On the Cahn–Hilliard equation with dynamic boundary conditions and a dominating boundary potential *

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Abstract

The Cahn–Hilliard and viscous Cahn–Hilliard equations with singular and possibly nonsmooth potentials and dynamic boundary condition are considered and some well-posedness and regularity results are proved.

Key words: Cahn–Hilliard equation, dynamic boundary conditions, phase separation, irregular potentials, well-posedness.

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1 Introduction

The classical Cahn–Hilliard equation and the so-called viscous Cahn–Hilliard equation (see [4, 10, 11]) in their simplest forms read

$$\partial_t y - \Delta w = 0 \quad \text{and} \quad w = \tau \partial_t y - \Delta y + \beta(y) + \pi(y) - g \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

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according to the case $\tau = 0$ or $\tau > 0$, respectively. Some significant extensions and a comparative discussion on the modeling approach for phase separation and atom mobility between cells can be found in [12, 15, 18, 8]).

In (1.1), y denotes the order parameter and w represents the chemical potential. Moreover, β and π are the derivatives of the convex part $\widehat{\beta}$ and of the concave perturbation $\widehat{\pi}$ of a double well potential $\mathcal{W}$, and g is a source term. Important examples of $\mathcal{W}$ are the everywhere defined regular potential $\mathcal{W}_{reg}$ and the logarithmic double-well potential $\mathcal{W}_{log}$ given by

$$\mathcal{W}_{reg}(r) = \frac{1}{4}(r^2-1)^2\,, \quad r \in \mathbb{R} \tag{1.2}$$

$$\mathcal{W}_{log}(r) = \big((1+r)\ln(1+r) + (1-r)\ln(1-r)\big) - cr^2\,, \quad r \in (-1,1) \tag{1.3}$$

where $c > 0$ in the latter is large enough in order to kill convexity. Another important example refers to the so-called double-obstacle problem and corresponds to the nonsmooth potential $\mathcal{W}_{2obst}$ specified by

$$\mathcal{W}_{2obst}(r) = \begin{cases} c(1-r^2) & \text{if } |r| \le 1 \\ +\infty & \text{if } |r| > 1 \end{cases} . \tag{1.4}$$

In this case β is no longer a derivative, but it represents the subdifferential $\partial I_{[-1,1]}$ of the indicator function of the interval $[-1,1]$, that is,

$$s \in \partial I_{[-1,1]}(r) \quad \text{if and only if} \quad s \begin{cases} \le 0 & \text{if } r=-1 \\ = 0 & \text{if } -1<r<1 \\ \ge 0 & \text{if } r=1 \end{cases} . \tag{1.5}$$

We are interested to the coupling of (1.1) with the usual no-flux condition for the chemical potential

$$(\partial_n w)_{|\Gamma} = 0 \quad \text{on } \Gamma \times (0,T) \tag{1.6}$$

and with the dynamic boundary condition

$$(\partial_n y)_{|\Gamma} + \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + \beta_\Gamma(y_\Gamma) + \pi_\Gamma(y_\Gamma) = g_\Gamma \quad \text{on } \Gamma \times (0,T) \tag{1.7}$$

where y_Γ denotes the trace $y_{|\Gamma}$ on the boundary Γ of Ω, $-\Delta_\Gamma$ stands for the Laplace–Beltrami operator on Γ, β_Γ and π_Γ are nonlinearities playing the same role as β and π but now acting on the boundary value of the order parameter, and finally g_Γ is a boundary source term with no relation with g acting on the bulk.

The physical meaning and free energy derivation of the boundary value problem given by (1.1) and (1.6)–(1.7) have been discussed specifically in [13]. The Cahn–Hilliard equation (1.1), endowed with the dynamic boundary condition (1.7), has drawn much attention in recent years: we quote, among other contributions, [6, 17, 19, 20, 22]. In particular, the existence and uniqueness of solutions as well as the behavior of the solutions as time goes to infinity have been studied for regular potentials $\mathcal{W}$ and $\mathcal{W}_\Gamma = \widehat{\beta}_\Gamma + \widehat{\pi}_\Gamma$. Moreover, a wide class of potentials, including especially singular potentials like (1.3) and (1.4), has been considered in [13, 14]: in these two papers the authors were able to overcome the difficulties due to singularities and to show well-posedness results along with the long-time behavior of solutions. The approach of [13, 14] is based on a set of assumptions for β, π and β_Γ, π_Γ that gives the role of the dominating potential to $\mathcal{W}$ instead of to $\mathcal{W}_\Gamma$ and entails some technical difficulties.

