Global strong solutions of the full Navier-Stokes and Q-tensor system for nematic liquid crystal flows in two dimensions

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Liquid crystals

- Fourth state of matter besides gas, liquid and solid
- Intermediate state between crystalline and isotropic
- The molecules do not possess a positional order (even partial) but display an orientational order
- Different liquid crystal phases, depending on the amount of order in the material
- nematic *(thread)*, smectic *(soap)*, cholesteric *(bile - solid)*
Nematic liquid crystals

- Simplest liquid crystal phase, *close* to the liquid one
- The molecules float around as in a liquid phase, but have the tendency to align along a preferred direction due to their orientation
The state variables $u$ and $Q$

- Beris & Edward model (1994)
- $u$ velocity field of the flow
- $Q$ symmetric traceless $d \times d$ tensor describing the local orientation and degree of ordering of the molecules ($d = 2, 3$ spatial dimension)

- coupled PDE system describing the interaction between the fluid velocity and the alignment of liquid crystal molecules

- incompressible Navier-Stokes equations for $u$, with highly nonlinear anisotropic force terms

- nonlinear convection-diffusion parabolic equation for $Q$
The Navier-Stokes and $Q$-tensor system

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= \lambda \nabla \cdot (\tau + \sigma) \\
\text{div } u &= 0 \\
\partial_t Q + u \cdot \nabla Q - S(\nabla u, Q) &= \Gamma H(Q)
\end{aligned}
\]

$u(x, t) : \mathbb{T}^d \times (0, \infty) \rightarrow \mathbb{R}^d$  
$Q(x, t) : \mathbb{T}^d \times (0, \infty) \rightarrow S_{0}^{(d)}$

$\mathbb{T}^d$ periodic box with period $a_i = 1$ in the $i$th direction

$S_{0}^{(d)} = \{ Q \in \mathbb{R}^{d \times d} | Q_{ij} = Q_{ji}, \text{tr}(Q) = 0, i, j = 1, \ldots, d \}$

$\tau, \sigma$ symmetric and skew-symmetric parts of the stress tensor

$P$ pressure

$\nu > 0$ viscosity

$\lambda$ competition between kinetic and elastic potential energy

$\Gamma$ Deborah number
The tensor $H$ and the matrix $S$

$H$ variational derivative of the free energy $\mathcal{F}(Q)$ w.r.t. $Q$

$S$ describes the rotating and stretching effects on $Q$

\[
H(Q) = L\Delta Q - aQ + b\left(Q^2 - \frac{1}{d}\text{tr}(Q^2)I\right) - cQ\text{tr}(Q^2)
\]

\[
S(\nabla u, Q) = (\xi D + \Omega)\left(Q + \frac{1}{d}I\right) + \left(Q + \frac{1}{d}I\right)(\xi D - \Omega)
\]

\[
-2\xi\left(Q + \frac{1}{d}I\right)\text{tr}(Q\nabla u)
\]

$I$ identity matrix

$D$, $\Omega$ symmetric and skew-symmetric part of the strain tensor

\[
D = \frac{\nabla u + \nabla^T u}{2} \quad \Omega = \frac{\nabla u - \nabla^T u}{2}
\]
The stress tensors $\tau$ and $\sigma$

\[
\tau = \xi \left( Q + \frac{1}{d} \mathbb{I} \right) H(Q) - \xi H(Q) \left( Q + \frac{1}{d} \mathbb{I} \right) + 2\xi \left( Q + \frac{1}{d} \mathbb{I} \right) \text{tr}(QH(Q)) - L \nabla Q \circ \nabla Q
\]

where

\[
(\nabla Q \circ \nabla Q)_{ij} = \sum_{k,l=1}^{d} \nabla_i Q_{kl} \nabla_j Q_{kl}
\]

\[
\sigma = QH(Q) - H(Q)Q
\]
Further features

\( \xi \in \mathbb{R} \) depends on the molecular shapes of the liquid crystal and measures the ratio between the tumbling and the aligning effect that a shear flow exerts on the liquid crystal directors

