Non-isothermal cyclic fatigue in an oscillating elastoplastic material with phase transition

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Elastoplastic materials subject to cyclic loading exhibit increasing fatigue, which is manifested by material softening, heat release and material failure in finite time.

In the uniaxial processes there is a qualitative and quantitative relationship between

- accumulated fatigue (by the rainflow algorithm, which counts closed hysteresis loops in the loading history and with each closed loop associates a number depending on its amplitude – the contribution of the loop to the total damage)
- dissipated energy (the number associated with a closed loop is its area).
In *multiaxial loading processes*

- the concept of closed loop is meaningless
- reliable counterpart of the rainflow algorithm?
- the notion of energy dissipation is a purely thermodynamic one – independent of the experimental setting

We propose a thermodynamic model for material fatigue accumulation based on the hypothesis that there exists a qualitative and quantitative relation between accumulated fatigue and dissipated energy.

We demonstrate our model on the example of a transversally oscillating elastoplastic beam.
Plan of the talk

Constitutive laws of elastoplasticity, stop operator, Prandtl-Ishlinskii energy balance

First idea of the model

Model with fatigue

Model with fatigue and temperature

Model with fatigue, temperature and phase transition

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A classical hysteresis-type model for one-dimensional elastoplasticity by L. Prandtl and A. Yu. Ishlinskii - the relation between strain $\varepsilon$ and stress $\sigma$ given by the formula

$$\sigma = \mathcal{P}[\varepsilon](t) = \int_0^\infty s_r[\varepsilon](t) \varphi(r) \, dr$$

for $\varepsilon \in W^{1,1}(0, T; \mathbb{R})$. Here, $\varphi(r) > 0$ is a weight function, and $s_r[\varepsilon](t)$ represents the elastic-ideally plastic element or stop operator with the threshold $r > 0$. 
The stop operator

Given a parameter $r > 0$, a function $\varepsilon : [0, T] \to \mathbb{R}$, and an initial condition $\sigma^0 \in [-r, r]$, we look for functions $\sigma, \xi : [0, T] \to \mathbb{R}$ such that $\sigma(0) = \sigma^0$, and

\[
\begin{align*}
\sigma(t) + \xi(t) &= \varepsilon(t) \\
|\sigma(t)| &\leq r \\
\dot{\xi}(t)(\sigma(t) - \tilde{\sigma}) &\geq 0 \quad \forall \tilde{\sigma} \in [-r, r]
\end{align*}
\]

For every $\varepsilon \in W^{1,1}(0, T)$ and $\sigma^0 \in [-r, r]$, the problem has a unique solution $\sigma \in W^{1,1}(0, T)$. The solution mapping

$$s_r : [-r, r] \times W^{1,1}(0, T) \to W^{1,1}(0, T), \quad \sigma = s_r[\sigma^0, \varepsilon],$$

is called the stop (or elastoplastic element).
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\mathcal{S}_r : [-r, r] \times W^{1,1}(0, T) \to W^{1,1}(0, T), \quad \sigma = \mathcal{S}_r[\sigma^0, \varepsilon],
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For every \( \varepsilon \in W^{1,1}(0, T) \) and \( \sigma^0 \in [-r, r] \), the problem has a unique solution \( \sigma \in W^{1,1}(0, T) \). The solution mapping is Lipschitz continuous and admits Lipschitz continuous extension to \( \mathcal{s}_r : [-r, r] \times C[0, T] \rightarrow C[0, T] \).
Prandtl-Ishlinskii energy balance

For a single stop, the energy balance reads
\[ \dot{\varepsilon} \] 
\[ s_r[\varepsilon] - d \left( \frac{1}{2} s_r[\varepsilon]^2 \right) = r \left| d \left( \varepsilon - s_r[\varepsilon] \right) \right|. \]

For the Prandtl-Ishlinskii operator
\[ P[\varepsilon] = \int_0^\infty \phi(r) s_r[\varepsilon] \, dr \]
we define the potential energy operator
\[ V[\varepsilon] = \frac{1}{2} \int_0^\infty \phi(r) s_r[\varepsilon]^2 \, dr \]
and the dissipation operator
\[ D[\varepsilon] = \int_0^\infty r \phi(r) \left( \varepsilon - s_r[\varepsilon] \right) \, dr. \]
For a single stop, the energy balance reads

\[ \dot{\varepsilon} s_r[\varepsilon] - \frac{d}{dt} \left( \frac{1}{2} s_r^2[\varepsilon] \right) = r \left| \frac{d}{dt} (\varepsilon - s_r[\varepsilon]) \right| . \]
Prandtl-Ishlinskii energy balance