In this note, we follow a strategy developed in [5] to investigate the Allen–Cahn equation with dynamic boundary condition, which consists in letting $\mathcal{W}_\Gamma$ be the leading potential between the two. This approach simplifies the analysis and allows for a unified treatment of the initial value problem for (1.1), (1.6), (1.7) and for a linearized version thereof. This was a main motivation for this paper, namely to complement and improve the results of [13]. Another input for the realization of this article was the related project of investigating the optimal control problem for the Cahn–Hilliard equation with dynamic boundary condition. In view of the already realized contributions for the corresponding Allen–Cahn equation (see [9] and [7]), a work program for the more difficult Cahn–Hilliard setting appeared to be natural and worth pursuing. This will be the subject of a forthcoming contribution, which will make intense use of the results established here.

Concerning the optimal control problems, let us mention that in [9] both the cases of distributed and boundary controls have been adressed for logarithmic-type potentials as in (1.3): after showing the existence of optimal controls and checking that the control-to-state mapping is twice continuously Fréchet differentiable, first-order necessary optimality conditions were established in terms of a variational inequality and the adjoint state equation, and second-order sufficient optimality conditions were proved. The related paper [7] deals with (non-differentiable) double obstacle potentials (see (1.4)) and contains the proofs of the existence of optimal controls and the derivation of first-order necessary conditions of optimality. Using the results from [9] for the case of (differentiable) logarithmic potentials, a so-called “deep quench limit” is performed to derive first-order necessary optimality conditions.

With the above motivation in mind, we study here the initial and boundary value problem

$$\partial_t y - \Delta w = 0 \quad \text{in } Q := \Omega \times (0,T) \tag{1.8}$$

$$w = \tau\, \partial_t y - \Delta y + \beta(y) + \pi(y) - g \quad \text{in } Q \tag{1.9}$$

$$\partial_n w = 0 \quad \text{on } \Sigma := \Gamma \times (0,T) \tag{1.10}$$

$$y_\Gamma = y_{|_\Gamma} \quad \text{and} \quad \partial_t y_\Gamma + (\partial_n y)_{|_\Gamma} - \Delta_\Gamma y_\Gamma + \beta_\Gamma(y_\Gamma) + \pi_\Gamma(y_\Gamma) = g_\Gamma \quad \text{on } \Sigma \tag{1.11}$$

$$y(0) = y_0 \quad \text{in } \Omega \tag{1.12}$$

as well as a linearization thereof, in which $\beta(y)+\pi(y)$ and $\beta_\Gamma(y_\Gamma)+\pi_\Gamma(y_\Gamma)$ are replaced by $\lambda\, y$ and $\lambda_\Gamma y_\Gamma$, for some given and a.e. bounded functions λ and λ_Γ on Q and Σ, respectively. We investigate both the viscous case $\tau > 0$ and the pure case $\tau = 0$, making the necessary distinctions and specifications. We show existence, uniqueness and regularity results, which are already introduced and made precise in the next section. Section 3 develops the details of the continuous dependence estimate that is also leading to uniqueness. The final Section 4 is concerned with the proofs of existence and of the various regularity results presented in our contribution: in particular, we prove the global boundedness of both y and y_Γ in our general framework for potentials $\mathcal{W}$, $\mathcal{W}_\Gamma$ and graphs β, β_Γ.

2 Main results

In this section, we describe the problem under study and state our results. As in the Introduction, Ω is the body where the evolution takes place. Being clear that just minor changes are needed to treat the lower-dimensional cases, we assume $\Omega \subset \mathbb{R}^3$ to be open, bounded, connected, and smooth and we write $|\Omega|$ for its Lebesgue measure. Moreover, we still denote the boundary of Ω, the outward normal derivative, the surface gradient and the Laplace–Beltrami operator by Γ, ∂_n, ∇_Γ and Δ_Γ, respectively. Given a finite final time T, we set for convenience

