Analysis of a degenerating PDE system for phase transitions and damage

Riccarda Rossi
(Università di Brescia)

in collaboration with
Elisabetta Rocca (Università di Milano)

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The PDE system for phase transitions/damage

$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathrm{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega\times(0,T),$$
$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega\times(0,T),$$
$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta\chi + W'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega\times(0,T),$$

Variables

- ϑ $\rightsquigarrow$ absolute temperature;
- $\mathbf{u}$ $\rightsquigarrow$ (small) displacements;
- $\chi\in[0,1]$ $\rightsquigarrow$ phase/damage parameter

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Nonlinearities & data

- c $\rightsquigarrow$ specific heat & K $\rightsquigarrow$ heat conductivity, $\rho \in \mathbb{R}$;
- $\mathrm{R}_e \rightsquigarrow$ elasticity tensor & $\mathrm{R}_v \rightsquigarrow$ viscosity tensor;
- $a,\ b \in \mathrm{C}^1([0,1];[0,+\infty))$;
- $W = \widehat{\beta} + \gamma$, $\widehat{\beta} : [0,1] \to \mathbb{R}$ convex & γ smooth; $\mu \in \{0,1\}$;
- $\mathbf{f}$ volume force & g heat source.

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Variables

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- $\chi \in [0,1]$ $\rightsquigarrow$ phase/(irreversible) **damage** parameter

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- c $\rightsquigarrow$ specific heat & K $\rightsquigarrow$ heat conductivity, $\rho \in \mathbb{R}$;
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- $W = \widehat{\beta} + \gamma$, $\widehat{\beta} : [0,1] \to \mathbb{R}$ convex with $\beta := \partial\widehat{\beta}$, & γ Lipschitz; $\mu = 1$;
- $\mathbf{f}$ volume force & g heat source.

The momentum balance equation (I)

$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$

Physical meaning: the case of phase transitions

Phase parameter $\chi \in [0,1]$ with

$$\begin{cases} \chi = 0 & \text{e.g. } \textbf{solid} \text{ phase} \\ \chi = 1 & \text{e.g. } \textbf{liquid} \text{ phase} \\ \chi \in (0,1) & \textbf{“mushy region”} \end{cases}$$

The momentum balance equation (I)

$$\mathbf{u}_{tt} - \mathrm{div}(\chi \mathrm{R}_v \varepsilon(\mathbf{u}_t) + (1-\chi)\mathrm{R}_e \varepsilon(\mathbf{u}) - \rho\vartheta \mathbf{1}) = \mathbf{f}$$

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We have

$$\begin{cases} a(\chi) = \chi & \rightsquigarrow \quad \textbf{viscous} \text{ contribution in the } \textbf{liquid} \text{ phase} \\ b(\chi) = 1-\chi & \rightsquigarrow \quad \textbf{elastic} \text{ contribution in the } \textbf{solid} \text{ phase} \end{cases}$$

The momentum balance equation (II)

$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$

Physical meaning: the case of damage

Damage parameter $\chi \in [0,1]$ with

$$\begin{cases} \chi = 0 & \textbf{complete damage} \\ \chi = 1 & \textbf{undamaged} \text{ state} \\ \chi \in (0,1) & \textbf{partial damage} \end{cases}$$

The momentum balance equation (II)

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We have, e.g.

$$\begin{cases} a(\chi) = \chi & \rightsquigarrow \quad \text{material stiffness decreases as } \chi \searrow 0 \\ b(\chi) = \chi & \rightsquigarrow \quad \text{material stiffness decreases as } \chi \searrow 0 \end{cases}$$

The momentum balance equation (II)

$$\mathbf{u}_{tt} - \operatorname{div}(a(\mathcal{X})\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\mathcal{X})\mathrm{R}_e\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega \times (0, T),$$

Analytical features

- **Elliptic degeneracy** of the momentum equation when $a(\mathcal{X})$ & $b(\mathcal{X}) \to 0$
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Analytical features

- **Elliptic degeneracy** of the momentum equation when $a(\chi) = \chi$ & $b(\chi) = \chi \to 0$: e.g., in the case of **complete damage**
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Analytical features

- **Elliptic degeneracy** of the momentum equation when $a(\chi)$ & $b(\chi) \to 0$
- This affects the **whole** PDE system

