Optimal Control of the Fokker-Planck equation

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Motivation

Consider an Optimal Control Problem (OCP)

$$\min_u \tilde{J}(X, u)$$

constrained to a Itô Stochastic Differential Equation (SDE)

$$dX_t = b(X_t, t; u)dt + \sigma(X_t, t)dW_t, \quad X(0) = x_0$$
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where
- \( t \in [0, T_E] \) for a fixed terminal time \( T_E > 0 \) and
- \( X_t \) is a random variable representing the state of the SDE
- The control \( u \) acts through the drift term \( b \)
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- $X_t$ is a random variable representing the state of the SDE
- The control $u$ acts through the drift term $b$

$X_t$ random $\Rightarrow$ The cost functional $\tilde{J}(X, u)$ is a random variable
Choice of the cost functional

**Standard approach**: consider the averaged objective

$$\min_u \mathbb{E}[\tilde{J}(X, u)] = \min_u \mathbb{E} \left[ \int_0^{T_E} L(t, X_t, u(t)) \, dt + \psi(X_{T_E}) \right]$$
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**Alternative approach**: express the objective in terms of the Probability Density Function (PDF) associated with the state $X_t$, which characterize the shape of its statistical distribution.

Related works: deterministic objectives defined by

- the Kullback-Leibler distance (G. Jumarie 1992, M. Kárnya 1996)

However, stochastic models are needed to obtain the PDF by averaging or by interpolation.

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Related works: deterministic objectives defined by

- the Kullback-Leibler distance (G. Jumarie 1992, M. Kárný 1996) or

between the state PDF and a desired one.

However, stochastic models are needed to obtain the PDF by averaging or by interpolation.
New approach pursued by Annunziato and Borzì (2010, 2013): Reformulate the objective using the underlying PDF

\[ y(x, t) := \int_\Omega \tilde{y}(x, t; z, 0)\rho(z, 0)\,dz \]

\( t > 0, \rho(z, 0) \) given initial density probability, \( \tilde{y} \) transition density probability distribution function

\[ \tilde{y}(x, t; z, s) := \mathbb{P}\{X(t) \in (x, x + dx) : X(s) = z\}, \quad t > s \]

and control the PDF directly.

The next essential step: the PDF evolves according to the Fokker-Planck Equation (or forward Kolmogorov Equation)
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The Fokker-Planck Equation

\[
\begin{align*}
\frac{\partial}{\partial t} y(x, t) - \frac{1}{2} \frac{\partial^2}{\partial x^2} \left( \sigma(x, t)^2 y(x, t) \right) + \frac{\partial}{\partial x} \left( b(x, t; u) y(x, t) \right) &= 0 \\
y(\cdot, 0) &= y_0
\end{align*}
\]
The Fokker-Planck Equation

Fokker-Planck Equation

\[
\begin{cases}
    \partial_t y(x, t) - \frac{1}{2} \partial_{xx}^2 \left( \sigma(x, t)^2 y(x, t) \right) + \partial_x \left( b(x, t; u) y(x, t) \right) = 0 \\
    y(\cdot, 0) = y_0
\end{cases}
\]

where \( y : \mathbb{R} \times [0, \infty[ \rightarrow \mathbb{R}_{\geq 0} \) is the PDF constrained to \( \int_{\mathbb{R}} y(x, t) \, dx = 1 \quad \forall t > 0 \),

\( y_0 : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \) is the initial PDF \( (\int_{\mathbb{R}} y_0(x) \, dx = 1) \),

\( \sigma : \mathbb{R} \times [0, \infty[ \rightarrow \mathbb{R} \) and \( b : \mathbb{R} \times [0, \infty[ \times \mathbb{R} \rightarrow \mathbb{R} \)

are given by the SDE
### A Fokker-Planck control framework

#### Deterministic PDE-constrained Optimal Control Problem

\[
\begin{align*}
\min_u \mathbb{E}[\tilde{J}(X, u)] & \quad \leadsto \quad \min_u J(y, u) \\
\text{s.t. Itô SDE} & \quad \leadsto \quad \text{s.t. Fokker-Planck PDE} \\
\quad dX_t = b(u)dt + \sigma dW_t & \quad y_t - \frac{1}{2}(\sigma^2 y)_{xx} + (b(u)y)_x = 0
\end{align*}
\]
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dX_t = b(u)dt + \sigma dW_t,
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and
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y_t - \frac{1}{2}(\sigma^2 y)_{xx} + (b(u)y)_x = 0
\]