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for every } t \in (0,T] \tag{2.1}$$

$$Q := Q_T\,, \quad \text{and} \quad \Sigma := \Sigma_T\,. \tag{2.2}$$

Now, we make the assumptions on the structure of our system precise. However, besides the Cahn–Hilliard equations with or without viscosity, we are interested in solving the corresponding linearized problem around some solution as well. As the latter corresponds to replace $\beta(y)+\pi(y)$ and $\beta_\Gamma(y_\Gamma)+\pi_\Gamma(y_\Gamma)$ by $\lambda\, y$ and $\lambda_\Gamma y_\Gamma$ in (1.9) and (1.11), respectively, where λ and λ_Γ are some functions on Q and Σ, we consider a problem that is slightly more general, in order to unify the treatment. So, we assume that we are given structural functions $\widehat{\beta}$, $\widehat{\beta}_\Gamma$, π, π_Γ, two functions λ, λ_Γ and a constant τ satisfying the conditions listed below.

$$\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to [0,+\infty] \text{ are convex, proper, and l.s.c. and } \widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0 \tag{2.3}$$

$$\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R} \text{ are Lipschitz continuous with } \pi(0) = \pi_\Gamma(0) = 0 \tag{2.4}$$

$$\lambda \in L^\infty(Q) \quad \text{and} \quad \lambda_\Gamma \in L^\infty(\Sigma) \tag{2.5}$$

$$\tau \geq 0 \tag{2.6}$$

$$\lambda \in L^\infty(0,T;W^{1,3}(\Omega)) \quad \text{if } \tau = 0\,. \tag{2.7}$$

We define the graphs β and β_Γ in $\mathbb{R} \times \mathbb{R}$ by

$$\beta := \partial\widehat{\beta} \quad \text{and} \quad \beta_\Gamma := \partial\widehat{\beta}_\Gamma \tag{2.8}$$

and note that both β and β_Γ are maximal monotone with some effective domains $D(\beta)$ and $D(\beta_\Gamma)$. Due to (2.3), we have $\beta(0) \ni 0$ and $\beta_\Gamma(0) \ni 0$. In the sequel, for any maximal monotone graph $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$, we use the notation (see, e.g., [2, p. 28])

$$\gamma^\circ(r) \text{ is the element of } \gamma(r) \text{ having minimum modulus} \tag{2.9}$$

$$\gamma^Y_\varepsilon \text{ is the Yosida regularization of } \gamma \text{ at level } \varepsilon, \text{ for } \varepsilon > 0. \tag{2.10}$$

Moreover, we still write the symbol γ (and, e.g., γ^Y_ε as a particular case) for the maximal monotone operator induced by γ on the space $L^2(Q)$. For the graphs β and β_Γ we assume the following compatibility condition

$$D(\beta_\Gamma) \subseteq D(\beta) \quad \text{and} \quad |\beta^\circ(r)| \leq \eta|\beta^\circ_\Gamma(r)| + C \\ \text{for some } \eta,\, C > 0 \text{ and every } r \in D(\beta_\Gamma) \tag{2.11}$$

and note that, roughly speaking, it is opposite to the one postulated in [13]. On the contrary, condition (2.11) is the same as the one introduced in the paper [5], which however deals with the Allen–Cahn equation. For those reasons, the results we obtain are completely new. Next, in order to simplify the notation, we set

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad H_\Gamma := L^2(\Gamma), \quad V_\Gamma := H^1(\Gamma) \tag{2.12}$$
$$\mathcal{V} := \{v \in V : \ v_{|\Gamma} \in V_\Gamma\} \tag{2.13}$$

and endowe the former spaces with their usual norms and the latter with the graph norm. For the norms in the generic Banach space X and in any power of it, we write $\|\cdot\|_X$. However, simpler symbols are used in particular cases. For instance, for $1 \le p \le \infty$, $\|\cdot\|_p$ is the usual norm in $L^p(\Omega)$ and $\|\cdot\|_*$ denotes a precise norm in V^* introduced later on (see the forthcoming (2.54)). Such a norm is equivalent to the standard dual norm. Furthermore, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between V^* and V. In the sequel, it is understood that H is embedded in V^* in the usual way, i.e., in order that $\langle u, v\rangle = (u, v)$, the inner product of H, for every $u \in H$ and $v \in V$. Finally, we define the generalized mean value of any $v^* \in V^*$ by setting

$$v^*_\Omega := \frac{1}{|\Omega|}\,\langle v^*, 1\rangle \quad \text{for } v^* \in V^*. \tag{2.14}$$

Clearly, (2.14) gives the usual mean value when applied to elements of H.