\( L > 0 \) elastic constant

\( a, b, c \in \mathbb{R} \) material and temperature dependent coefficients

we assume \( c > 0 \) \( \Rightarrow \) free energy \( \mathcal{F}(Q) \) is bounded from below

\[
\mathcal{F}(Q) = \int_{\mathbb{T}^d} \left( \frac{L}{2} |\nabla Q|^2 + f_b(Q) \right)
\]

\[
f_b(Q) = \frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2) \quad \text{bulk functional}
\]

\[
\frac{L}{2} |\nabla Q|^2 \quad \text{elastic functional}
\]
Initial and boundary conditions

- periodic boundary conditions

\[ u(x + e_i, t) = u(x, t) \quad \text{for} \ (x, t) \in \mathbb{T}^d \times \mathbb{R}^+ \]

\[ Q(x + e_i, t) = Q(x, t) \quad \text{for} \ (x, t) \in \mathbb{T}^d \times \mathbb{R}^+ \]

where \( \{ e_i \}_{i=1}^d \) is the canonical orthonormal basis of \( \mathbb{R}^d \)

- spatially 1-periodic initial data

\[ u(x, 0) = u_0(x), \ x \in \mathbb{T}^d, \ \text{with} \ \nabla \cdot u_0 = 0 \]

\[ Q(x, 0) = Q_0(x), \ x \in \mathbb{T}^d, \ \text{with} \ Q_0 \in S_0^{(d)} \]

Remark

The system preserves for all time both the symmetry and tracelessness of any solution \( Q \) associated to an initial datum with the same properties.
Literature: $\xi = 0$

- Paicu & Zarnescu (2012)
  - $\exists$ of global weak solutions in 2D and 3D
  - Higher order global regularity in 2D
  - Weak-strong uniqueness in 2D

- Dai, Feireisl, Rocca, Schimperna & Shonbek (2014)
  - Asymptotic behavior in 3D in particular cases

- Abels, Dolzmann & Liu (2015)

  - $\exists$ of global weak solutions in 2D and 3D
  - $\exists!$ of local strong solutions in 2D and 3D
  - Regularity results in 2D and 3D
Paicu & Zarnescu (2011)
- ∃ of global weak solutions in 2D and 3D for small $|\xi|$.

De Anna & Zarnescu (2015)
- uniqueness of weak solutions in 2D.

Abels, Dolzmann & Liu (2014)
- local in time well-posedness and ∃ of global weak solutions in 2D and 3D with nonhomogeneous Dirichlet/Neumann boundary conditions.
Ericksen (1976) and Leslie (1978) developed the hydrodynamics theory of liquid crystals based on the evolution of the \textit{velocity field} $u$ and the \textit{director field} $d$

- rigorous derivation of the classical Ericksen-Leslie system from the Beris-Edwards system

- Wang, Zhang & Zhang (2013)
- Wu, Xu & Liu (2013)
- Cavaterra & Rocca (2013)
- Huang, Lin & Wang (2014)
- Cavaterra, Rocca & Wu (2014)
(E-L) system with Leslie coefficients

Existence of suitably defined weak solutions in 3D without any restriction on the size of the fluid viscosity and of the initial data

Main difficulty: the lack of control of $\|d\|_{L^\infty}$ (no maximum principle for $d$)
Navier-Stokes and $Q$-tensor system + (PBC) +(IC)

\[
\begin{aligned}
\begin{cases}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= \lambda \nabla \cdot (\tau + \sigma), \quad \mathbb{T}^d \times \mathbb{R}^+ \\
\text{div } u &= 0, \quad \mathbb{T}^d \times \mathbb{R}^+ \\
\partial_t Q + u \cdot \nabla Q - S(\nabla u, Q) &= \Gamma H(Q), \quad \mathbb{T}^d \times \mathbb{R}^+ \\
u(x + e_i, t) &= u(x, t), \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}^+ \\
Q(x + e_i, t) &= Q(x, t), \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}^+ \\
u(x, 0) &= u_0(x) \text{ with } \nabla \cdot u_0(x) = 0, \quad x \in \mathbb{T}^d \\
Q(x, 0) &= Q_0(x) \in S_0^{(d)}, \quad x \in \mathbb{T}^d 
\end{cases}
\end{aligned}
\]
Our main results on problem (NSQ)

- $\exists !$ of a global strong solution $(u, Q)$ with periodic boundary conditions in 2D for arbitrary $\xi$

