(Non)local phase transitions and minimal perimeter interfaces

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ADMAT 2012

PDEs for multiphase advanced materials

Outline of the talk

The fractional Laplacian operator

Back to the Laplacian case
- The Allen-Cahn equation
- A conjecture of De Giorgi

Some research lines for $(-\Delta)^s$
- The symmetry problem for $(-\Delta)^s$
- Density estimates for the level sets
- Γ-convergence for $(-\Delta)^s$
- Few words on the nonlocal perimeter

Uniform regularity of s-minimal surfaces
- Further research needed

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$$(-\Delta)^s u = \mathcal{F}^{-1}\big(|\xi|^{2s}(\mathcal{F}u)\big),$$

where $s \in (0, 1)$ and $\mathcal{F}$ is the Fourier transform.
This definition is consistent with the case $s = 1$:

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An equivalent definition may be given by integrating against a singular kernel, which suitably averages a second-order incremental quotient:

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy.$$

Up to a factor 2, this is the same as defining the operator as an integral in the principal value sense

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Motivation: the fractional Laplacian naturally surfaces in probability, water waves, and lower dimensional obstacle problems (among others). In statistical mechanics it is a way to take into account long-range particle interactions.

Difficulty: The operator is nonlocal, hence one needs to estimate also the contribution coming from far. Also, integrating is usually harder than differentiating.

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Goal: Understand the geometric properties of the solutions of

$$(-\Delta)^s u = u - u^3.$$

When $s = 1$, the equation

$$-\Delta u = u - u^3$$

is named after Allen-Cahn (or Ginzburg-Landau, or Modica-Mortola...) and it is a model for phase transitions.
The pure phases correspond to $u \sim +1$ and $u \sim -1$.
The set in which $u \sim 0$ is the interface which separates the pure phases.

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Let u be a smooth bounded solution of

$$-\Delta u = u - u^3$$

in $\mathbb{R}^n$, with

$$\partial_{x_n} u > 0.$$

Is it true that u depends only on one Euclidean variable?

I.e. $\exists u_o : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^{n-1}$ such that $u(x) = u_o(\omega \cdot x)$?

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The answer is YES when $n \leq 3$ and NO when $n \geq 9$.

The answer is also YES when $n \leq 8$ and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

The answer is also YES for any n if

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Let $s \in (0,1)$ and u be a smooth bounded solution of

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In this case, Cabré and Solà-Morales proved that the answer is YES when $n = 2$ and $s = 1/2$.

Also YES when $n = 2$ and any $s \in (0, 1)$ (Cabré, Sire and V.) and when $n = 3$ and $s \in [1/2, 1)$ (Cabré and Cinti).

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For the fractional Laplacian, here is a density estimate (Savin and V.):

If u_ϵ minimizes $\mathcal{F}_\epsilon$ in B_r and $u(0) = 0$ then

$$|\{u_\epsilon > 1/2\} \cap B_r| \geq c\, r^n$$

provided that $\epsilon \leq cr$.

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The scaling of the energy functional $\mathcal{F}_\epsilon$ is chosen to satisfy the following asymptotics:

If $s \in [1/2, 1)$, the functional $\mathcal{F}_\epsilon$ Γ-converges to the perimeter functional.

If $s \in (0, 1/2)$, the functional $\mathcal{F}_\epsilon$ Γ-converges to the nonlocal perimeter functional (as introduced by Caffarelli, Roquejoffre and Savin).

A minimizer u_ϵ converges a.e. to a step function $\chi_E - \chi_{\mathbb{R}^n \setminus E}$, and the level sets of u_ϵ converge to ∂E locally uniformly.

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For Γ-convergence of nonlocal functionals related with phase transitions, see also Alberti, Bellettini, Bouchitté, Garroni, González, Seppecher, etc.