A few words on the model derivation

- cf. **[M. Frémond, Non-smooth thermomechanics, 2002]**
- cf. **[M. Frémond, Phase change in mechanics, 2012]**

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♣ **The equations:**

for $\mathbf{u}$: momentum balance with inertia: σ stress tensor

$$\mathbf{u}_{tt} - \mathrm{div}\sigma = \mathbf{f}$$

for χ: equation for **microscopic motions**: B & $\mathbf{H}$ $\rightsquigarrow$ microscopic forces

$$B - \mathrm{div}(\mathbf{H}) = 0$$

for ϑ: internal energy balance: e internal energy & $\mathbf{q}$ heat flux

$$e_t + \mathrm{div}\,\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H} \cdot \nabla\chi_t$$

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♣ The expressions of σ, B, $\mathbf{H}$, e and $\mathbf{q}$ are recovered from

- free energy (with $f = f(\vartheta)$ concave s.t. c: Legendre transf. of $-f$)

$$\mathcal{F}(\vartheta, \varepsilon(\mathbf{u}), \chi, \nabla\chi) = \int_\Omega \left(f(\vartheta) + b(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \frac{1}{2}|\nabla\chi|^2 + W(\chi) - \vartheta\chi + \rho\vartheta\mathrm{tr}(\varepsilon(\mathbf{u}))\right)\mathrm{d}x$$

- pseudo-potential of dissipation

$$\mathcal{P}(\nabla\vartheta, \chi_t, \varepsilon(\mathbf{u}_t)) = \frac{\mathsf{K}(\vartheta)}{2}|\nabla\vartheta|^2 + \frac{1}{2}|\chi_t|^2 + \mu I_{(-\infty,0]}(\chi_t) + a(\chi)\frac{\varepsilon(\mathbf{u}_t)\mathrm{R}_v\varepsilon(\mathbf{u}_t)}{2}$$

via standard **constitutive relations**.

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via standard **constitutive relations**.

♣ Small perturbation assumptions: neglect **quadratic** terms $|\chi_t|^2 + a(\chi)\varepsilon(\mathbf{u}_t)\mathrm{R}_v\varepsilon(\mathbf{u}_t)$

Analytical difficulties

$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega\times(0,T),$$
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$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta\chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)b\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega\times(0,T),$$

- highly nonlinear coupling with **quadratic** terms
-
-
-

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- **low regularity** of ϑ
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- **doubly nonlinear** character of the χ-equation
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- highly nonlinear coupling with **quadratic** terms
- **low regularity** of ϑ
- **doubly nonlinear** character of the χ-equation
- **elliptic degeneracy** of the momentum equation

From a nondegenerate to a degenerate system

- To handle the **elliptic degeneracy**,

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From a nondegenerate to a degenerate system

- To handle the **elliptic degeneracy**, consider the approximate **nondegenerate** system

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From a nondegenerate to a degenerate system

- To handle the elliptic degeneracy, consider the approximate nondegenerate system

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$$
\text{limit as } \delta \downarrow 0 \ \Rightarrow \begin{cases} \text{notion of } \textit{weak} \text{ solution} \\ \text{with evolution for } \chi \text{ given by} \\ \text{variational inequality + total energy inequality} \end{cases}
$$

cf. for rate-independent complete-damage evolution

- **[G. Bouchitté & A. Mielke & T. Roubíček, ZAMP 2009]**
- **[A. Mielke & T. Roubíček & J. Zeman, CMAME 2010]**

From a nondegenerate to a degenerate system

- To handle the **elliptic degeneracy**, consider the approximate **nondegenerate** system

$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega\times(0,T),$$
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♣ Alternative approach for **complete damage** in (cf. **[E. Bonetti & C. Kraus & A. Segatti, work in progress 2012]**)

A scheme of the results (I)

$$
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&\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathrm{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega\times(0,T),\\
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The reversible system $\mu = 0$

- **existence** for $\rho = 0$ and $\rho \neq 0$
- **uniqueness** in the isothermal case

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The reversible system $\mu = 0$

- **existence** for $\rho = 0$ and $\rho \neq 0$
- **uniqueness** in the isothermal case

The IRreversible system $\mu = 1$

- **existence** for $\rho = 0$
- **enhanced regularity** in the isothermal case
- **degenerate limit** as $\delta \downarrow 0$ for $\rho = 0$

A scheme of the results (II)

$$
\begin{aligned}
&\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathrm{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega\times(0,T),\\
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Common features for $\mu = 0$ and $\mu = 1$