- convergence of the unique global strong solution to a single steady state

- For $\xi \neq 0$ the $Q$-equation in (NSQ) does not satisfy a maximum principle

- The highly nonlinear terms are very difficult to handle
Functional spaces

\[ L^p(\mathbb{T}^d) = L^p(\mathbb{T}^d, M), \quad M = \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \]
\[ W^{m,p}(\mathbb{T}^d) = W^{m,p}(\mathbb{T}^d, M), \quad M = \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \]
\[ H^m(\mathbb{T}^d) = W^{m,2}(\mathbb{T}^d) \]
\[ H = \left\{ u \in L^2(\mathbb{T}^d), \nabla \cdot u = 0, \int_{\mathbb{T}^d} u \, dx = 0 \right\} \]
\[ V = \left\{ u \in H^1(\mathbb{T}^d), \nabla \cdot u = 0, \int_{\mathbb{T}^d} u \, dx = 0 \right\} \]

\[ L^2(\mathbb{T}^d, S_0^{(d)}) = \left\{ Q : \mathbb{T}^d \to S_0^{(d)}, \int_{\mathbb{T}^d} |Q|^2 < \infty \right\} \]
\[ H^1(\mathbb{T}^d, S_0^{(d)}) = \left\{ Q : \mathbb{T}^d \to S_0^{(d)}, \int_{\mathbb{T}^d} |\nabla Q|^2 + |Q|^2 < \infty \right\} \]
\[ H^2(\mathbb{T}^d, S_0^{(d)}) = \left\{ Q : \mathbb{T}^d \to S_0^{(d)}, \int_{\mathbb{T}^d} |\Delta Q|^2 + |\nabla Q|^2 + |Q|^2 < \infty \right\} \]
\[ |Q| = \sqrt{\text{tr}(Q^2)} = \sqrt{Q_{ij}Q_{ij}} \]
\[ |\nabla Q|^2 = \nabla_k Q_{ij} \nabla_k Q_{ij} \quad |\Delta Q|^2 = \Delta Q_{ij} \Delta Q_{ij} \]
Preliminary results

\[ \mathcal{E}(t) = \lambda \mathcal{F}(Q(t)) + \frac{1}{2} \int_{\mathbb{T}^d} |u|^2, \quad \text{total energy} \]

\[ \mathcal{E}(t) = \text{sum of free energy and kinetic energy for } u \]

\( c > 0 \Rightarrow \mathcal{E}(t) \text{ is bounded from below} \)

Lemma

Let \((u, Q)\) be a smooth solution to problem (NSQ).

Then, for \(d = 2, 3\), we have

\[ \frac{d}{dt} \mathcal{E}(t) = -\nu \int_{\mathbb{T}^d} |\nabla u|^2 - \lambda \Gamma \int_{\mathbb{T}^d} |H(Q)|^2 \leq 0, \quad \forall \ t > 0 \]

- energy dissipation of the liquid crystal flow
- \( \mathcal{E} \) is a Lyapunov functional for (NSQ)
- \( \mathcal{E}(t) + \int \int_{(0,T) \times \mathbb{T}^d} (\nu |\nabla u|^2 + \lambda \Gamma |H(Q)|^2) = \mathcal{E}(0) \)
Lemma

Assume \((u_0, Q_0) \in H \times H^1(\mathbb{T}^d, S_0^{(d)})\).
Let \((u, Q)\) be a smooth solution to the problem (NSQ).
Then, for \(d = 2, 3\), we have

\[
\|u(t)\|_{L^2} + \|Q(t)\|_{H^1} \leq C, \quad \forall \ t > 0
\]

where \(C > 0\) depends on \(\|u_0\|_{L^2}, \|Q_0\|_{H^1}, L, \lambda, a, b, c\).
Moreover, it holds

\[
\int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 + |\Delta Q|^2 < C_T, \quad \forall \ T > 0
\]

where \(C_T > 0\) may further depend on \(\nu, \Gamma\) and \(T\).
Preliminary results

Existence of global weak solutions for $d = 2, 3$
(Abels, Dolzmann & Liu (2014))

**Lemma**

Let $d = 2, 3$ and $\xi \in \mathbb{R}$.
Assume $(u_0, Q_0) \in H \times H^1(\mathbb{T}^d, S_0^{(d)})$.

