SOME EQUATIONS WITH LOGARITHMIC NONLINEAR TERMS

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Dedicated to Gianni Gilardi on the occasion of his 65th birthday

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Three equations :

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(x,t)$$

$$\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) = g(x,t), \ \epsilon > 0$$

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = g(x,t)$$

$g \in L^\infty(\Omega \times (0,T)), \ \forall T > 0$
Ω : bounded and regular domain of R^n, $n = 1$, 2 or 3

Motivations :

• Caginalp phase-field system :

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + f(u) = \theta \\ \frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \end{cases}$$

u : order parameter
θ : relative temperature

Generalization :

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t} \\ \frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -u - \frac{\partial u}{\partial t} \end{cases}$$

$\alpha = \int_0^t \theta \, ds + \alpha_0$: thermal displacement variable

Based on the Maxwell-Cattaneo law

• Hyperbolic relaxation of the Caginalp model :

$$\begin{cases} \epsilon\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) = \theta, \ \epsilon > 0 \\ \frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \end{cases}$$

Models rapid phase transitions in certain classes of materials (P. Galenko et al.)

• Cahn-Hilliard equation

Models phase separation processes in binary alloys

In general :

$$f = F',\ F(s) = \frac{1}{4}(s^2 - 1)^2,\, f(s) = s^3 - s$$

Approximation of logarithmic potentials :

$$\begin{array}{l} F(s) = -\frac{\kappa_0}{2}s^2 + \kappa_1[(1+s)\ln(1+s) + (1-s)\ln(1-s)] \\ f(s) = -\kappa_0 s + \kappa_1 \ln\frac{1+s}{1-s} \\ s \in (-1,1),\ 0 < \kappa_1 < \kappa_0 \end{array}$$

Logarithmic terms : entropy of mixing

Regular nonlinear terms : the problems are well understood

The first model problem :

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(x,t) \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$
$$u|_{t=0} = u_0$$

Γ : boundary of Ω

Assumptions :

• g, $\frac{\partial g}{\partial t} \in L^\infty(\Omega \times (0,T))$, $\forall T > 0$

• $f \in \mathcal{C}^1(-1,1), f(0) = 0, f$ is odd
• $\lim_{s\to\pm 1} f(s) = \pm\infty$, $\lim_{s\to\pm 1} f'(s) = +\infty$
• $f' \geq -c_0$, $F \geq -c_1$, c_0, $c_1 \geq 0$, $F(s) = \int_0^s f(\tau)\, d\tau$

• $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\|u_0\|_{L^\infty(\Omega)} < 1$

Existence of a solution :

Take $\delta \in (0,1)$ s.t.

$$\|u_0\|_{L^\infty(\Omega)} \leq \delta,\ \|g\|_{L^\infty(\Omega\times(0,T))} - f(\delta) \leq 0$$

(recall that $\lim_{s\to 1} f(s) = +\infty$)

Set $U = u - \delta$:

$$\frac{\partial U}{\partial t} - \Delta U + f(u) - f(\delta) = g - f(\delta)$$

Multiply the equation by $U^+ = \max(U,0)$ ($f' \geq -c_0$) :

$$\frac{d}{dt}\|U^+\|^2_{L^2(\Omega)} \leq c\|U^+\|^2_{L^2(\Omega)}$$

Gronwall's lemma ($U^+(0) = 0$) :

$$u(x,t) \leq \delta, \text{ a.e. } (x,t) \in \Omega \times (0,T)$$

f is odd :

$$\|u(t)\|_{L^\infty(\Omega)} \leq \delta \in (0,1), \ \forall t \in [0,T]$$

$\rightarrow$ We essentially have a problem with a regular nonlinear term

Additional regularity : $u(t) \in H^2(\Omega), \forall t \in [0,T]$

Let u_1 and u_2 be 2 solutions with initial data $u_{0,1}$ and $u_{0,2}$ and set $u = u_1 - u_2$ and $u_0 = u_{0,1} - u_{0,2}$:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) = 0 \\
&u = 0 \text{ on } \Gamma \\
&u|_{t=0} = u_0
\end{aligned}$$

Multiply the equation by u ($f' \geq -c_0$, $c_0 \geq 0$) :

$$\|u_1(t) - u_2(t)\|_{L^2(\Omega)} \leq e^{ct}\|u_{0,1} - u_{0,2}\|_{L^2(\Omega)}$$

Consequences :