For a single stop, the energy balance reads

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For the Prandtl-Ishlinskii operator

\[ \mathcal{P}[\varepsilon] = \int_0^\infty \varphi(r) \sigma_r[\varepsilon] dr \]
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Prandtl-Ishlinskii energy balance
\[
\dot{\varepsilon} P[\varepsilon] - V[\varepsilon]_t = |D[\varepsilon]_t|.
\]
Kinematic hardening:

\[ \sigma = B\varepsilon + \mathcal{P}[\varepsilon] \]  

(2)

with \( B \) positive.

The momentum balance equation:

\[ \rho u_{tt} - \text{div} \sigma = f, \]

in \( \Omega \times (0, T) \) and with suitable initial and boundary conditions. Here \( u \) is the displacement, \( \varepsilon = u_x \), \( f \) is a given volume force and \( \rho \) is the mass density.
First idea of the model

Prandtl- Ishlinskii operators are easily understood and rather intuitive, but their use in physical and engineering literature is still nonstandard.
Disadvantages:
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- The density function $\varphi$ is a priori unknown and must be identified.
- Other 3D plasticity models like von Mises or Tresca models are available.

In P. Krejčí, J. Sprekels: Elastic-ideally plastic beams and Prandtl-Ishlinskii hysteresis operators, Math. Meth. Appl. Sci. 30 (2007), 2371–2393 they showed that in the modeling of the one-dimensional transversal vibrations of an elastoplastic beam the tree-dimensional von Mises model leads to a scalar Prandtl-Ishlinskii model whose density function is a priori given.
The model is based on the idea that the Euler-Bernoulli dimensional reduction applied to transversal oscillations of an elastoplastic beam leads, as a result of averaging over the thickness of the beam, to a Prandtl-Ishlinskii constitutive law:

\[ \sigma = \mathcal{P}[\varepsilon](t) = \int_{0}^{\infty} s_r[\varepsilon](t) \varphi(r) \, dr \]

for \( \varepsilon \in W^{1,1}(0, T; \mathbb{R}) \) and to the momentum balance equation: (after rescaling all constants to 1)

\[ w_{tt} - w_{xxtt} + \sigma_{xx} = f, \]

where \( w \) is the transversal displacement, \( \varepsilon = w_{xx} \).
The resulting prototypical system of partial differential equations is of the form

\[ w_{tt} - w_{xxtt} + \mathcal{P}[w_{xx}]_{xx} = f \]

with boundary conditions

\[ w(0, t) = w(L, t) = \mathcal{P}[w_{xx}](0, t) = \mathcal{P}[w_{xx}](L, t) = 0. \]

Prandtl-Ishlinskii operators are not differentiable in general; hence, for the existence and uniqueness analysis, we rewrite the PDE as a system

\[ u_t = \mathcal{P}[w_{xx}] \]
\[ w_t - w_{xxt} = -u_{xx} + g \]

with boundary conditions

\[ u(0, t) = w(0, t) = u(L, t) = w(L, t) = 0. \]
Our basic modeling assumption consists in replacing the elastoplastic constitutive law (2) by

\[ \sigma = B(m)\varepsilon + \int_{0}^{\infty} s_t[\varepsilon](t) \varphi(r) \, dr \quad (3) \]

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where \( m \) is the fatigue parameter and the momentum balance equation becomes

\[ w_{tt} - w_{xxtt} + [B(m(w_{xx}))w_{xx} + P[w_{xx}]_{xx}] = f. \]
The fatigue equation

We complete the system by an evolution equation for the fatigue parameter $m$:

$$
\left( \frac{1}{C} + \frac{1}{2} B'(m) \varepsilon^2 \right) m_t = \int_0^\infty \partial_t (\varepsilon - s_r[\varepsilon]) s_r[\varepsilon] \varphi(r) \, dr
\]

$$

assuming that the rate of fatigue $m_t$ is proportional to the dissipation rate $D$

$$
D = -B'(m) \varepsilon^2 m_t + |D[\varepsilon]_t|
\]

with a proportionality factor $C$. 