♣ All existence results proved by passing to the limit in **time-discretization** scheme

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Common features for $\mu = 0$ and $\mu = 1$

- ♣ All existence results proved by passing to the limit in **time-discretization** scheme
- ♣ In the eqn. for χ, $-\Delta\chi$ replaced by

$$A_p\chi = -\mathrm{div}(|\nabla\chi|^{p-2}\nabla\chi) \text{ with } p > d, \text{ or}$$

$$A_s\chi \ \leftrightsquigarrow\ a_s(\chi_1,\chi_2) := \int_\Omega\int_\Omega \frac{(\nabla\chi_1(x) - \nabla\chi_1(y))\cdot(\nabla\chi_2(x) - \nabla\chi_2(y))}{|x-y|^{d+2(s-1)}}\mathrm{d}x\mathrm{d}y \text{ with } s > \frac{d}{2}$$

hence χ is estimated either in $W^{1,p}(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$, or $H^s(\Omega) \subset \mathrm{C}^0(\overline{\Omega})$.

A scheme of the results (II)

$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}(\mathbf{u}_t) - \mathrm{div}(\mathsf{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega\times(0,T),$$
$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi)+\delta)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \quad \text{in } \Omega\times(0,T),$$
$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta\chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega\times(0,T),$$

Common features for $\mu = 0$ and $\mu = 1$

♣ All existence results proved by passing to the limit in **time-discretization** scheme

♣ In the eqn. for χ, $-\Delta\chi$ replaced by

$$A_p\chi = -\mathrm{div}(|\nabla\chi|^{p-2}\nabla\chi) \text{ with } p > d, \text{ or}$$

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hence χ is estimated either in $W^{1,p}(\Omega)\subset \mathrm{C}^0(\overline{\Omega})$, or $H^s(\Omega)\subset\mathrm{C}^0(\overline{\Omega})$.

- **Gradient theories for damage:**
  - p-Laplacian in rate-dependent & rate-independent models, see e.g. **[Bonetti, Mielke, Roubíček, Segatti, Schimperna, Thomas.....]**
  - s-Laplacian **[D. Knees & R. R. & C. Zanini, *M^3AS* to appear]**

A scheme of the results (II)

$$\begin{aligned}
&c(\vartheta)\vartheta_t + \chi_t\vartheta - \mathrm{div}(\mathrm{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega\times(0,T),\\
&\mathbf{u}_{tt} - \mathrm{div}((a(\chi)+\delta)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u})) = \mathbf{f} \quad \text{in } \Omega\times(0,T),\\
&\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \vartheta \quad \text{in } \Omega\times(0,T),
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hence χ is estimated either in $W^{1,p}(\Omega)\subset \mathrm{C}^0(\overline{\Omega})$, or $H^s(\Omega)\subset \mathrm{C}^0(\overline{\Omega})$.

- We focus on the **irreversible case** $\mu = 1$, $\rho = 0$
  - existence $\rightsquigarrow$ OK for $A_p\chi$, $A_s\chi$
  - degenerate limit as $\delta\downarrow 0$ $\rightsquigarrow$ OK for $A_s\chi$

Outline

- The existence theorem for $\delta > 0$
- Sketch of the proof
- Two ideas on the degenerate limit $\delta \downarrow 0$
- The existence theorem for $\delta = 0$

Preliminary: "enthalpy" transformation

In view of the time-discretization,

$$\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \mathrm{div}(\mathrm{K}(\vartheta)\nabla\vartheta) = g \quad \text{in } \Omega \times (0,T),$$
$$\downarrow$$
$$w_t + \chi_t\Theta(w) - \mathrm{div}(K(w)\nabla w) = g \quad \text{in } \Omega \times (0,T),$$

via the enthalpy transformation (cf. **[T. Roubíček, SIMA 2010]**)

$$w_t = \mathrm{c}(\vartheta)\vartheta_t \quad \text{i.e.} \quad w = \int_0^\vartheta \mathrm{c}(s)\mathrm{d}s \ \Rightarrow \begin{cases} \vartheta \ \rightsquigarrow \ \Theta(w), \\ \mathrm{K}(\vartheta) \ \rightsquigarrow \ K(w) = \frac{\mathrm{K}(\Theta(w))}{\mathrm{c}(\Theta(w))} \end{cases}$$

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Hence,

$$w_t + \chi_t\Theta(w) - \mathrm{div}(K(w)\nabla w) = g \quad \text{in } \Omega\times(0,T),$$
$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi)+\delta)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u})) = \mathbf{f} \quad \text{in } \Omega\times(0,T),$$
$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \quad \text{in } \Omega\times(0,T),$$

+ homogeneous Dirichlet bdry cond. for u + no-flux bdry cond. for w, χ

The weak formulation of the equation for χ

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \quad \text{in } \Omega \times (0,T) \qquad (\mathrm{eq}\chi)$$

Why do we need a weak formulation?

