega \underline{\boldsymbol{\sigma}}(t) : \mathrm{DIV}^T \vec{v}\, dx = \int_\Omega \vec{\mathbf{f}}^m(t) \vec{v}\, dx \quad \forall \vec{v} \in H_0^1(\Omega, \Gamma_m)$$

$$\int_\Omega \vec{\mathbf{D}}(t) \cdot \mathrm{grad} v\, dx = \int_\Omega \mathbf{f}^e(t) v\, dx \quad \forall v \in H_0^1(\Omega, \Gamma_e)$$

$$\vec{\mathbf{u}}(t) \in H^1_{\vec{\mathbf{u}}^m(t)}(\Omega, \Gamma_m) = \{\vec{v} \in H^1(\Omega)^3 \ : \ \vec{v}_{\Gamma^m} = \vec{\mathbf{u}}^m(t)\}$$

$$\boldsymbol{\phi}(t) \in H^1_{\boldsymbol{\phi}^e(t)}(\Omega, \Gamma_e) = \{v \in H^1(\Omega) \ : \ v_{\Gamma^e} = \boldsymbol{\phi}^e(t)\}$$

$$\underline{\boldsymbol{\sigma}}(t) = \mathcal{P}^m[\mathrm{DIV}^T \vec{\mathbf{u}}, \mathrm{grad}\boldsymbol{\phi}](t)$$

$$\vec{\mathbf{D}}(t) = \mathcal{P}^e[\mathrm{DIV}^T \vec{\mathbf{u}}, \mathrm{grad}\boldsymbol{\phi}](t)$$

$$\vec{\mathbf{u}}(0) = u_0$$

$$\boldsymbol{\phi}(0) = \phi_0$$

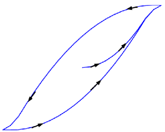
Wellposedness in the Elliptic Case: Result

F_i hysteresis operators (rate independent),
$U_i \geq 0$ clockwise hysteresis potential for F_i,
Assumption 0: some invertibility condition on F_1
Assumption 1: some monotonicity condition on F_1, F_2[1]
Assumption 2: some continuity condition on F_1, F_2

Theorem

For $\vec{\mathbf{f}}^m \in H^1(0,T;H^{-1}(\Omega,\Gamma_m))$, $\mathbf{f}^e \in H^1(0,T;H^{-1}(\Omega,\Gamma_e))$, $\vec{\mathbf{u}}^m(t) \in H^1(0,T;H^{1/2}(\Gamma_m)$, $\vec{\mathbf{u}}^e(t) \in H^1(0,T;H^{1/2}(\Gamma_e)$, *there exists a solution*
$(\vec{\mathbf{u}},\phi) \in H^1(0,T;H^1_{\vec{\mathbf{u}}^m(t)}(\Omega,\Gamma_m)) \times H^1(0,T;H^1_{\phi^e(t)}(\Omega,\Gamma_e))$

[1] F_1, F_2 are not monotone operators!



Idea of Proof I

a simpler model problem "electrostatics with remanent polarization"

$$
\begin{aligned}
-\mathrm{div}\vec{\mathbf{D}}(t) &= \mathbf{f}^e(t) \text{ in } \Omega \qquad \boldsymbol{\phi}(t) = 0 \text{ on } \partial\Omega \\
\vec{\mathbf{D}}(t) &= \mathcal{P}[\mathrm{grad}\boldsymbol{\phi}](t) \\
& \quad \text{for all } t \in [0, T] \\
\boldsymbol{\phi}(0) &= \phi_0
\end{aligned}
$$

time discretization $t_1, \ldots, t_N,\ t_i = i\tau,\ \tau = \frac{T}{N},\ \phi_i \approx \boldsymbol{\phi}(t_i),\ \vec{D}_i \approx \vec{\mathbf{D}}(t_i) \ldots$

$$
\begin{aligned}
\int_\Omega \vec{D}_i \cdot \mathrm{grad} v \, dx &= \int_\Omega f_i^e v \, dx \quad \forall v \in H_0^1(\Omega)\,, \text{ and } \phi_i \in H_0^1(\Omega) \\
\vec{D}_i &= \mathcal{P}_f[\mathrm{grad}\phi_0, \ldots \mathrm{grad}\phi_i] \\
& \quad \text{for all } i \in \{1, \ldots, N\}
\end{aligned}
$$