At this point, we can describe our problem, which consists in the variational formulation of system (1.8)–(1.12). To prepare the assumptions we need for our main existence result (the problem is meaningful and a uniqueness type theorem holds under much weaker hypotheses, indeed), we give the data g, g_Γ, and y_0 satisfying

$$g \in L^2(0,T;H) \quad \text{and} \quad g_\Gamma \in L^2(0,T;H_\Gamma) \tag{2.15}$$
$$g \in H^1(0,T;H) \quad \text{if } \tau = 0 \tag{2.16}$$
$$y_0 \in \mathcal{V}, \quad \widehat{\beta}(y_0) \in L^1(\Omega) \quad \text{and} \quad \widehat{\beta}_\Gamma(y_{0|\Gamma}) \in L^1(\Gamma) \tag{2.17}$$
$$m_0 := (y_0)_\Omega \quad \text{lies in the interior of } D(\beta_\Gamma). \tag{2.18}$$

Our problem consists in looking for a quintuplet $(y, y_\Gamma, w, \xi, \xi_\Gamma)$ such that

$$y \in H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \ \text{ and } \ \tau\,\partial_t y \in L^2(0,T;H) \tag{2.19}$$
$$y_\Gamma \in H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;H^2(\Gamma)) \tag{2.20}$$
$$y_\Gamma(t) = y(t)_{|\Gamma} \quad \text{for a.a. } t \in (0,T) \tag{2.21}$$
$$w \in L^2(0,T;V) \tag{2.22}$$
$$\xi \in L^2(0,T;H) \quad \text{and} \quad \xi \in \beta(y) \quad \text{a.e. in } Q \tag{2.23}$$
$$\xi_\Gamma \in L^2(0,T;H_\Gamma) \quad \text{and} \quad \xi_\Gamma \in \beta_\Gamma(y_\Gamma) \quad \text{a.e. on } \Sigma \tag{2.24}$$

and satisfying for a.a. $t \in (0,T)$ the variational equations

$$\langle \partial_t y(t), v\rangle + \int_\Omega \nabla w(t) \cdot \nabla v = 0 \tag{2.25}$$

$$\begin{aligned} \int_\Omega w(t) v = {} & \int_\Omega \tau\,\partial_t y(t)\,v + \int_\Gamma \partial_t y_\Gamma(t)\,v + \int_\Omega \nabla y(t)\cdot\nabla v + \int_\Gamma \nabla_\Gamma y_\Gamma(t)\cdot\nabla_\Gamma v \\ & + \int_\Omega \bigl(\xi(t) + \lambda(t)\,\pi(y(t)) - g(t)\bigr)\,v + \int_\Gamma \bigl(\xi_\Gamma(t) + \lambda_\Gamma(t)\,\pi_\Gamma(y_\Gamma(t)) - g_\Gamma(t)\bigr)\,v \end{aligned} \tag{2.26}$$

for every $v \in V$ and every $v \in \mathcal{V}$, respectively, and the Cauchy condition

$$y(0) = y_0 \,. \tag{2.27}$$

For simplicity, we have used the same symbol v for both the test function and its trace on the boundary, and we do so in the sequel if no misunderstanding can arise. Moreover, we write products by τ (like the pointwise value $\tau\partial_t y(t)$ in (2.26) which might be meaningless, in principle) for the sake of conciseness, also in the sequel. In such cases, is understood that the product vanishes for $\tau = 0$. We note that an equivalent formulation of (2.25)–(2.26) is given by

$$\int_0^T \langle \partial_t y(t), v(t) \rangle \, dt + \int_Q \nabla w \cdot \nabla v = 0 \tag{2.28}$$

$$\begin{aligned} \int_Q wv &= \int_Q \tau\partial_t y \, v + \int_\Sigma \partial_t y_\Gamma \, v + \int_Q \nabla y \cdot \nabla v + \int_\Sigma \nabla_\Gamma y_\Gamma \cdot \nabla_\Gamma v \\ &\quad + \int_Q \bigl(\xi + \lambda \, \pi(y) - g\bigr) \, v + \int_\Sigma \bigl(\xi_\Gamma + \lambda_\Gamma \pi_\Gamma(y_\Gamma) - g_\Gamma\bigr) \, v \end{aligned} \tag{2.29}$$

for every $v \in L^2(0,T;V)$ and every $v \in L^2(0,T;\mathcal{V})$, respectively.