- For $(\mathrm{eq}\chi)$ to make sense **a.e. in** $\Omega \times (0,T)$, we need to estimate **separately** $A_s\chi$ and $\beta(\chi)$
- This could be done by testing $(\mathrm{eq}\chi)$ by $\partial_t(A_s\chi + \beta(\chi))$
- This would involve an integration by parts in time of the term

$$\iint \Theta(w)\partial_t(A_s\chi + \beta(\chi))$$

NOT doable, because of the low regularity in time of w.

The weak formulation of the equation for χ

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \quad \text{in } \Omega\times(0,T) \tag{eqχ}$$

- **Weak formulation of** (eqχ) for $\beta = \partial I_{[0,1]} \rightsquigarrow \beta = \partial I_{[0,+\infty)}$ (cf. **[C. Heinemann & C. Kraus, AMSA, to appear]**):

$$\chi_t(x,t) \le 0 \quad \text{for a.a. } (x,t) \in \Omega\times(0,T), +$$

$$\int_0^T\int_\Omega \left(\chi_t\varphi + a_s(\chi,\varphi) + \xi\varphi + \gamma'(\chi)\varphi + b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2}\varphi - \Theta(w)\varphi\right)\mathrm{d}x\,\mathrm{d}t \ge 0$$

$$\text{with } \xi \in \partial I_{[0,+\infty)}(\chi), \text{for all } \varphi \in L^2(0,T;H^s(\Omega)) \cap L^\infty(Q) \text{ with } \varphi \le 0, +$$

$$\begin{aligned}
&\int_s^t\int_\Omega |\chi_t|^2\mathrm{d}x\mathrm{d}r + \frac{1}{2}a_s(\chi(t),\chi(t)) + \int_\Omega W(\chi(t))\mathrm{d}x \\
&\le \frac{1}{2}a_s(\chi(s),\chi(s)) + \int_\Omega W(\chi(s))\mathrm{d}x + \int_s^t\int_\Omega \chi_t\left(-b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w)\right)\mathrm{d}x\mathrm{d}r \\
&\qquad \forall\, t \in (0,T], \ \text{for a.a. } 0 < s \le t.
\end{aligned}$$

The weak formulation of the equation for χ

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \quad \text{in } \Omega\times(0,T) \qquad (\mathrm{eq}\chi)$$

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with $\xi\in\partial I_{[0,+\infty)}(\chi)$, for all $\varphi\in L^2(0,T;H^s(\Omega))\cap L^\infty(Q)$ with $\varphi\le 0$, +

$$\int_s^t\int_\Omega |\chi_t|^2\mathrm{d}x\mathrm{d}r + \frac{1}{2}a_s(\chi(t),\chi(t)) + \int_\Omega W(\chi(t))\mathrm{d}x$$
$$\le \frac{1}{2}a_s(\chi(s),\chi(s)) + \int_\Omega W(\chi(s))\mathrm{d}x + \int_s^t\int_\Omega \chi_t\left(-b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w)\right)\mathrm{d}x\mathrm{d}r$$
$$\forall\, t\in(0,T],\ \text{for a.a. } 0 < s \le t.$$

- **Consistent** with the formulation of $(\mathrm{eq}\chi)$ a.e. in $\Omega\times(0,T)$

The existence result for the nondegenerate system for damage

Theorem I [Rocca & R., arXiv preprint 2012]

Under suitable assumptions, there exist

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*) \quad \forall\, 1 \leq r < \frac{d+2}{d+1},$$

$$\mathbf{u} \in H^1(0,T;H^2_0(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d)),$$