- Uniqueness

• We can define solutions in

$$\Phi = \{v \in L^\infty(\Omega),\ \|v\|_{L^\infty(\Omega)} \leq 1\}$$

(we can consider initial data containing the pure states)

We have

$$\|u(t)\|_{L^\infty(\Omega)} < 1,\ t > 0$$

(the phases mix instantaneously)

Remark : End of proof of existence : set

$$f_\delta(s) = f(s),\ |s| \leq \delta$$
$$f_\delta(s) = f(\delta) + f'(\delta)(s-\delta),\ s > \delta$$
$$f_\delta(s) = f(-\delta) + f'(-\delta)(s+\delta),\ s < -\delta$$

δ : as above

We have : $f'_\delta \geq -c_0$, $F_\delta \geq -c_1$

This yields : $\|u_\delta(t)\|_{L^\infty(\Omega)} \leq \delta \in (0,1)$, $t \in [0,T]$

Since $f_\delta = f$ in $[-\delta, \delta]$, we deduce the existence and uniqueness of the solution

Remark : Dissipative estimates : we take

$$g = g(x),\; g \in L^\infty(\Omega)$$

Key step : dissipative $L^\infty(\Omega)$-estimate

Consider the ODE's

$$y'_\pm + f(y_\pm) = h_\pm := \pm\|g\|_{L^\infty(\Omega)},\; y_\pm(0) = \pm\|u_0\|_{L^\infty(\Omega)}$$

We have

$$\begin{aligned}
&|y_\pm(t)| \le 1 - \delta(D(u_0) + |h_\pm|),\; t \in [0,1] \\
&|y_\pm(t)| \le 1 - \delta(|h_\pm|),\; t \ge 1 \\
&D(v) = \tfrac{1}{1-\|v\|_{L^\infty(\Omega)}}
\end{aligned}$$

Comparison principle :

$$y_-(t) \leq u(x,t) \leq y_+(t), \text{ a.e. } (x,t) \in \Omega \times R^+$$

This yields

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1-\delta, \ t \geq 0$$
$$\|u(t)\|_{L^\infty(\Omega)} \leq 1-\delta(\|g\|_{L^\infty(\Omega)}), \ t \geq 1$$

$\to$ Dissipative estimate

$\to$ Existence of finite-dimensional attractors

$\to$ Convergence of trajectories to steady states

Remark : Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

ν : unit outer normal

Similar results

Remark : Dynamic boundary conditions

Account for the interactions with the walls in confined systems

$$\frac{\partial u}{\partial t} - \Delta_\Gamma u + f_\Gamma(u) + \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma$$

Δ_Γ : Laplace-Beltrami operator
f_Γ : regular surface nonlinear term

Main feature : nonexistence of classical solutions

Counterexample :

$$y'' - f(y) = 0,\ x \in (-1,1),\ y'(\pm 1) = K,\ K > 0$$

(stationary 1D problem, $f \equiv -K$, $g \equiv 0$)

No classical solution for K large : critical value K_0 s.t.

• If $K < K_0$: existence of the unique solution s.t. $|y(x)| \leq \delta \in (0,1)$

• If $K > K_0$: no classical solution

The approximate solution y_δ converges to the solution to

$$y'' - f(y) = 0,\ y(\pm 1) = \pm 1$$

$\rightarrow$ The boundary condition is lost

More generally : the approximate solution u_δ converges to u s.t.

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + f(u) = g(x,t) \text{ in } \Omega \\ &\frac{\partial u}{\partial t} - \Delta_\Gamma u + f_\Gamma(u) + h(u) = 0 \text{ on } \Gamma \end{aligned}$$

In general : $h(u) \neq \frac{\partial u}{\partial \nu}$

u is the unique solution to a variational inequality

The equality holds when :

• f has a growth of the form $\frac{u}{(1-u^2)^p}$, $p > 1$, close to ± 1

• $\pm f_\Gamma(\pm 1) > 0$

Coupled systems : the situation can be more complicated

Caginalp system : similar results

Generalized Caginalp system based on the Maxwell-Cattaneo law :

$$
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t} \\
\frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -u - \frac{\partial u}{\partial t} \\
u = \alpha = 0 \text{ on } \Gamma \\
u|_{t=0} = u_0,\ \alpha|_{t=0} = \alpha_0,\ \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1
\end{cases}
$$

By approximating f as above : existence of a solution s.t.