$\mathcal{P}_f \ldots$ Nemitskii operator induced by final value mapping

Idea of Proof II

$$\int_\Omega \vec{D}_i \cdot \mathrm{grad} v \, dx = \int_\Omega f_i^e v \, dx \quad \forall v \in H_0^1(\Omega), \text{ and } \phi_i \in H_0^1(\Omega)$$

$$\vec{D}_i = \mathcal{P}_f[\mathrm{grad}\phi_0, \ldots \mathrm{grad}\phi_i]$$

$$\text{for all } i \in \{1, \ldots, N\}$$

Assumption 1:

$$\forall n \in \mathbb{N} \; \forall i \in \{1, \ldots N\} \; \forall(\vec{\xi}_0, \ldots, \vec{\xi}_{i-1}) \in (\mathbb{R}^3)^i \; :$$

$$I_i^N : \mathbb{R}^3 \to \mathbb{R}^3 \quad \in C^1(\mathbb{R}^3) \text{ and strictly monotone}$$

$$\vec{\lambda} \mapsto \mathcal{P}_f(\vec{\xi}_0, \ldots, \vec{\xi}_{i-1}, \vec{\lambda})$$

$$0 < \underline{\mu} \leq \frac{(I_i^N(\vec{\lambda}_1) - I_i^N(\vec{\lambda}_2)) \cdot (\vec{\lambda}_1 - \vec{\lambda}_2)}{|\vec{\lambda}_1 - \vec{\lambda}_2|^2} \quad \text{unif. wrt. } i, N, (\vec{\xi}_0, \ldots, \vec{\xi}_{i-1})$$

Browder-Minty $\Rightarrow$ **existence and uniqueness of time discrete solution** $\phi_i \in H_0^1(\Omega)$.

Idea of Proof III

"differentiate" wrt time $\Rightarrow$

$$\begin{aligned} \int_\Omega \frac{\vec{D}_i - \vec{D}_{i-1}}{\tau} \cdot \mathrm{grad} v \, dx &= \int_\Omega \frac{f_i^e - f_{i-1}^e}{\tau} v \, dx \quad \forall v \in H_0^1(\Omega)\,, \\ \vec{D}_i &= \mathcal{P}_f[\mathrm{grad}\phi_0, \dots \mathrm{grad}\phi_i] \\ & \quad \text{for all } i \in \{1, \dots, N\} \end{aligned}$$

multiplier $v = \frac{\phi_i - \phi_{i-1}}{\tau}$, monotonicity estimate

$$\frac{\vec{D}_i - \vec{D}_{i-1}}{\tau} \cdot \frac{\mathrm{grad}\phi_i - \mathrm{grad}\phi_{i-1}}{\tau} \geq \underline{\mu} \left| \frac{\mathrm{grad}\phi_i - \mathrm{grad}\phi_{i-1}}{\tau} \right|^2$$

$\Rightarrow$ **uniform estimate** for piecewise linear interpolates
$\boldsymbol{\phi}^N$ in $H^1(0, T; H_0^1(\Omega))$ and $\vec{\mathbf{D}}^N$ in $H^1(0, T; L^2(\Omega))$

Idea of Proof IV

$$\int_\Omega \vec{D}_i \cdot \mathrm{grad} v \, dx = \int_\Omega f_i^e v \, dx \quad \forall v \in H_0^1(\Omega), \qquad \phi_i \in H_0^1(\Omega)$$

$$\vec{D}_i = \mathcal{P}_f[\mathrm{grad}\phi_0, \ldots \mathrm{grad}\phi_i]$$

$((\vec{\mathbf{D}}^N, \boldsymbol{\phi}^N)$ uniformly bounded in $H^1(0,T;L^2(\Omega)) \times H^1(0,T;H_0^1(\Omega))$
$\Rightarrow \exists$ weakly convergent subsequence whose weak limit $(\vec{\mathbf{D}}, \boldsymbol{\phi})$ satisfies

$$\int_\Omega \vec{\mathbf{D}}(t) \cdot \mathrm{grad} v \, dx = \int_\Omega \mathbf{f}^e(t) v \, dx \quad \forall v \in H_0^1(\Omega), \quad \text{for all } t \in [0,T]$$