Then problem (NSQ) admits at least one global-in-time weak solution $(u, Q)$ such that

\[ u \in L^\infty(0, T; H) \cap L^2(0, T; V) \]
\[ Q \in L^\infty(0, T; H^1(\mathbb{T}^d, S_0^{(d)})) \cap L^2(0, T; H^2(\mathbb{T}^d, S_0^{(d)})) \]

Moreover the following energy inequality holds:

\[ \mathcal{E}(t) + \int_0^t \int_{\mathbb{T}^d} \nu |\nabla u|^2 + \lambda \Gamma |H(Q)|^2 \leq \mathcal{E}(0), \quad \text{a.e. } t \in (0, T) \]
The case $d = 2$

- $\text{tr}(Q^3) = 0$
- $f_B(Q) = \frac{a}{2}\text{tr}(Q^2) + \frac{c}{4}\text{tr}^2(Q^2)$
- $H(Q) = L\Delta Q - aQ - cQ\text{tr}(Q^2)$

**Definition**

Assume $d = 2$ and $(u_0, Q_0) \in V \times H^2(\mathbb{T}^2, S_0^{(2)})$.

$(u, Q)$ is called global strong solution to problem (NSQ) if

- $u \in C([0, +\infty); V) \cap L^2_{loc}(0, +\infty; H^2(\mathbb{T}^2, \mathbb{R}^2))$
- $Q \in C([0, +\infty); H^2(\mathbb{T}^2, S_0^{(2)}) \cap L^2_{loc}(0, +\infty; H^3(\mathbb{T}^2, S_0^{(2)}))$
- the equations for $u$ and $Q$ in (NSQ) are satisfied a.e.
Main result: \( \exists ! \) of global strong solutions

**Theorem**

Assume \( d = 2 \) and \( \xi \in \mathbb{R} \).

Then, for any \((u_0, Q_0) \in \mathbf{V} \times H^2(\mathbb{T}^2, S_0^{(2)})\), problem (NSQ) admits a unique global strong solution \((u, Q)\) such that

\[
\|u(t)\|_{H^1} + \|Q(t)\|_{H^2} \leq C, \quad \forall \, t \geq 0
\]

where \( C \) is a positive constant depending on \( \|u_0\|_{H^1}, \|Q_0\|_{H^2}, \nu, \Gamma, L, \lambda, a, c \) and \( \xi \), but it is independent of \( t \).
Sketch of the proof

- semi-Galerkin approximation scheme $\Rightarrow$ problem $(NSQ)_N$
- existence of approximate solutions $(u^N, Q^N)$ to $(NSQ)_N$
- lower order estimates for $(u^N, Q^N)$, independent of $N$ and $t$
- higher order estimates for $(u^N, Q^N)$, independent of $N$ and $t$
- passage to the limit for $N \to \infty$
- existence of solutions for $(NSQ)$
- uniqueness of solution for $(NSQ)$
Semi-Galerkin approximation scheme

- \( \{ \mathbf{v}_n \}_{n=1}^{\infty} \) eigenvectors of Stokes operator \( S \) with zero mean

\[ S \mathbf{v}_n = \kappa_n \mathbf{v}_n, \quad \nabla \cdot \mathbf{v}_n = 0, \quad \int_{\mathbb{T}^2} \mathbf{v}_n(x) \, dx = 0, \quad \text{in } \mathbb{T}^2 \]

\[ \mathbf{v}_n(x + e_i) = \mathbf{v}_n(x), \quad x \in \mathbb{T}^2 \]

\( 0 < \kappa_1 \leq \kappa_2 \leq \ldots \uparrow +\infty \) corresponding eigenvalues

- \( \{ \mathbf{v}_n \}_{n=1}^{\infty} \) orthogonal basis of \( H \) and \( V \)

- \( V_N = \text{span}\{ \mathbf{v}_n \}_{n=1}^{N} \)

- \( \Pi_N : H \rightarrow V_N \) orthogonal projection operators
Semi-Galerkin approximation scheme