$$|u(x,t)| < 1 \text{ a.e. } (x,t) \in \Omega \times (0,T),\ T > 0$$

The uniqueness is not straightforward : we need to estimate

$$\int_{\Omega} (f(u_1) - f(u_2)) \frac{\partial}{\partial t}(u_1 - u_2)\, dx$$

Idea : prove the strict separation property

$$\|u(t)\|_{L^\infty(\Omega)} \leq \delta(T) \in (0,1),\ t \in [0,T],\ T > 0$$

One possibility : prove an $L^\infty(\Omega)$-estimate on $\frac{\partial \alpha}{\partial t}$

The best we can have in general :

$$\|\frac{\partial \alpha}{\partial t}\|_{L^\infty(0,T;H_0^1(\Omega))} \leq c(T),\ T > 0$$

Here : $u_0 \in H_0^1(\Omega) \times H^3(\Omega)$, $\alpha_0 \in H_0^1(\Omega) \times H^3(\Omega)$, $\alpha_1 \in H_0^1(\Omega) \times H^2(\Omega)$

In one space dimension : we can conclude with the continuous injection $H^1(\Omega) \subset L^\infty(\Omega)$

We can also prove the strict separation in two space dimensions

We need ans estimate of the form

$$\|f'(u)\|_{L^p(\Omega\times(0,T))} \leq c(p,T),\ p \geq 1,\ T > 0$$

($p = 4$ is sufficient)

Lemma : We have

$$\int_{\Omega\times(0,T)} e^{L|f(u)|}\,dx\,dt \leq c(T),\ L > 0,\ T > 0.$$

Multiply the equation by $f(u)e^{L|f(u)|}$

Use the young's inequality

$$ab \leq \phi(a) + \psi(b),\ a,\ b \geq 0$$

$$\phi(s) = e^s - s - 1,\ \psi(s) = (1+s)\ln(1+s) - s,\ s \geq 0$$

$\rightarrow$ We obtain

$$\int_{\Omega\times(0,T)} |f(u)|^2 e^{L|f(u)|}\,dx\,dt \leq c$$
$$+2\int_{\Omega\times(0,T)} e^{c'|\frac{\partial\alpha}{\partial t}|}\,dx\,dt$$

We conclude by using the Orlicz embedding

$$\int_\Omega e^{c|v|}\,dx \leq e^{c'(\|v\|^2_{H^1(\Omega)}+1)},\ v \in H^1(\Omega)$$

We assume that

$$|f'| \leq e^{c|f|+c'}$$

(True for the logarithmic nonlinear terms)

$\rightarrow f'(u) \in L^p(\Omega \times (0,T)),\ T > 0,\ p \geq 1$

This yields, differentiating the equation for u w.r.t. t

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;H_0^1(\Omega))$$

Inject in the equation for α

$\rightarrow \frac{\partial \alpha}{\partial t} \in L^\infty(0,T;H^2(\Omega))$

In three space dimensions : we need

$$f'(u) \in L^6(\Omega \times (0,T)),\ T > 0$$

We can conclude when $|f'| \leq c|f|^{\frac{6}{5}} + c'$

$\rightarrow$ Not satisfied by the logarithmic nonlinear terms

Satisfied when f has a growth of the form

$$\frac{c}{(1-s^2)^r},\ r \geq 5,\ c > 0$$

close to ± 1

The second model problem :

$$\epsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u + f(u) = g, \ \epsilon > 0$$
$$u = 0 \text{ on } \Gamma$$
$$u|_{t=0} = u_0, \ \frac{\partial u}{\partial t}|_{t=0} = u_1$$

For simplicity : $g = g(x) \in L^\infty(\Omega)$

Here : $u_0 \in L^\infty(\Omega)$, $\|u_0\|_{L^\infty(\Omega)} < 1$

Existence of strong solutions only (when $\epsilon > 0$ is small and the initial data are not too large)

We are not able to prove the existence of weak solutions

Main ingredients :

• Perturbation argument : the solutions remain close to those of the limit parabolic problem

• Dissipativity provided by the equation

Theorem : There exists $\epsilon_0 > 0$ and a monotone decreasing function $R : (0, \epsilon_0] \to R^+$ satisfying

$$\lim_{\epsilon \to 0^+} R(\epsilon) = +\infty$$

s.t., for every initial data satisfying

$$D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon \|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq R(\epsilon),$$

there exists a unique global solution s.t.