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The associated system

\[ u_t = B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}] \quad \text{in } Q_T, \]

\[ w_t - w_{xxt} = -u_{xx} + g(x, t) \quad \text{in } Q_T, \]

\[ u(1, t) = u_x(1, t) = 0 \quad 0 \leq t \leq T, \]

\[ w(0, t) = w_x(0, t) = 0 \quad 0 \leq t \leq T, \]

\[ u(x, 0) = w^1(x) \quad 0 \leq x \leq 1, \]

\[ w(x, 0) = w^0(x) \quad 0 \leq x \leq 1, \]

where we put

\[ u(x, t) = w^1(x) + \int_0^t [B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}]](x, s) \, ds, \]

\[ g(x, t) = w^1(x) + \int_0^t f(x, s) \, ds \]

is well posed on some a priori unknown time interval \([0, T^*]\).
The fatigue model with temperature

Our basic modeling assumption consists in replacing the elastoplastic constitutive law (5) by

\[ \sigma = B(m)\varepsilon + \int_{0}^{\infty} s_r[\varepsilon](t) \varphi(r) \, dr - \beta(\theta - \theta_c) + \nu \varepsilon_t, \quad (4) \]

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where \( m \) is the fatigue parameter, \( \theta \) is the absolute temperature, \( \beta > 0 \) is the thermal dilation coefficient, \( \theta_c > 0 \) is a fixed reference temperature and \( \nu \) a viscosity parameter.

The momentum balance equation becomes

\[
w_{tt} - w_{xxtt} + [B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}]]_{xx} - \beta \theta_{xx} + \nu w_{xxxxxt} = f.
\]
With the constitutive law (4) we associate the specific entropy

\[ S[\theta, \varepsilon] = c_V \log(\theta/\theta_c) + \beta\varepsilon \]

and the specific internal energy

\[ U[\theta, \varepsilon] = c_V \theta + \frac{1}{2} B(m)\varepsilon^2 + \frac{1}{2} \int_0^\infty \varphi(r)s_r[\varepsilon](t)dr + \beta\theta_c\varepsilon. \]

We have the energy balance

\[ U_t + q_x = \sigma \varepsilon_t, \]

and again we assume that the fatigue rate \( m_t \) is proportional to the dissipation rate \( D \) with a proportionality factor \( C(\theta) \):

\[ \left( \frac{1}{C(\theta)} + \frac{1}{2} B'(m)\varepsilon^2 \right) m_t = \int_0^\infty \partial_t(\varepsilon - s_r[\varepsilon])s_r[\varepsilon]\varphi(r)dr \]

\[ = |D[\varepsilon]_t| \]
The second Principle of Thermodynamics (Claudius-Duhem inequality) states for the entropy production $\psi$

$$\psi := S[\theta, \varepsilon]_t + \left( \frac{q}{\theta} \right)_x \geq 0.$$ 

We rewrite it in the form

$$\theta \psi := \sigma \varepsilon_t + \theta S[\theta, \varepsilon]_t - U[\theta, \varepsilon]_t - \frac{q \theta_x}{\theta} \geq 0,$$

use the Fourier law $q = -k \theta_x$ and get that the dissipation rate

$$D = -B'(m) \varepsilon^2 m_t + |D[\varepsilon]_t|$$

has to be nonnegative. The fatigue accumulation rate $m_t$ should be nonnegative, so we need to assume that $B'(m)$ is negative.
Oscillating elastoplastic beam

For the oscillating elastoplastic beam problem

\begin{align*}
    u_t &= B(m) w_{xx} + P[w_{xx}] + w_{xxt} - (\theta - \theta_c), \\
    w_t - w_{xxt} &= -u_{xx} + g \theta_t = \theta_{xx} - \frac{1}{2} B'(m) w_{xx} m_t + |D[w_{xx}]_t| + w_{xxt} - \theta w_{xxt}, \\
    m_t &= \int_1^0 \lambda (x - y) \left( -\frac{1}{2} B'(m) w_{xx} m_t + |D[w_{xx}]_t| \right) (y, t) \, dy.
\end{align*}

With a spatially regularized fatigue equation, and with zero initial and boundary conditions for \(w\) and \(u\), and homogeneous Neumann boundary conditions for \(\theta\), we find an efficient lower bound for the existence and uniqueness time \(T^*\).

Singularity occurs when \(m_t \to +\infty\).

Extension to temperature and fatigue dependent plasticity is straightforward.
For the oscillating elastoplastic beam problem

\[ u_t = B(m)w_{xx} + P[w_{xx}] + w_{xxt} - (\theta - \theta_c) , \]
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\[ \theta_t = \theta_{xx} - \frac{1}{2}B'(m)w_{xx}^2m_t + |D[w_{xx}]_t| + w_{xxt}^2 - \theta w_{xxt} , \]
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Singularity occurs when \( m_t \to +\infty \).