- \( u^N = \sum_{i=1}^{N} h_i(t) v_i(x) \) approximation of velocity, solution to

\[
\int_{\mathbb{T}^2} (u^N)_t \cdot v_k - \int_{\mathbb{T}^2} (u^N \otimes u^N) : \nabla v_k + \nu \int_{\mathbb{T}^2} \nabla u^N : \nabla v_k \]

\[
= - \int_{\mathbb{T}^2} (\sigma^N + \tau^N) : \nabla v_k, \quad \text{in } (0, T), \quad \forall k = 1, \ldots, N
\]

- \( Q^N \) is determined in terms of \( u^N \) and solves uniquely

\[
Q^N_t + u^N \cdot \nabla Q^N - S^N(\nabla u^N, Q^N) = \Gamma H^N(Q^N), \quad \mathbb{T}^2 \times \mathbb{R}^+
\]

- \( \tau^N, \sigma^N, S^N, H^N \) approximation of \( \tau, \sigma, S, H \), where

\[
u \Rightarrow u^N, \quad Q \Rightarrow Q^N
\]

\[
D \Rightarrow D^N = \frac{\nabla u^N + \nabla^T u^N}{2}, \quad \Omega \Rightarrow \Omega^N = \frac{\nabla u^N - \nabla^T u^N}{2}
\]
The approximation problem \((\text{NSQ})_N\)

\[
\begin{align*}
\int_{\mathbb{T}^2} (u^N)_t \cdot v_k - \int_{\mathbb{T}^2} (u^N \otimes u^N) : \nabla v_k + \nu \int_{\mathbb{T}^2} \nabla u^N : \nabla v_k &= - \int_{\mathbb{T}^2} (\sigma^N + \tau^N) : \nabla v_k, \quad \text{in } (0, T), \quad \forall k = 1, \ldots, N \\
Q^N_t + u^N \cdot \nabla Q^N - S^N(\nabla u^N, Q^N) &= \Gamma H^N(Q^N), \quad \mathbb{T}^2 \times \mathbb{R}^+ \\
u^N(x + e_i, t) &= u^N(x, t), \quad \mathbb{T}^2 \times \mathbb{R}^+ \\
Q^N(x + e_i, t) &= Q^N(x, t), \quad \mathbb{T}^2 \times \mathbb{R}^+ \\
u^N|_{t=0} &= \Pi_N u_0, \quad Q^N|_{t=0} = Q_0, \quad \mathbb{T}^2
\end{align*}
\]
Existence of approximate solutions \((u^N, Q^N)\)

**Proposition**

Assume \(u_0 \in V, Q_0 \in H^2(\mathbb{T}^2, S_0^{(2)})\).

For any \(N \in \mathbb{N}\), there exists \(T_N > 0\) depending on \(N\), \(\|u_0\|_{H^1}\) and \(\|Q_0\|_{H^2}\) s.t. problem \((NSQ)_N\) admits a solution \((u^N, Q^N)\) satisfying

\[
\begin{align*}
u^N &\in L^\infty(0, T_N; V) \cap L^2(0, T_N; H^2(\mathbb{T}^2, \mathbb{R}^2)) \cap H^1(0, T_N; H) \\
Q^N &\in L^\infty(0, T_N; H^2(\mathbb{T}^2, S_0^{(2)})) \cap L^2(0, T_N; H^3(\mathbb{T}^2, S_0^{(2)})) \cap H^1(0, T_N; H^1(\mathbb{T}^2, S_0^{(2)}))
\end{align*}
\]
given \( \tilde{u} \in C([0, T]; V_N) \), then there exists a unique \( Q = Q[\tilde{u}] \) solution to the equation for \( Q_N \) in (NSQ)_N with \( u^N \) replaced by \( \tilde{u} \)

inserting \( Q[\tilde{u}] \) in the equation for \( u_N \) in (NSQ)_N, the solution \( u \) to the resulting ODE system defines a mapping \( \mathcal{T} : \tilde{u} \mapsto \mathcal{T}[\tilde{u}] = u \)