$$\begin{aligned} &D(u(t)) + \|u(t)\|^2_{H^2(\Omega)} + \epsilon \|\tfrac{\partial u}{\partial t}(t)\|^2_{H^1(\Omega)} + \|\tfrac{\partial u}{\partial t}(t)\|^2_{L^2(\Omega)} \\ &+ \int_0^t e^{-\alpha(t-s)} \|\tfrac{\partial u}{\partial t}(s)\|^2_{H^1(\Omega)}\, ds \\ &\leq Q(D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon \|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}}) e^{-\alpha t} \\ &+ Q(\|g\|_{L^\infty(\Omega)}),\ \alpha > 0, \end{aligned}$$

where α and Q are independent of ϵ and $D(v) = \frac{1}{1 - \|v\|_{L^\infty(\Omega)}}$.

Uniqueness : standard ($f' \geq -c_0$, $c_0 \geq 0$)

Existence : follows the following steps :

Step 1 : Dissipative estimate in $H^1(\Omega) \times L^2(\Omega)$:

$$\|u(t)\|^2_{H^1(\Omega)} + \epsilon\|\frac{\partial u}{\partial t}(t)\|^2_{L^2(\Omega)} + \int_0^t e^{-\alpha(t-s)}\|\frac{\partial u}{\partial t}(s)\|^2_{H^1(\Omega)}\,ds$$
$$\leq Q(D(u_0) + (\|u_0\|^2_{H^1(\Omega)} + \epsilon\|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}})e^{-\alpha t}$$
$$+Q(\|g\|_{L^\infty(\Omega)}),\ \alpha > 0$$

α and Q independent of ϵ

Step 2 : Consider the limit parabolic problem ($\epsilon = 0$) :

$$\begin{aligned} &\frac{\partial u^0}{\partial t} - \Delta u^0 + f(u^0) = g \\ &u^0 = 0 \text{ on } \Gamma \\ &u^0|_{t=0} = u_0 \end{aligned}$$

We have :

$$D(u^0(t)) + \|u^0(t)\|^2_{H^2(\Omega)} \leq Q(D(u_0) + \|u_0\|^2_{H^2(\Omega)})e^{-\alpha t} + Q(\|g\|_{L^\infty(\Omega)}),\ \alpha > 0$$

Step 3 : Compare the solution to the hyperbolic problem to that to the limit parabolic problem :

$$\|u(t) - u^0(t)\|^2_{L^2(\Omega)} \leq \epsilon(Q(D(u_0) + \|u_0\|^2_{H^1(\Omega)} + \epsilon\|u_1\|^2_{L^2(\Omega)})e^{-\alpha t} + Q(\|g\|_{L^\infty(\Omega)}))$$

$\alpha > 0$ and Q independent of ϵ

Step 4 : Multiply the equation by $-\Delta(\beta u + \frac{\partial u}{\partial t})$, $\beta > 0$ small enough :

$$\frac{dE_u(t)}{dt} + \beta E_u(t) + \frac{\beta}{2}(\|\Delta u(t)\|^2_{L^2(\Omega)} + \|\nabla \frac{\partial u}{\partial t}(t)\|^2_{L^2(\Omega)}) \leq c\|f(u(t))\|^2_{H^1(\Omega)}$$

where

$$E_u(t) = \epsilon\|\nabla \tfrac{\partial u}{\partial t}(t)\|^2_{L^2(\Omega)} + \beta\|\nabla u(t)\|^2_{L^2(\Omega)} + \|\Delta u(t)\|^2_{L^2(\Omega)}$$
$$-2((g, \Delta u(t)))_{L^2(\Omega)} + \beta\epsilon((\nabla u(t), \nabla \tfrac{\partial u}{\partial t}(t)))_{L^2(\Omega)}$$

Step 5 : Estimate $\|f(u(t))\|_{H^1(\Omega)}$

We have :

$$\|f(u(t)) - f(u^0(t))\|^2_{H^1(\Omega)} \leq M_f\left(\frac{1}{1-\|u^0(t)\|_{L^\infty(\Omega)} - \|u(t)-u^0(t)\|_{L^\infty(\Omega)}}\right)\times$$
$$\times(1 + \|u^0(t)\|^2_{H^2(\Omega)} + \|u(t)\|^2_{H^2(\Omega)})\|u(t) - u^0(t)\|^2_{H^1(\Omega)}$$