Remark 2.1. Even though what we say is completely standard for Cahn–Hilliard equations, it is worth to note it. By testing (2.25) by the constant $1/|\Omega|$, we obtain

$$(\partial_t y(t))_\Omega = 0 \quad \text{for a.a. } t \in (0,T) \quad \text{and} \quad y(t)_\Omega = m_0 \quad \text{for every } t \in [0,T] \tag{2.30}$$

with the notations (2.14) and (2.18).

As far as uniqueness and continuous dependence are concerned, we have

Theorem 2.2. *Assume* (2.3)–(2.8) *and let* $(g_i, g_{\Gamma,i}, y_{0,i})$, $i = 1,2$, *be two sets of data satisfying* (2.15) *and such that* $y_{0,1}, y_{0,2}$ *belong to* V *and have the same mean value. Then, if* $(y_i, y_{\Gamma,i}, w_i, \xi_i, \xi_{\Gamma,i})$ *are any two corresponding solutions to problem* (2.19)–(2.27), *the inequality*

$$\begin{aligned} &\|y_1 - y_2\|^2_{L^\infty(0,T;V^*)} + \tau\|y_1 - y_2\|^2_{L^\infty(0,T;H)} + \|y_{\Gamma,1} - y_{\Gamma,2}\|^2_{L^\infty(0,T;H_\Gamma)} \\ &\quad + \|\nabla(y_1 - y_2)\|^2_{L^2(0,T;H)} + \|\nabla_\Gamma(y_{\Gamma,1} - y_{\Gamma,2})\|^2_{L^2(0,T;H_\Gamma)} \\ &\leq c\Bigl\{ \|y_{0,1} - y_{0,2}\|^2_* + \tau\|y_{0,1} - y_{0,2}\|^2_H + \|y_{0,1}|_\Gamma - y_{0,2}|_\Gamma\|^2_{H_\Gamma} \\ &\qquad + \|g_1 - g_2\|^2_{L^2(0,T;H)} + \|g_{\Gamma,1} - g_{\Gamma,2}\|^2_{L^2(0,T;H_\Gamma)} \Bigr\} \end{aligned} \tag{2.31}$$

holds true with a constant c *that depends only on* Ω, T, *the Lipschitz constants of* π *and* π_Γ *and on the norms* $\|\lambda\|_{L^\infty(Q)}$ *and* $\|\lambda_\Gamma\|_{L^\infty(\Sigma)}$. *In particular, any two solutions to problem* (2.19)–(2.27) *have the same components* y, y_Γ *and* ξ_Γ. *Moreover, even the components* w *and* ξ *of such solutions are the same if* β *is single-valued.*

The above theorem is quite similar to the results stated in [13, Thm. 1 and Rem. 9]. In the same paper (see [13, Rem. 4 and Rem. 8]), it is also shown that partial uniqueness and conditionally full uniqueness as in the above statement are the best one can prove. At this point, we are maily interested in existence and regularity and what we prove is new with respect to [13], as already observed. Here is our first result in that direction.

Theorem 2.3. *Assume* (2.3)–(2.8), (2.11) *and* (2.15)–(2.18). *Then, there exists a quintuplet* $(y, y_\Gamma, w, \xi, \xi_\Gamma)$ *satisfying* (2.19)–(2.24) *and solving problem* (2.25)–(2.27).