\( \mathcal{T} \) admits a fixed point by means of fixed point Schauder’s theorem on \( (0, T_N) \), for some \( T_N > 0 \) depending on \( \|u_0\|_{H^1}, \|Q_0\|_{H^2} \) and \( N \)
\[ \| u^N(t) \|_{L^2} + \| Q^N(t) \|_{H^1} \leq C, \quad \forall t \in [0, T_N) \]

\[ \int_0^t \int_{\mathbb{T}^d} \left( |\nabla u^N(\tau)|^2 + |\Delta Q^N(\tau)|^2 \right) < C(1 + t), \quad \forall t \in [0, T_N) \]

where the constant \( C > 0 \) depends on \( \| u_0 \|_{L^2}, \| Q_0 \|_{H^1}, L, \lambda, \nu, \Gamma, a, c \), but it is independent of \( N \) and \( t \).
Uniform-in-time higher-order estimates

\[ \|u^N(t)\|_{H^1} + \|Q^N(t)\|_{H^2} \leq C, \quad \forall t \in [0, T_N) \]

\[ \int_0^t \int_{\mathbb{T}^2} \left| \Delta u^N(\tau) \right|^2 + \left| \nabla \Delta Q^N(\tau) \right|^2 < C(1 + t), \quad \forall t \in [0, T_N) \]

where the constant \( C > 0 \) depends on \( \|u_0\|_{H^1}, \|Q_0\|_{H^2}, L, \lambda, \nu, \Gamma, a, c \), but it is independent of \( N \) and \( t \)
Extension of \((u^N, Q^N)\) to \([0, T]\), for any \(T > 0\)

- uniform-in-time lower-order and higher-order estimates 
  \[ \Rightarrow (u^N, Q^N) \text{ does not blow up in finite time} \]

- every approximate solution \((u^N, Q^N)\) can be extended to the time interval \([0, T]\), for any \(T > 0\)

- since the uniform estimates are also independent of \(N \Rightarrow \)
  \[ u^N \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\mathbb{T}^2, \mathbb{R}^2)) \cap H^1(0, T; H) \]
  \[ Q^N \in L^\infty(0, T; H^2(\mathbb{T}^2, S_0^{(2)})) \cap L^2(0, T; H^3(\mathbb{T}^2, S_0^{(2)})) \]
  \[ \cap H^1(0, T; H^1(\mathbb{T}^2, S_0^{(2)})) \]
Limit for $N \to \infty$ and existence of solutions for (NSQ)

- uniform-in-time estimates
- standard weak compactness results
- Aubin-Lions compactness Lemma
- $(u^N, Q^N) \to (u, Q)$ up to a subsequence
- $(u, Q)$ is a global strong solution to problem (NSQ)
Let \((u_i, Q_i)\) be two global strong solutions to (NSQ), corresponding to the initial data \((u_{i0}, Q_{i0}), \ i = 1, 2\).

Then we can prove, for any \(\xi \in \mathbb{R}\),

\[
\| (u_1 - u_2)(t) \|_{L^2}^2 + \| (Q_1 - Q_2)(t) \|_{H^1}^2 \\
+ \int_0^t \left( \| \nabla (u_1 - u_2)(\tau) \|_{L^2}^2 + \| \Delta (Q_1 - Q_2)(\tau) \|_{L^2}^2 \right) d\tau \\
\leq C e^{\int_0^t h(\tau) d\tau} \left( \| u_{01} - u_{02} \|_{L^2}^2 + \| Q_{01} - Q_{02} \|_{H^1}^2 \right), \quad \forall \ t \in (0, T)
\]

where \(h \in L^1(0, T)\).

Therefore, the global strong solution to (NSQ) is unique.
Continuous dependence result

- Previous estimate shows that the map $(u_0, Q_0) \rightarrow (u(t), Q(t))$, where $(u, Q)$ is the global strong solution to (NSQ), is continuous from $V \times H^2 \rightarrow H \times H^1$.

- The continuous dependence from $V \times H^2 \rightarrow V \times H^2$ is an open problem.

**Remark**

*In the paper De Anna & Zarnescu the authors prove uniqueness of weak solutions by means of a continuous dependence estimate showing a similar gap in the functional spaces.*
Theorem

Assume \( d = 2 \) and \( \xi \in \mathbb{R} \).

For any \( (u_0, Q_0) \in V \times H^2(T^2, S_0^{(2)}) \), the unique global strong solution to problem (NSQ) converges to a single steady state \((0, Q_\infty)\) as time tends to infinity.

