On some Cahn-Hilliard models with nonlinear diffusion

Giulio Schimperna

Dipartimento di Matematica “F. Casorati”,
Università di Pavia, Italy
giusch04@unipv.it

Gianni’s session at ADMAT 2012
Cortona, September 19, 2012

References

G. Schimperna and I.Pawłow, *On a Cahn-Hilliard model with nonlinear diffusion*, ArXiv:1106.1581 (2011)

G. Schimperna and I.Pawłow, *On a Cahn-Hilliard model with singular diffusion*, ArXiv:1206.5604 (2012), JDE, to appear (2013).

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The system

- We are interested in the following class of systems:

$$u_t - \Delta w = 0, \tag{1}$$

$$w = -a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u) - \lambda u + \varepsilon u_t + \delta\Delta^2 u \tag{2u}$$

  representing a variant of the Cahn-Hilliard model for phase separation.

- We add no flux b.c.: $\partial_{\mathbf{n}} u = \partial_{\mathbf{n}} w = \delta \partial_{\mathbf{n}} \Delta u = 0$ (and, of course, the initial conditon $u|_{t=0} = u_0$).

- We will just consider the 4th order case $\delta = 0$.

- We will set $\mathcal{A}(u) := -a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2$.

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Motivation and related work

- Phase separation in water-oil mixtures: [Gompper-Kraus, 1992], [Gompper-Zschocke, 1993].
- Phase-field crystal growth: [Elder-Grant, 2004].
- Faceting of a growing crystalline surface: [Savina et al., 2003], [Korzec-Rybka, 2012].
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- A lot of mathematical work in recent contributions by [Pawłow and Zajączkowski] (especially related to the sixth order case with smooth potential).

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The u-formulation

u-System:

$$u_t - \Delta w = 0, \tag{1}$$

$$w = -a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + f(u) - \lambda u + \varepsilon u_t \tag{2u}$$

- $f(u) = F'(u) = \log(1+u) - \log(1-u)$
    - either a smooth, $0 < \underline{a} \leq a(\cdot) \leq \overline{a} < \infty$ (smooth nonlinear diffusion)
    - or $a(u) = f'(u) = \dfrac{2}{1-u^2}$ (singular diffusion).

Outcome of the basic a-priori estimates

- Energy estimate:

$$u \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^\infty(Q).$$

- Estimate on time derivatives (only for convex energies):

$$u \in W^{1,\infty}(\tau,T;V'), \quad 0 < \tau < T$$

- Estimate of $f(u)$:

$$f(u) \in L^2(0,T;H) \cap L^\infty(\tau,T;H)$$

The z-formulation

- Set

$$z := \phi(u) := \int_0^u \left(\frac{a(s)}{2}\right)^{1/2}$$
$$= \int_0^u \frac{1}{(1-s^2)^{1/2}} = \arcsin u \quad \text{(singular diffusion)}$$

z-System:

$$u_t - \Delta w = 0, \tag{1}$$
$$w = -2\phi'(u)\Delta z + f(u) - \lambda u + \varepsilon u_t \tag{2z}$$

- Note that $\phi'(u) = \dfrac{1}{(1-u^2)^{1/2}}$ (singular diffusion)

The v-formulation (singular diffusion)

- Set
$$v := f(u) = \log(1+u) - \log(1-u),$$
- so that
$$u := j(v) := f^{-1}(v) := \frac{1}{2}\tanh\frac{v}{2}$$

v-System:

$$u_t - \Delta w = 0, \tag{1}$$

$$w = -\Delta v + \frac{j(v)}{2}|\nabla v|^2 + v - \lambda u + \varepsilon u_t \tag{2v}$$

An integration by parts formula

Lemma (Dal Passo, Garcke & Grün, 1998)

Let $h \in W^{2,\infty}(\mathbb{R})$, $y \in H^2(\Omega)$, $\partial_{\mathbf{n}} y = 0$ on $\Gamma = \partial\Omega$. Then,

$$\int_\Omega h'(y)|\nabla y|^2 \Delta y = -\frac{1}{3}\int_\Omega h''(y)|\nabla y|^4 + \frac{2}{3}\int_\Omega h(y)\left(|D^2 y|^2 - |\Delta y|^2\right) + \frac{2}{3}\int_\Gamma h(y) II(\nabla y),$$

where $II(\cdot)$ denotes the second fundamental form of Γ.

The key estimate (singular diffusion)

Recall that

$$w = -\Delta v + \frac{j(v)}{2}|\nabla v|^2 + v - \lambda u + \varepsilon u_t \qquad \text{(2v)}$$

- Compute (2v) $\times - \mathrm{div}(m(v)\nabla v)$.
- Applying the Lemma, we get, on the left hand side,

$$\frac{1}{3}\int_\Omega \Big(- m''(v) - m'(v)j(v) + \frac{m(v)j'(v)}{2}\Big)|\nabla v|^4$$
$$+ \frac{2}{3}\int_\Gamma \Big(m(v) - \frac{\widehat{mj}(v)}{2} + K\Big) II(\nabla v),$$

- together with several lower order terms.

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Choose now:

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