Our next goal is regularity, and we present several results. First, we want to prove that the unique solution to problem (2.25)–(2.27) given by the above theorems also satisfies

$$y \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)) \quad \text{and} \quad \tau\partial_t y \in L^\infty(0,T;H) \tag{2.32}$$

$$y_\Gamma \in W^{1,\infty}(0,T;H_\Gamma) \cap H^1(0,T;V_\Gamma) \cap L^\infty(0,T;H^2(\Gamma)) \tag{2.33}$$

whence also

$$y \in L^\infty(Q) \quad \text{and} \quad y_\Gamma \in L^\infty(\Sigma). \tag{2.34}$$

To this aim, besides (2.5) and (2.7) we suppose that

$$\lambda \in W^{1,\infty}(0,T;H) \quad \text{and} \quad \lambda_\Gamma \in W^{1,\infty}(0,T;H_\Gamma)\,. \tag{2.35}$$

As far as the data are concerned, we also assume

$$g \in H^1(0,T;H) \quad \text{and} \quad g_\Gamma \in H^1(0,T;H_\Gamma) \tag{2.36}$$

$$y_0 \in H^2(\Omega), \quad \partial_n y_{0|_\Gamma} = 0 \quad \text{and} \quad y_{0|_\Gamma} \in H^2(\Gamma) \tag{2.37}$$

$$\text{there exists } \xi_0 \in H \text{ such that } \xi_0 \in \beta(y_0) \text{ a.e. in } Q \tag{2.38}$$

$$\text{there exists } \xi_{\Gamma,0} \in H_\Gamma \text{ such that } \xi_{\Gamma,0} \in \beta_\Gamma(y_{0|_\Gamma}) \text{ a.e. on } \Sigma \tag{2.39}$$

and, if $\tau = 0$, we reinforce (2.38) by requiring that

$$\text{the family} \quad \{-\Delta y_0 - \beta_\varepsilon(y_0) - g(0) :\ \varepsilon \in (0,\varepsilon_0)\} \quad \text{is bounded in } V \tag{2.40}$$

for some $\varepsilon_0 > 0$. Clearly, in order to ensure (2.40), one can assume that $\Delta y_0 + g(0) \in V$ and that $\beta_\varepsilon(y_0)$ remains bounded in V for ε small enough, and a sufficient condition for the latter is the following: there exist $r_\pm, r'_\pm \in \mathbb{R}$ such that $r'_- < r_- \le y_0 \le r_+ < r'_+$ a.e. in Ω, $(r'_-, r'_+) \subset D(\beta)$ and the restriction of β to (r'_-, r'_+) is a single-valued Lipschitz continuous function.

Here is our first regularity result.

Theorem 2.4. *Assume* (2.3)–(2.8), (2.11) *and* (2.35) *on the structure and suppose that the data satisfy* (2.36)–(2.39) *and* (2.18). *Then, the unique solution to problem* (2.25)–(2.27) *given by Theorems 2.2 and 2.3 also satisfies* (2.32)–(2.34). *Moreover, we have that*

$$w \in L^\infty(0,T;V), \quad \xi \in L^\infty(0,T;H), \quad \xi_\Gamma \in L^\infty(0,T;H_\Gamma). \tag{2.41}$$

We present at once a consequence of our results, which is obtained by simply taking

$$\widehat{\beta}(r) = \widehat{\beta}_\Gamma(r) = 0 \quad \text{and} \quad \pi(r) = \pi_\Gamma(r) = r \quad \text{for every } r \in \mathbb{R}.$$

Corollary 2.5. *Assume* $\tau > 0$ *and* (2.5). *Moreover, assume* (2.36) *and* (2.37). *Then, there exists a unique triplet* (y, y_Γ, w) *satisfying the regularity requirements* (2.32)–(2.34), (2.22) *and the Cauchy condition* (2.27), *and solving the variational equations* (2.25) *for every* $v \in V$ *and*

$$\int_\Omega w(t) v = \int_\Omega \tau\, \partial_t y(t)\, v + \int_\Gamma \partial_t y_\Gamma(t)\, v + \int_\Omega \nabla y(t) \cdot \nabla v + \int_\Gamma \nabla_\Gamma y_\Gamma(t) \cdot \nabla_\Gamma v$$
$$+ \int_\Omega \bigl(\lambda(t)\, y(t) - g(t)\bigr)\, v + \int_\Gamma \bigl(\lambda_\Gamma(t)\, y_\Gamma(t) - g_\Gamma(t)\bigr)\, v$$

for every $v \in \mathcal{V}$.