Extension to temperature and fatigue dependent plasticity is straightforward.
(i) $\mathcal{P}$ is a Prandtl-Ishlinskii operator and $\mathcal{D}$ is its associated dissipation operator. We assume that its distribution function $\varphi \in L^1(0, \infty)$ is such that $\varphi \geq 0$ a.e., and $\int_0^\infty r\varphi(r)dr < \infty$. 

(ii) $B: [0, \infty) \to (0, \infty)$ is a $C^2$ function, $B'(0) = 0$, $-1 \leq B''(m) \leq 0$ for all $m > 0$. 

(iii) $\lambda: \mathbb{R} \to [0, \infty)$ is a $C^1$ function with compact support, $L := \max\{\lambda(x) + |\lambda'(x)|, x \in \mathbb{R}\}$. 

(iv) $g \in L^2(\Omega_T)$ is a given function for some fixed $T > 0$, such that $g_{tt}, g_{xx} \in L^2(\Omega_T)$. 

(v) $\theta_0 \in L^\infty(0, 1)$ is such that $\theta_0 \geq \theta^* > 0$, $\theta_0_{xx} \in L^2(0, 1)$. 

(vi) $\theta_c$ is a given positive constant.
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Hypotheses

(i) \( \mathcal{P} \) is a Prandtl-Ishlinskii operator and \( \mathcal{D} \) is its associated dissipation operator. We assume that its distribution function \( \varphi \in L^1(0, \infty) \) is such that \( \varphi \geq 0 \) a.e., and \( \int_0^\infty r\varphi(r)dr < \infty \).

(ii) \( B : [0, \infty) \to (0, \infty) \) is a \( C^2 \) function, \( B'(0) = 0 \), \(-1 \leq B''(m) \leq 0 \) for all \( m > 0 \).

(iii) \( \lambda : \mathbb{R} \to [0, \infty) \) is a \( C^1 \) function with compact support, 
\[ L := \max\{\lambda(x) + |\lambda'(x)|, \ x \in \mathbb{R}\}. \]
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(iv) $g \in L^2(\Omega_T)$ is a given function for some fixed $T > 0$, such that $g_{tt}, g_{xx} \in L^2(\Omega_T)$. 

(v) $\theta_0 \in L^\infty(0, 1)$ is such that $\theta_0 \geq \theta^* > 0$, $\theta_0 xx \in L^2(0, 1)$. 

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Hypotheses

(i) $P$ is a Prandtl-Ishlinskii operator and $\mathcal{D}$ is its associated dissipation operator. We assume that its distribution function $\varphi \in L^1(0, \infty)$ is such that $\varphi \geq 0$ a.e., and $\int_0^\infty r\varphi(r)dr < \infty$.

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- On the interval \([0, T^R]\), we derive estimates for the approximate solutions which enable us to show that the truncation never becomes active if \(R\) is sufficiently large, and can be removed.
- We select a convergent subsequence indexed by \(n\) and pass to the limit as \(n \to \infty\) to obtain the solution.
Energy identity

The solution is constructed by passing to the limit in a space-semidiscrete scheme. Higher order estimates are obtained by successive testing by higher and higher order terms and imply the following regularity:

\[ w_{xxx}, w_{xxt}, \theta_t, \theta_{xx}, u_{tt}, u_{xxt} \in L^2(0, T^*; L^2(\Omega)) \]
\[
\frac{d}{dt} \int_{\Omega} \left( \theta + \frac{1}{2} (w_t^2 + w_{xt}^2) + B(m)w_{xx}^2 \right) \ dx \\
= \int_{\Omega} g_t w_t \ dx.
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\[
\frac{d}{dt} \int_{\Omega} \left( \theta + \frac{1}{2} (w_t^2 + w_{xt}^2 + B(m)w_{xx}^2) + V[w_{xx}] + \theta_c w_{xx} \right) dx
\]