For any $s \in (0, 1/2)$ the s-perimeter of a set E inside a given domain Ω is defined by

$$\begin{aligned} \mathrm{Per}_s(E,\Omega) \quad &:= \int_{E\cap\Omega}\int_{(\mathcal{C}E)\cap\Omega} \frac{1}{|x-y|^{n+2s}}\,dy\,dx \\ &+ \int_{E\cap\Omega}\int_{(\mathcal{C}E)\cap(\mathcal{C}\Omega)} \frac{1}{|x-y|^{n+2s}}\,dy\,dx \\ &+ \int_{E\cap(\mathcal{C}\Omega)}\int_{(\mathcal{C}E)\cap\Omega} \frac{1}{|x-y|^{n+2s}}\,dy\,dx, \end{aligned}$$

where $\mathcal{C}$ means the complement (see Caffarelli, Roquejoffre and Savin).

Then (see Caffarelli and V.), if E is a smooth set,

$$\lim_{s\to(1/2)^-} s(1-2s)\mathrm{Per}_s(E,B_r) = \mathrm{Per}(E,B_r)$$

for a dense set of r's.

Also, if E_k are minimal for Per_{s_k} and $s_k \to (1/2)^-$, then E_k converges to some set E which is minimal for Per.

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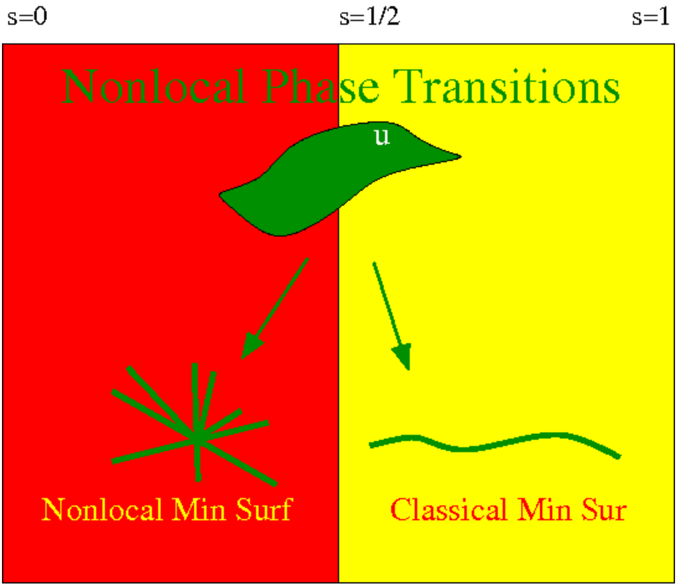
s=0
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Nonlocal Phase Transitions
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Nonlocal Min Surf
Classical Min Sur

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$$\partial E \cap B_1 \subseteq \{|x_n| \le \varepsilon_\star\}$$

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Differently from Caffarelli, Roquejoffre and Savin, here $\varepsilon_\star$ does not depend on s.

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There exists $\varepsilon \in (0, 1/2)$ such that, if

$$\frac{1}{2} - \varepsilon < s < \frac{1}{2},$$

then we have:

- if $n \leq 7$, any s-minimal cone is a hyperplane and any s-minimal surface is $C^{1,\alpha}$;
- in any dimension, any s-minimal surface is $C^{1,\alpha}$ possibly outside a closed set Σ, with $\mathcal{H}^d(\Sigma) = 0$ for any $d > n - 8$.

Open problem: find ε! (no estimate available!)

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Open problem: find ε! (no estimate available!)

The only full-regularity result available is in dimension 2, where $\varepsilon = 1/2$:

(Savin and V.) Let $s \in (0, 1/2)$. If $n = 2$ any s-minimal cone is a straight line and any s-minimal surface is $C^{1,\alpha}$.

Further regularity:

(Barrios Barrera, Figalli and V.) $C^{1,\alpha} \Longrightarrow C^{\infty}$.

It would be desirable to better understand the behavior of nonlocal minimal perimeter sets and to exploit their rigid (?) geometry in order to obtain information on the level sets of u...