1) the class of objectives described by \(\min_u J(y, u)\) is larger than that expressed by \(\min_u \mathbb{E}[\tilde{J}(X, u)]\), indeed

\[
\mathbb{E} \left[ \int_0^{T_E} L(t, X_t, u(t)) + \psi(X_{T_E}) \right] = \int \int \int L(t, x, u) y(t, x) + \int \psi(x) y(T_E, x)
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\]

2) Bilinear control through the coefficient of the divergence term
In order to build a real-time sub-optimal control feedback law we apply a Model Predictive Control scheme, also known as Receding Horizon scheme, to the Fokker-Planck equation.

**Basic idea of MPC:**

- OCP on a long (possibly infinite) time horizon
- Several iterative OCPs on (shorter) finite time horizons
Related References

- FP-MPC approach: Annunziato and Borzì 2010, 2013, ...
- Existence of optimal controls for bilinear controls: Addou and Benbrik 2002, for a control $u = u(t)$
- Controllability of the FPE: Blaquiè re 1992, Porretta 2014
- Connection with mean field type control: Bensoussan, Frehse, Yam 2013
Existing Work by Annunziato and Borzì, 2010, 2013

Track a desired PDF over a given time interval.

Optimal Control Problem

\(\Omega \subset \mathbb{R}\) open, \(u_a, u_b \in \mathbb{R}\) with \(u_a < u_b\), \(y_d \in L^2(\Omega)\) and \(\lambda > 0\).

Consider the following OCP on \([0, T]\):

\[
\min_u J(y, u) := \frac{1}{2} \|y(\cdot, T) - y_d(\cdot, T)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} |u|^2
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s.t.

\[
\begin{aligned}
\partial_t y - \frac{1}{2} \partial_{xx} (\sigma^2 y) + \partial_x (b(u)y) &= 0 & \text{in } \Omega \times (0, T) \\
y(\cdot, 0) &= y_n & \text{in } \Omega \\
y &= 0 & \text{in } \partial \Omega \times (0, T)
\end{aligned}
\]

\(u \in U_{ad} := \{u \in \mathbb{R} | u_a \leq u \leq u_b\}\)

\(b(x; u) := \gamma(x) + u\) with \(\gamma \in C^1(\Omega)\)
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Collaboration with Arthur Fleig and Lars Grüne, Univ. Bayreuth

Consider a control function $u(x, t)$

**Proposition (Aronson)**

Let the following assumptions hold:

- $\sigma \in C^1(\Omega)$ such that
  \[ \sigma(x, t) \geq \theta \quad \forall (x, t) \in Q, \text{ for some constant } \theta > 0 \]

- $b \in L^q(0, T_E; L^p(\Omega))$ with $2 < p, q \leq \infty$ and $\frac{d}{2p} + \frac{1}{q} < \frac{1}{2}$.

- $y_0 \in L^2(\Omega)$ is nonnegative and bounded.

Then there exists a unique *nonnegative* weak solution $y$ to the Fokker-Planck IBVP (1)
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Towards existence of Optimal Solutions

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $H := L^2(\Omega)$, $V := H^1_0(\Omega)$ and $V'$ dual of $V$. The Fokker-Planck equation $\mathcal{E}(y_0, u, f)$ can be rewritten as

$$\begin{cases}
\dot{y}(t) + Ay(t) + \text{div}(b(t, u(t))y(t)) = f(t) & \text{in } V', \ t \in (0, T) \\
y(0) = y_0,
\end{cases}$$

where $y_0 \in H$, $A : V \rightarrow V'$ linear and continuous, $f \in L^2(0, T; V')$, $b : \mathbb{R}^{d+1} \times \mathcal{U} \rightarrow \mathbb{R}^d$, $(x, t; u) \mapsto b(x, t; u(x, t))$ satisfies

$$\sum_{i=1}^d |b_i(x, t; u)|^2 \leq M(1 + |u(x, t)|^2) \quad \forall x \in \Omega, \ \forall t \in [0, T],$$

and $\forall u \in \mathcal{U} := L^q(0, T; L^\infty(\Omega; \mathbb{R}^d))$, for some $q > 2$.