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The solution is constructed by passing to the limit in a space-semidiscrete scheme. Higher order estimates are obtained by successive testing by higher and higher order terms and imply the following regularity:

\[w_{xxtt}, w_{xxtt}, \theta_t, \theta_{xx}, u_{tt}, u_{xxt} \in L^2(0, T^*; L^2(\Omega)).\]
The idea of this model is to consider also decreasing fatigue. Our present model takes into account the possibility to repair a partially damaged material by the effects of partial melting, so that fatigue can also decrease in time. The constitute law will be the same as before

\[ \sigma = B(m)\varepsilon + \int_0^\infty s_r[\varepsilon](t) \varphi(m,r) \, dr - \beta(\theta - \theta_c) + \nu \varepsilon_t, \quad (5) \]

and also the momentum balance equation stays the same:

\[ w_{tt} - w_{xxtt} + [B(m(w_{xx}))w_{xx} + \mathcal{P}[w_{xx}]]_{xx} = f. \]
With the constitutive law (5) we associate the specific entropy

\[ S[\theta, \varepsilon, \chi] = c_V \log(\theta/\theta_c) + \beta \varepsilon + \frac{L}{\theta_c} \chi \]

and the specific internal energy

\[ U[\theta, \varepsilon, \chi] = c_V \theta + \frac{1}{2} B(m) \varepsilon^2 + \frac{1}{2} \int_0^\infty \varphi(m, r) s_r^2[\varepsilon](t) dr + \beta \theta_c \varepsilon \]

\[ + L \chi + I_{[0,1]}(\chi), \]

where \( L \) is the constant latent heat, \( \chi \) is the space and time dependent phase variable and \( I_A \) is the indicator function of the set \( A \). We have the energy balance

\[ U_t + q_x = \sigma \varepsilon_t, \]

and the second Principle of Thermodynamics (Claudius-Duhem inequality) for the entropy production \( \psi \)

\[ \psi := S[\theta, \varepsilon]_t + \left( \frac{q}{\theta} \right)_x \geq 0. \]
We assume as before that $B'(m)$ is negative and we have an equation for the phase variable $\chi$:

$$-\gamma \chi_t \in \partial l_{[0,1]}(\chi) - \frac{L}{\theta_c}(\theta - \theta_c),$$

and the evolution equation for the fatigue rate we assume in the form

$$\frac{1}{C(\theta)} m_t + \frac{1}{2} B'(m) \varepsilon^2 m^- = -h(m) \chi_t |\chi_t| + \int_0^\infty \partial_t (\varepsilon - s_{r}[\varepsilon]) s_{r}[\varepsilon] \varphi(r) \, dr.$$
The system

For the problem

\[ u(t) = B(m)w_{xx} + P[w_{xx}] + w_{xxt} - (\theta - \theta_c), \]

\[ w_t - w_{xxt} = -u_{xx} + g, \]

\[ \theta_t = \theta_{xx} - \frac{1}{2} B'(m)w_{xx}^2 + |D[w_{xx}]_t| + w_{xxt} - \theta w_{xxt}, \]

\[ -L\chi_t - \frac{1}{2} m_t \int_0^\infty \phi(m, r) s r[\varepsilon] dr, \]

\[ -\gamma \chi_t \in \partial I[0, 1](\chi) - L\theta_c (\theta - \theta_c), \]

\[ C(\theta) m_t + \frac{1}{2} B'(m) \varepsilon^2 m - t = -h(m) \chi_t |\chi_t| + \int_0^\infty \partial s r[\varepsilon] s r[\varepsilon] \phi(r) dr. \]

with zero initial and boundary conditions for \( w \) and \( u \), and homogeneous Neumann boundary conditions for \( \theta \), we expect existence of a global solution.
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For the problem

\[ u_t = B(m)w_{xx} + \mathcal{P}[w_{xx}] + w_{xxt} - (\theta - \theta_c), \]
\[ w_t - w_{xxt} = -u_{xx} + g, \]
\[ \theta_t = \theta_{xx} - \frac{1}{2}B'(m)w_{xx}^2 m_t + |\mathcal{D}[w_{xx}]_t| + w_{xxt} - \theta w_{xxt} - L\chi_t, \]
\[ -\frac{1}{2}m_t \int_0^\infty \varphi(m, r)s_r[\varepsilon]dr, \]
\[ -\gamma\chi_t \in \partial I_{[0,1]}(\chi) - \frac{L}{\theta_c}(\theta - \theta_c), \]
\[ \frac{1}{C(\theta)}m_t + \frac{1}{2}B'(m)\varepsilon^2 m_t^- = -h(m)\chi_t|\chi_t| + \int_0^\infty \partial_t(\varepsilon - s_r[\varepsilon])s_r[\varepsilon]\varphi(r) dr. \]

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