$\boldsymbol{\phi}^N$ uniformly bounded in $H^1(0,T;H_0^1(\Omega)) \hookrightarrow L^2(\Omega; C[0,T])$
Assumption 2: $\mathcal{P} : C[0,T] \to C[0,T]$ continuous
$\Bigg\} \Rightarrow$

$\Rightarrow (\vec{\mathbf{D}}, \boldsymbol{\phi})$ satisfies $\quad \vec{\mathbf{D}}(t) = \mathcal{P}[\mathrm{grad}\boldsymbol{\phi}](t) \quad$ for all $t \in [0,T]$

Idea of Proof IV

$$\int_\Omega \vec{D}_i \cdot \mathrm{grad} v \, dx = \int_\Omega f_i^e v \, dx \quad \forall v \in H_0^1(\Omega)\,, \qquad \phi_i \in H_0^1(\Omega)$$
$$\vec{D}_i = \mathcal{P}_f[\mathrm{grad}\phi_0, \dots \mathrm{grad}\phi_i]$$

$((\vec{\mathbf{D}}^N, \boldsymbol{\phi}^N)$ uniformly bounded in $H^1(0,T;L^2(\Omega)) \times H^1(0,T;H_0^1(\Omega))$
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(similar to wellposedness proof for heat equation with hysteresis, see, e.g., [Brokate&Sprekels 1996], [Visintin 1994])

Idea of Proof IV

$$\begin{aligned} \int_\Omega \vec{D}_i \cdot \mathrm{grad} v \, dx &= \int_\Omega f_i^e v \, dx \quad \forall v \in H_0^1(\Omega)\,, \qquad \phi_i \in H_0^1(\Omega) \\ \vec{D}_i &= \mathcal{P}_f[\mathrm{grad}\phi_0, \ldots \mathrm{grad}\phi_i] \end{aligned}$$

$((\vec{\mathbf{D}}^N, \boldsymbol{\phi}^N)$ uniformly bounded in $H^1(0,T;L^2(\Omega)) \times H^1(0,T;H_0^1(\Omega))$
$\Rightarrow \exists$ weakly convergent subsequence whose weak limit $(\vec{\mathbf{D}}, \boldsymbol{\phi})$ satisfies

$$\int_\Omega \vec{\mathbf{D}}(t) \cdot \mathrm{grad} v \, dx = \int_\Omega \mathbf{f}^e(t) v \, dx \quad \forall v \in H_0^1(\Omega)\,, \quad \text{for all } t \in [0,T]$$

$\boldsymbol{\phi}^N$ uniformly bounded in $H^1(0,T;H_0^1(\Omega)) \hookrightarrow L^2(\Omega; C[0,T])$
Assumption 2: $\mathcal{P} : C[0,T] \to C[0,T]$ continuous
$\Bigg\} \Rightarrow$

$\Rightarrow (\vec{\mathbf{D}}, \boldsymbol{\phi})$ satisfies $\quad \vec{\mathbf{D}}(t) = \mathcal{P}[\mathrm{grad}\boldsymbol{\phi}](t) \qquad$ for all $t \in [0,T]$
(similar to wellposedness proof for heat equation with hysteresis, see, e.g., [Brokate&Sprekels 1996], [Visintin 1994])

Conclusions and Outlook

- thermodynamic consistent modelling via Preisach operators
- includes ferroelectricity, ferroelasticity as well as their coupling
- wellposedness in elliptic case by monotonicity
- ◦ FEM computations
- ◦ hysteresis identification

→ creep

→ vector hysteresis (cf. talk by Olaf Klein, Monday)

→ wellposedness in hyperbolic case by convexity?
  ⇝ wave equation with hysteresis [Krejčí'96])

Thank you for your attention!

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26th IFIP TC7 Conference 2013 on System Modelling and Optimization

September 9-13, 2013, Klagenfurt, Austria

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