$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

fulfilling initial conditions +

$$\int_\Omega \varphi(t)\, w(t)(\mathrm{d}x) - \int_0^t \int_\Omega w\varphi_t \mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega \chi_t \Theta(w)\varphi \mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega K(w)\nabla w \nabla\varphi \mathrm{d}x\mathrm{d}s$$
$$= \int_0^t \int_\Omega g\varphi + \int_\Omega w_0 \varphi(0)\mathrm{d}x$$

for all $\varphi \in \mathscr{F} := \mathrm{C}^0([0,T];W^{1,r'}(\Omega)) \cap W^{1,r'}(0,T;L^{r'}(\Omega))$ and for all $t \in (0,T]$, +

$\mathbf{u}_{tt} - \mathrm{div}((a(\chi)+\delta)\mathrm{R}_v\varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e\varepsilon(\mathbf{u})) = \mathbf{f}$ in $H^{-1}(\Omega;\mathbb{R}^d)$, a.e. in $(0,T)$, +

weak formulation of equation for χ

Furthermore, **positivity** of the temperature $\vartheta = \Theta(w)$.

(Formal) A priori estimates

♡ All calculations rigorous on the time-discrete level.

♣ **First estimate: energy estimate**

$$\iint \Big(w_t + \chi_t \Theta(w) - \mathrm{div}(K(w)\nabla w) = g \Big) \times 1 +$$

$$\iint \Big(\mathbf{u}_{tt} - \mathrm{div}((a(\chi)+\delta)\mathrm{R}_v \varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e \varepsilon(\mathbf{u})) = \mathbf{f} \Big) \times \mathbf{u}_t +$$

$$\iint \Big(\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s\chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \Big) \times \chi_t$$

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$$\Rightarrow \quad \|w\|_{L^\infty(0,T;L^1(\Omega))} + \|\mathbf{u}\|_{H^1(0,T;H^1_0(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} + \|\chi\|_{L^\infty(0,T;H^s(\Omega))\cap H^1(0,T;L^2(\Omega))} \leq C$$

(Formal) A priori estimates

♣ **Second estimate:** **regularity estimate** on equation for $\mathbf{u}$ cf. **[E. Bonetti & G. Schimperna & A. Segatti, JDE 2005]**

$$\iint \Big(\mathbf{u}_{tt} - \mathrm{div}((a(\mathcal{X})+\delta)\mathrm{R}_v \varepsilon(\mathbf{u}_t) + b(\mathcal{X})\mathrm{R}_e \varepsilon(\mathbf{u})) = \mathbf{f} \Big) \times (-\,\mathrm{div}(\varepsilon(\mathbf{u}_t)))$$

for this, we need $\|\mathcal{X}\|_{L^\infty(0,T;H^s(\Omega))} \leq C$ with $s > d/2$

$$\Rightarrow \quad \|\mathbf{u}\|_{H^1(0,T;H^2_0(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega;\mathbb{R}^d)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C\,.$$

(Formal) A priori estimates

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♣ **Third estimate:** Boccardo-Gallouët estimate on equation for w

$$\Rightarrow \|w\|_{L^r(0,T;W^{1,r}(\Omega))\cap\cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)} \leq C \quad \forall\, 1 \leq r < \frac{d+2}{d+1},$$

How to pass to the limit as $\delta \downarrow 0$

- Nondegenerate system:

$$w_t + \chi_t \Theta(w) - \mathrm{div}(K(w)\nabla w) = g \quad \textbf{(weak formulation)}$$

$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi)+\delta)\mathrm{R}_v \varepsilon(\mathbf{u}_t) + b(\chi)\mathrm{R}_e \varepsilon(\mathbf{u})) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \quad \text{a.e. in } (0,T)$$

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s \chi + \beta(\chi) + \gamma'(\chi) \ni -b'(\chi)\frac{\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})}{2} + \Theta(w) \quad \textbf{(weak formulation)}$$

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Main idea (cf. [A. Mielke & T. Roubíček & J. Zeman, CMAME 2010])

- Let $(w_\delta, \mathbf{u}_\delta, \chi_\delta)_\delta$ be solutions: the **energy estimate** in particular yields

$$\|\sqrt{\chi_\delta + \delta}\,\mathrm{R}_v\,\varepsilon(\partial_t \mathbf{u}_\delta)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} + \|\sqrt{\chi_\delta + \delta}\,\mathrm{R}_e\,\varepsilon(\mathbf{u}_\delta)\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} \leq C,$$

$$\|\partial_t^2 \mathbf{u}_\delta\|_{L^2(0,T;H^{-1}(\Omega;\mathbb{R}^d))} + \|\partial_t \mathbf{u}_\delta\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C$$