M_f : smooth monotone increasing function only depending on f and satisfying

$$\lim_{z\to+\infty} M_f(z) = +\infty$$

u^0 : solution to the limit parabolic problem

Consider the interpolation inequalities

$$\|u(t) - u^0(t)\|_{H^1(\Omega)} \leq c\|u(t) - u^0(t)\|^{\frac{1}{2}}_{L^2(\Omega)}\|u(t) - u^0(t)\|^{\frac{1}{2}}_{H^2(\Omega)}$$
$$\|u(t) - u^0(t)\|_{L^\infty(\Omega)} \leq c\|u(t) - u^0(t)\|^{\frac{1}{4}}_{L^2(\Omega)}\|u(t) - u^0(t)\|^{\frac{3}{4}}_{H^2(\Omega)}$$

This yields

$$\|f(u(t))\|^2_{H^1(\Omega)} \leq Q_0\epsilon^{\frac{1}{2}}(1+E_u(t))^2 M_f\left(\frac{1}{(\overline{Q}+Q_0)^{-1}-\epsilon^{\frac{1}{8}}(\overline{Q}+Q_0)(1+E_u(t))}\right)$$
$$+Q_0e^{-\alpha t}+\overline{Q}$$

$Q_0 = Q_0(D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon\|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}})$

$\overline{Q} = \overline{Q}(\|g\|_{L^\infty(\Omega)})$

$\alpha > 0$, $Q_0, \overline{Q}$: independent of ϵ

Finally :

$$\frac{dE_u(t)}{dt} + \beta E_u(t) \leq Q_0\epsilon^{\frac{1}{2}}(1+E_u(t))^2 M_f\left(\frac{1}{(\overline{Q}+Q_0)^{-1}-\epsilon^{\frac{1}{8}}(\overline{Q}+Q_0)(1+E_u(t))}\right)$$
$$+2Q_0e^{-\alpha t}+2\overline{Q} \qquad (\beta < \alpha)$$

Step 6 : Assume that

$$D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon\|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq R(\epsilon)$$

$R = R(\epsilon)$ solves

$$\overline{Q} = Q_0(R)\epsilon^{\frac{1}{2}}(1 + 2(\beta-\alpha)^{-1}Q_0(R) + 3\beta^{-1}\overline{Q})^2 \times \\ \times M_f\left(\frac{1}{(\overline{Q}+Q_0(R))^{-1} - \epsilon^{\frac{1}{8}}(\overline{Q}+Q_0(R))(1+2(\beta-\alpha)^{-1}Q_0(R)+3\beta^{-1}\overline{Q})}\right)$$

Then :

$$E_u(t) \leq E_0(t)$$

where

$$E_0(t) = 2(\beta-\alpha)^{-1}Q_0(D(u_0) + (\|u_0\|^2_{H^2(\Omega)} + \epsilon\|u_1\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)})^{\frac{1}{2}}) \times \\ \times e^{-\alpha t} + 3\beta^{-1}\overline{Q}$$

Consequence of the comparison principle :

E_0 satisfies

$$\frac{dE_0(t)}{dt} + \beta E_0(t) \geq Q_0 \epsilon^{\frac{1}{2}} (1 + E_0(t))^2 M_f \big(\frac{1}{(\overline{Q}+Q_0)^{-1} - \epsilon^{\frac{1}{8}} (\overline{Q}+Q_0)(1+E_0(t))} \big)$$
$$+ 2Q_0 e^{-\alpha t} + 2\overline{Q}$$

We can take

$$E_u(0) \leq E_0(0)$$

We conclude by noting that

$$\lim_{\epsilon \to 0^+} R(\epsilon) = +\infty$$

Further results :

Additional regularity

Existence of finite-dimensional attractors

Extension : hyperbolic relaxation of the Caginalp phase-field system

More difficult : hyperbolic relaxation of the generalized Caginalp phase-field system

The third model problem :

$$\begin{aligned}&\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = g\\&u = \Delta u = 0 \text{ on } \Gamma\\&u|_{t=0} = u_0\end{aligned}$$

$g = g(x,t) \in L^\infty(\Omega \times (0,T))$

Approximation of f : existence and uniqueness of the solution s.t.

$$|u(x,t)| < 1 \text{ a.e. } (x,t) \qquad (\|u_0\|_{L^\infty(\Omega)} < 1)$$

Strict separation :

• In one space dimension :

$$\|u(t)\|_{L^\infty(\Omega)} \leq \delta \in (0,1),\ t \in (0,T),\ T > 0$$

(continuous embedding $H^1(\Omega) \subset L^\infty(\Omega)$)