Rmk: optimization problem in a Banach space
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**Rmk:** optimization problem in a Banach space
A-priori estimates

Let $y_0 \in H$, $f \in L^2(0, T; V')$ and $u \in U$. Then a solution $y$ of the Fokker-Planck equation satisfies the estimates

\[
|y|_{L^\infty(0, T; H)}^2 \leq C e^c |u|_U^2 \left( |y(0)|_H^2 + |f|_{L^2(0, T; V')}^2 \right),
\]
\[
|y|_{L^2(0, T; V)}^2 \leq C (1 + |u|_U^2 e^c |u|_U^2) \left( |y(0)|_H^2 + |f|_{L^2(0, T; V')}^2 \right),
\]
\[
|\dot{y}|_{L^2(0, T; V')}^2 \leq C (1 + |u|_U^2 e^c |u|_U^2) \left( |y(0)|_H^2 + |f|_{L^2(0, T; V')}^2 \right),
\]

for some positive constants $c$, $C$. 

Existence of Optimal Controls

Theorem (Fleig - G.)

Let \( y_0 \in V, \ y_d \in H, \ u_a, u_b \in L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^d)) \),
consider \( \min_{u \in \mathcal{U}_{ad}} J(y, u) \), where \( y \) is the unique solution to
\[
\begin{aligned}
\dot{y}(t) + Ay(t) + \text{div}(b(t, u(t)), y(t)) &= 0 \quad \text{in } V', \ t \in (0, T) \\
y(0) &= y_0 ,
\end{aligned}
\]
\( \mathcal{U}_{ad} := \{ u \in \mathcal{U} : u_a(x, t) \leq u(x, t) \leq u_b(x, t) \text{ for a.e. } (x, t) \in Q \} \).
Then there exists a pair
\[
(\bar{y}, \bar{u}) \in C([0, T], H) \times \mathcal{U}_{ad}
\]
such that \( \bar{y} \) is a solution of \( \mathcal{E}(y_0, \bar{u}, 0) \) and \( \bar{u} \) minimizes \( J \) in \( \mathcal{U}_{ad} \).
Necessary Optimality Conditions

Let \( y_d \in L^2(0, T; H) \), \( y_\Omega \in H \), \( \alpha, \beta, \lambda \geq 0 \) with \( \max\{\alpha, \beta\} > 0 \).

\[
J(u) := \frac{\alpha}{2} \|y - y_d\|^2_{L^2(0, T; H)} + \frac{\beta}{2} \|y(T) - y_\Omega\|^2_H + \frac{\lambda}{2} \|u\|^2_{L^2(0, T; H)}.
\]

We derive the first-order necessary optimality system for \( \bar{u}(x, t) \)

\[
\begin{align*}
\partial_t \bar{y} &- \sum_{i,j=1}^d \partial_{ij}^2 (a_{ij} \bar{y}) + \sum_{i=1}^d \partial_i (\bar{u}_i \bar{y}) = 0, \quad \text{in } Q, \\
-\partial_t \bar{p} &- \sum_{i,j=1}^d a_{ij} \partial_{ij}^2 \bar{p} - \sum_{i=1}^d \bar{u}_i \partial_i \bar{p} = \alpha [\bar{y} - y_d], \quad \text{in } Q, \\
\bar{y} & = \bar{p} = 0 \quad \text{on } \Sigma, \\
\bar{y}(0) & = y_0, \quad \bar{p}(T) = \beta [\bar{y}(T) - y_\Omega], \quad \text{in } \Omega, \\
\int\int_Q [\bar{y} \partial_i \bar{p} + \lambda \bar{u}_i] (u_i - \bar{u}_i) \, dx \, dt & \geq 0 \quad \forall u \in \mathcal{U}_{ad}, i = 1, \ldots, d.
\end{align*}
\]
Consider the Ornstein-Uhlenbeck process with
\[
\sigma(x, t) \equiv \bar{\sigma} = 0.8, \quad b(x, t, u) := u - x
\]
on \Omega := ] - 5, 5[ \text{ with } u_a = -10, u_b = 10, \lambda = 0.001, \text{ and } T_E = 5.
Consider the Ornstein-Uhlenbeck process with

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on \( \Omega := ]-5, 5[ \) with \( u_a = -10, u_b = 10, \lambda = 0.001 \), and \( T_E = 5 \).