More precisely, it holds

\[
\lim_{t \to +\infty} \left( \|u(t)\|_{H^1} + \|Q(t) - Q_\infty\|_{H^2} \right) = 0
\]

where \( Q_\infty \in S_0^{(2)} \) and solves the elliptic problem in \( T^2 \)

\[
\begin{align*}
L \nabla Q_\infty - aQ_\infty - c \text{tr}(Q_\infty^2)Q_\infty &= 0, & T^2 \\
Q_\infty(x + e_i) &= Q_\infty(x), & x \in T^2
\end{align*}
\]
Proof: main steps

- decay property of the solution
- characterization of the $\omega$-limit set
- convergence to equilibrium
Decay property

**Lemma**

Assume \((u_0, Q_0) \in V \times H^2(T^2, S_0^{(2)})\) and \(\xi \in \mathbb{R}\).

Then the unique global strong solution \((u, Q)\) to (NSQ) has the decay property

\[
\lim_{t \to +\infty} \left( \|u(t)\|_{H^1} + \|H(Q(t))\|_{L^2} \right) = 0
\]

so that

\[u(t) \to 0 \quad \text{in } H^1\]
Lemma

Assume \((u_0, Q_0) \in V \times H^2(\mathbb{T}^2, S_0^{(2)})\) and \(\xi \in \mathbb{R}\).

Then the \(\omega\)-limit set \(\omega(u_0, Q_0)\)

\[\omega(u_0, Q_0) = \{(u_\infty, Q_\infty) | \exists \{t_n\} \nearrow \infty : u(t_n) \to u_\infty \text{ in } L^2, \quad Q(t_n) \to Q_\infty \text{ in } H^1 \text{ as } n \to \infty}\]

is a non-empty bounded subset in \(V \times H^2(\mathbb{T}^2, S_0^{(2)})\), satisfying

\[\omega(u_0, Q_0) \subset \{(0, Q_\ast) : Q_\ast \in \mathcal{G}\}, \text{ where}\]

\[\mathcal{G} = \{Q_\ast : L\Delta Q_\ast - aQ_\ast - c \text{tr}(Q_\ast^2)Q_\ast = 0, \quad Q_\ast \in S_0^{(2)} \quad \text{and } Q_\ast(x + e_i) = Q_\ast(x) \text{ in } \mathbb{T}^2\}\]
Convergence to equilibrium

- Generalization of Simon’s results in the scalar case to the matrix case
- Gradient inequality of Łojasiewicz-Simon type for the matrix valued functions
- Due to the special structure of the $Q$-tensor in 2D, the problem reduces to the vector case

**Lemma**

Let $Q_* \in H^1(T^2, S_0^{(2)})$ be a critical point of free energy $F(Q)$.

Then there exist $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on $Q_*$ s.t., for any $Q \in H^1(T^2, S_0^{(2)})$ satisfying $\|Q - Q_*\|_{H^1} < \beta$, we have

$$\|L \Delta Q - a Q - c \text{tr}(Q^2) Q\|_{(H^1)'} \geq |F(Q) - F(Q_*)|^{1-\theta}$$

Here, $(H^1(T^2, S_0^{(2)}))'$ is the dual space of $H^1(T^2, S_0^{(2)})$. 
Main result: convergence rate to the asymptotic limit

Theorem

Assume $d = 2$, $\xi \in \mathbb{R}$ and $(u_0, Q_0) \in V \times H^2(T^2, \mathcal{S}_0^{(2)})$.

Let us consider the global strong solution $(u, Q)$ to problem (NSQ).

Then the following estimate holds

$$\|u(t)\|_{H^1} + \|Q(t) - Q_\infty\|_{H^2} \leq C(1 + t)^{-\frac{\theta}{1-2\theta}}, \quad \forall \ t \geq 0$$

where $C > 0$ is a constant depending on $\nu$, $\Gamma$, $L$, $\lambda$, $a$, $c$, $\xi$, $\|u_0\|_{H^1}$, $\|Q_0\|_{H^2}$, $\|Q_\infty\|_{H^2}$.

Here the constant $\theta \in (0, \frac{1}{2})$ depends on $Q_\infty$. 
Open issues

- generalization to the case of no-slip boundary condition for \( u \) and nonhomogeneous Dirichlet boundary data for \( Q \)

- regularization in finite time of weak solutions in the case of periodic boundary conditions or more general boundary conditions
THANKS FOR YOUR ATTENTION!