• In two space dimensions : Orlicz embedding

• In three space dimensions : growth assumption on f : f grows like

$$\frac{c}{(1-s^2)^r},\ r > \frac{3}{7}$$

close to ± 1

$\rightarrow$ Not satisfied by the logarithmic nonlinear terms

Remark : Viscous Cahn-Hilliard equation :

$$\frac{\partial u}{\partial t} - \epsilon\Delta\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = g,\ \epsilon \geq 0$$

$\epsilon = 0$: Cahn-Hilliard equation

$\epsilon > 0$: strict separation (even in three space dimensions)

Remark : Neumann boundary conditions ($g \equiv 0$) :

$$\begin{aligned} &\tfrac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0 \\ &\tfrac{\partial u}{\partial \nu} = \tfrac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma \\ &u\big|_{t=0} = u_0 \end{aligned}$$

Main feature : mass conservation

$$< u(t) >=< u_0 >,\ t \geq 0,\ < \cdot >= \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega \cdot\, dx$$

Same results as in the case of Dirichlet boundary conditions

Key step : $H^{-1}(\Omega)$-estimate

$\rightarrow$ We rewrite the equation as

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u) =< f(u) >$$

$(-\Delta)^{-1}$: acts on functions with null average

We need to deal with the nonlocal term

$$< f(u) >= \frac{1}{\mathrm{Vol}(\Omega)} \int_{\Omega} f(u)\, dx$$

$\rightarrow$ Additional mathematical difficulties

Remark : Dynamic boundary conditions :

$$\begin{aligned} &\frac{\partial}{\partial \nu}(-\Delta u + f(u)) = 0 \text{ on } \Gamma \\ &\frac{\partial u}{\partial t} - \Delta_\Gamma u + f_\Gamma(u) + \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \end{aligned}$$

f_Γ : affine

The situation is similar to what was said in the first model problem :

Nonexistence of classical solutions

Existence of classical solutions if f satisfies growth assumptions or f_Γ satisfies sign assumptions

The sequence u_δ converges to the solution to the Cahn-Hilliard equation with

$$\begin{aligned} &\frac{\partial}{\partial \nu}(-\Delta u + f(u)) = 0 \text{ on } \Gamma \\ &\frac{\partial u}{\partial t} - \Delta_\Gamma u + f_\Gamma(u) + h(u) = 0 \text{ on } \Gamma \end{aligned}$$

In general : $h(u) \neq \frac{\partial u}{\partial \nu}$

Existence of finite-dimensional attractors :

Neumann boundary conditions :

Main difficulty : no strict separation in three space dimensions

We can define the continuous (in $H^{-1}(\Omega)$) semigroup

$$S(t) : \Phi_m \to \Phi_m,\ u_0 \mapsto u(t),\ t \geq 0,\ m \in (-1, 1)$$

$S(0) = \mathrm{Id},\ S(t+s) = S(t) \circ S(s),\ t,\ s \geq 0$

$$\Phi_m = \{v \in L^\infty(\Omega),\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma,$$
$$\|v\|_{L^\infty(\Omega)} \leq 1,\ < v >= m\}$$

Definition : Let Φ be a Banach space and $S(t)$ be a semigroup acting on Φ. A set $\mathcal{A} \subset \Phi$ is called the global attractor for $S(t)$ if

(i) $\mathcal{A}$ is compact in Φ.

(ii) $S(t)\mathcal{A} = \mathcal{A},\ t \geq 0$.

(iii) $\forall \epsilon > 0,\ \forall B \subset \Phi$ bounded, $\exists t_0 = t_0(B, \epsilon) \geq 0$ s.t. $t \geq t_0$ implies $S(t)B \subset \mathcal{U}_\epsilon$, where $\mathcal{U}_\epsilon$ is the ϵ-neighborhood of $\mathcal{A}$.

The global attractor is unique

It is the smallest closed set satisfying (iii)

Dimension : fractal (entropy) dimension

First proof of existence of the global attractor : A. Debussche-L. Dettori

Finite-dimensionality : based on the differentiability of the semigroup

$\rightarrow$ The strict separation from ± 1 was necessary

$\rightarrow$ Could be proved only for small domains

Theorem : For every $m \in (0,1)$, $\exists \mathcal{A}_m \subset H^2(\Omega)$ s.t.