The target and initial PDF are given by

\[ y_d(x, t) := \exp \left( -\frac{[x-2\sin(\pi t/5)]^2}{2\cdot0.2^2} \right) \]

\[ \sqrt{2\pi \cdot 0.2^2} \]

and

\[ y_0(x) := y_d(x, 0) = \exp \left( -\frac{x^2}{2\cdot0.2^2} \right) \]

\[ \sqrt{2\pi \cdot 0.2^2} \],

respectively.
Ornstein–Uhlenbeck

$T = 0.5, N = 1, \lambda = 0.001, \alpha = 1, t = 0.$
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With space-dependent control, larger class of objectives possible:

- region avoidance, without prescribing the shape of the PDF, e.g. try to force the state PDF into $[0, 0.5]$.

- Try to track non-smooth targets, e.g.

$$y_d(x, t) := \begin{cases} 0.5 & \text{if } x \in [-1 + 0.15t, 1 + 0.15t] \\ 0 & \text{otherwise.} \end{cases}$$
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Consider the two-dimensional stochastic process, modeling the dispersion of substance in shallow water [Heemink, 1990]

\[
\sigma(x, t) := \begin{pmatrix} \sqrt{2D(x)} & 0 \\ 0 & \sqrt{2D(x)} \end{pmatrix}
\]

with \( D(x) := -\frac{1}{64} ((x_1 - 4)^2 + (x_2 - 4)^2) + \frac{3}{5} > 0 \) in \( \Omega \) and

\[
b(x, t; u) := \begin{pmatrix} u_1 + \frac{\partial D}{\partial x_1} - \frac{1}{10} \\ u_2 + \frac{\partial D}{\partial x_2} - \frac{1}{10} \end{pmatrix} = \begin{pmatrix} u_1 - \frac{x_1}{32} + \frac{1}{40} \\ u_2 - \frac{x_2}{32} + \frac{1}{40} \end{pmatrix}
\]

on \( Q := \Omega \times [0, 5] \) with \( \Omega := ]0, 8[ \times ]0, 8[ \).
Numerical Example (3)

- **Initial distribution** $y_0$: (smoothed) delta-Dirac located at $(4, 4)$.

- Target PDF is given by

  $$y_d(x_1, x_2) = m \left[ \frac{1}{x_1} \exp \left( \frac{2C_1}{\sigma^2} \log(x_1) \right) - \frac{2}{\sigma^2}(x_1 - 1) \right]$$

  $$\left[ \frac{1}{x_2} \exp \left( \frac{2C_2}{\sigma^2} \log(x_2) \right) - \frac{2}{\sigma^2}(x_2 - 1) \right]$$

  with $C_1 = 2.625$, $C_2 = 2.125$, $\sigma = 0.5$, $m \approx 0.00004591595108$.

  (Equilibrium PDF of a stochastic Lotka-Volterra two-species prey-predator model [Yeung and Stewart, 2007])

- Other parameters: sampling time $T = 0.5$, regularization parameter $\lambda = 0.001$, and control bounds $u_a = -10$, $u_b = 10$. 

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R. Guglielmi (RICAM), Optimal Control of the FP equation
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- Other parameters: sampling time $T = 0.5$, regularization parameter $\lambda = 0.001$, and control bounds $u_a = -10, u_b = 10$. 
$u \in \mathbb{R}^2$
$u \in \mathbb{R}^2$:
$u \in \mathbb{R}^2$: 
$u \in \mathbb{R}^2$:
$u \in \mathbb{R}^2$: 

$\mathbf{t} = 2.0$
$u \in \mathbb{R}^2$: 
\( u \in \mathbb{R}^2: \)
$u \in \mathbb{R}^2$:
$u \in \mathbb{R}^2$: 
$u \in \mathbb{R}^2$: 
\[ u \in \mathbb{R}^2: \]
$t = 5.0$
Some remarks

- The computed optimal control of the FPE is then applied to the stochastic process
- Right boundary conditions of Robin type (already embedded in the numerical scheme)
- Annunziato, Borzì et al. have applied the same Fokker-Planck Optimal Control framework to
  - the class of piecewise deterministic processes
  - optimal control of open quantum systems
  - subdiffusion processes
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Thank you for your attention!