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$$\mathbf{u}_{tt} - \mathrm{div}((\chi + \delta)\mathrm{R}_v \varepsilon(\mathbf{u}_t) + (\chi + \delta)\mathrm{R}_e \varepsilon(\mathbf{u})) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \quad \text{a.e. in } (0,T)$$

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$$\|\partial_t^2 \mathbf{u}_\delta\|_{L^2(0,T;H^{-1}(\Omega;\mathbb{R}^d))} + \|\partial_t \mathbf{u}_\delta\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \le C$$

- Hence, work with the elastic and viscous **quasi-stresses**

$$\boldsymbol{\eta}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\mathbf{u}_\delta), \qquad \boldsymbol{\mu}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\partial_t \mathbf{u}_\delta)$$

and pass to the limit as $\delta \downarrow 0$ in

$$\partial_t^2 \mathbf{u}_\delta - \mathrm{div}(\sqrt{\chi_\delta + \delta}\,\mathrm{R}_v\,\boldsymbol{\mu}_\delta) - \mathrm{div}(\sqrt{\chi_\delta + \delta}\,\mathrm{R}_e\,\boldsymbol{\eta}_\delta) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \quad \text{a.e. in } (0,T)\,.$$

The existence result for the nondegenerate system for damage

Theorem II [Rocca & R., arXiv preprint 2012]

Under suitable assumptions, up to a subseq. the functions $(w_\delta, \mathbf{u}_\delta, \boldsymbol{\eta}_\delta, \boldsymbol{\mu}_\delta, \chi_\delta)_\delta$ converge to $(w, \mathbf{u}, \boldsymbol{\eta}, \boldsymbol{\mu}, \chi)$ fulfilling initial conditions +

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t), \quad \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}) \text{ a.e. in any open set } A \subset \Omega \times (0,T) \text{ s.t. } \chi > 0 \text{ a.e. in } A,$$

$$\partial_t^2 \mathbf{u} - \operatorname{div}(\sqrt{\chi}\,\mathrm{R}_v\,\boldsymbol{\mu}) - \operatorname{div}(\sqrt{\chi}\mathrm{R}_e\,\boldsymbol{\eta})) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \text{ a.e. in } (0,T),$$

+ weak formulation of enthalpy equation +

+ weak formulation of equation for χ:

$$\int_0^T \int_\Omega (\partial_t \chi + \gamma'(\chi))\,\varphi \mathrm{d}x\mathrm{d}t + \int_0^T a_s(\chi,\varphi)\mathrm{d}t \geq \int_0^T \int_\Omega \left(-\frac{1}{2\chi}\boldsymbol{\eta}\,\mathrm{R}_e\,\boldsymbol{\eta} + \Theta(w) \right) \varphi \mathrm{d}x\mathrm{d}t$$

for all $\varphi \in L^2(0,T;H^s(\Omega)) \cap L^\infty(Q)$ with $\varphi \leq 0$ & $\operatorname{supp}(\varphi) \subset \{\chi > 0\}$,

$$\int_\Omega w(t)(\mathrm{d}x) + \int_0^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}\int_0^t \int_\Omega \boldsymbol{\mu}(r)\,\mathrm{R}_v\,\boldsymbol{\mu}(r)\,\mathrm{d}x\,\mathrm{d}r + \int_\Omega W(\chi(t))\,\mathrm{d}x + \mathcal{J}(t)$$

$$= \int_\Omega w_0\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,\mathrm{d}x + \frac{1}{2}a_{\mathrm{el}}(b(\chi_0)\mathbf{u}_0,\mathbf{u}_0) + \frac{1}{2}a_s(\chi_0,\chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x$$

$$+ \int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_0^t \int_\Omega g\,\mathrm{d}x\mathrm{d}r$$

with $\displaystyle\int_0^t \mathcal{J}(r)\mathrm{d}r \geq \frac{1}{2}\int_0^t \int_\Omega \left(|\mathbf{u}_t(r)|^2 + \boldsymbol{\eta}(r)\,\mathrm{R}_v\,\boldsymbol{\eta}(r) \right)\,\mathrm{d}x + a_s(\chi(r),\chi(r))\,\mathrm{d}r$ for all $0 \leq t \leq T$.