(i) $\mathcal{A}_m$ is compact in $L^\infty(\Omega)$ and $H^{-1}(\Omega)$.

(ii) $\mathcal{A}_m$ has finite fractal dimension in $L^\infty(\Omega)$ and $H^{-1}(\Omega)$.

(iii) $\mathcal{A}_m$ attracts Φ_m in $H^{-1}(\Omega)$.

$\rightarrow$ No restriction on the size of Ω

Existence of the global attractor : follows from classical results

Finite-dimensionality : construction of an exponential attractor

Exponential attractor : compact and positively invariant $(S(t)\mathcal{M}_m \subset \mathcal{M}_m,\ t \geq 0)$ set which contains the global attractor, has finite fractal dimension and attracts exponentially fast the trajectories

Main tool : find a proper set $\mathcal{C}$ s.t.

$$\|S(t)u_1 - S(t)u_2\|_{L^2(\Omega)} \leq c(t)\|u_1 - u_2\|_{H^{-1}(\Omega)}$$

for some $t > 0$, $\forall u_1, u_2 \in \mathcal{C}$

$\mathcal{A}_m$ is trivial if m is large : $\exists M \in (0,1)$ s.t.

$$\mathcal{A}_m = \{m\} \text{ if } |m| \geq M$$

Set $S(t)(\pm 1) = \pm 1$

Then

$$S(t)\Phi = \Phi,\ \Phi = \cup_{|m|\leq 1}\Phi_m = B_{L^\infty(\Omega)}(0,1)$$

Set $\mathcal{A}_{\pm 1} = \{\pm 1\}$

Theorem : The semigroup $S(t)$ possesses the finite-dimensional global attractor

$$\mathcal{A} = \cup_{|m|\leq 1}\mathcal{A}_m$$

on Φ.

Dynamic boundary conditions :

Main difficulty : the order parameter can reach the pure states on a set with nonzero measure on the boundary

We can define the continuous (in $H^{-1}(\Omega) \times L^2(\Omega)$) semigroup $S(t)$ acting on

$$\Phi_m = \{(u, u|_\Gamma) \in \Phi, \ < u >= m\}, \ m \in (-1, 1)$$

Here :

$$\Phi = \{(v, v|_\Gamma) \in L^\infty(\Omega) \times L^\infty(\Gamma), \ \|v\|_{L^\infty(\Omega)} \leq 1, \ \|v|_\Gamma\|_{L^\infty(\Gamma)} \leq 1\}$$

$S(t)$ is associated with the solutions obtained via the regularization of f

Theorem : For every $m \in (-1, 1)$, the semigroup $S(t)$ possesses the finite-dimensional global attractor $\mathcal{A}_m$ which is bounded in $\mathcal{C}^\alpha(\Omega) \times \mathcal{C}^\alpha(\Gamma)$, $0 < \alpha < \frac{1}{4}$.

Existence of the global attractor : follows from classical results

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We need some (asymptotically) compact smoothing property on the difference of 2 solutions

We have

$$\|u_1(t) - u_2(t)\|^2_{\Phi^{\mathrm{w}}} \leq ce^{-\beta t}\|u_1(0) - u_2(0)\|^2_{\Phi^{\mathrm{w}}} + c' \int_0^t \|\theta(u_1(s) - u_2(s))\|^2_{L^2(\Omega)}\, ds$$

$\beta > 0$, θ : smooth cut-off function

$\Phi^{\mathrm{w}} = H^{-1}(\Omega) \times L^2(\Gamma)$

$\rightarrow$ Contraction, up to $\|\theta(u_1 - u_2)\|_{L^2(0,t;L^2(\Omega))}$

Compactness : we work on spaces of trajectories and use the compactness of

$$L^2(0,t;H^1(\Omega)) \cap H^1(0,t;H^{-3}(\Omega)) \subset L^2(0,t;L^2(\Omega))$$

We have

$$\|\frac{\partial}{\partial t}[\theta(u_1 - u_2)]\|^2_{L^2(0,t;H^{-3}(\Omega))} + \|\theta(u_1 - u_2)\|^2_{L^2(0,t;H^1(\Omega))} \leq ce^{c't}\|u_1(0) - u_2(0)\|^2_{H^{-1}(\Omega)\cap L^2(\Gamma)}$$

$u_1(0), u_2(0) \in B_{H^{-1}(\Omega)\cap L^2(\Gamma)}(u_0, \epsilon)$, $\epsilon > 0$ small