Such a corollary, which is more significant if $\tau > 0$, as we have assumed, can be applied to the problem obtained by linearizing (2.19)–(2.27) around its solution. Therefore, it is useful in the control problem associated to (2.19)–(2.27) we are going to discuss in a forthcoming paper. Our second regularity result that deals with the general case is the following

Theorem 2.6. *In addition to the assumptions of Theorem 2.4, suppose that* $\tau > 0$ *and that*

$$g \in L^\infty(Q), \quad g_\Gamma \in L^\infty(\Sigma) \quad \textit{and} \quad \beta^\circ(y_0) \in L^\infty(Q). \tag{2.42}$$

Then, the solution to problem (2.25)*–*(2.27) *also satisfies*

$$w \in L^\infty(0,T;H^2(\Omega)) \subset L^\infty(Q) \quad \textit{and} \quad \xi \in L^\infty(Q). \tag{2.43}$$

The regularity result just stated has an interesting consequence in the case of operators β and β_Γ satisfying the following assumptions

$$D(\beta) \text{ is an open interval } I \quad \text{and} \quad D(\beta_\Gamma) = D(\beta). \tag{2.44}$$

The first (2.44) is fulfilled if $\widehat{\beta}$ is, for instance, the everywhere smooth potential (1.2) or the logarithmic potential (1.3). On the contrary, potentials whose convex part is an indicator function are excluded. We observe that, if I is not the whole of $\mathbb{R}$ and r_0 is an end-point of it, then β° has an infinite limit at r_0 since the interval I is open. Due to the second condition in (2.44), the same remarks hold for β_Γ°. The result we state easily follows from (2.44), on account of (2.34) (to be used if I is unbounded) and the second property in (2.43). Therefore, we do not prove it.

Corollary 2.7. *In addition to the assumptions of Theorem 2.4, suppose that* $\tau > 0$ *and that* (2.42) *and* (2.44) *are satisfied. Moreover, assume that* $\lambda = 1$ *and* $\lambda_\Gamma = 1$*. Then, the following conclusions hold true: i) for the solution* $(y, y_\Gamma, w, \xi, \xi_\Gamma)$ *to problem* (2.19)*–*(2.27) *we have*

$$y(x,t) \in K \quad \textit{for a.a. } (x,t) \in Q \textit{ and some compact subset } K \subset I. \tag{2.45}$$

In particular, even ξ_Γ *is bounded. ii) Assume that* β *and* β_Γ *are single-valued* C^1 *functions. Then, the functions* $\beta'(y)$ *and* $\beta'_\Gamma(y_\Gamma)$ *are bounded as well. iii) Assume that* β, β_Γ, π *and* π_Γ *are of class* C^2*, in addition. Then*

$$\beta'(y) + \pi'(y) \in L^\infty(0,T;V) \cap L^\infty(Q) \quad \textit{and} \quad \beta'_\Gamma(y) + \pi'_\Gamma(y) \in L^\infty(0,T;V_\Gamma) \cap L^\infty(\Sigma).$$

The rest of the section is devoted to recall some facts that are well known and to introduce some notation that is widely used in the sequel. First of all, we often owe to the Young inequality (mainly with $p = p' = 2$, thus with $\delta^{-p'/p} = \delta^{-1}$)

$$ab \le \frac{\delta}{p}\, a^p + \frac{\delta^{-p'/p}}{p'}\, b^{p'} \quad \text{for every } a, b \ge 0,\ \delta > 0 \text{ and } p > 1 \tag{2.46}$$

where $p' := p/(p-1)$, and to the Hölder inequality. Moreover, we account for the well-known embeddings and the related inequalities, as well as the Poincaré inequality, namely

$$\|v\|_\infty \le C\|v\|_{H^2(\Omega)} \quad \text{for every } v \in H^2(\Omega) \tag{2.47}$$

$$\|v\|_\infty \le C\|v\|_{H^2(\Gamma)} \quad \text{for every } v \in H^2(\Gamma) \tag{2.48}$$

$$\|v\|_V^2 \le C\big(\|\nabla v\|_H^2 + |v_\Omega|^2\big) \quad \text{for every } v \in V \tag{2.49}$$

where C depends only on Ω. Furthermore, we observe that the identity $\|v\|_H^2 = \langle v, v \rangle$ for $v \in V$ easily implies the inequality

$$\|v\|_H^2 \le \delta \|\nabla v\|_H^2 + c_\delta \|v\|_*^2 \quad \text{for every } v \in V \tag{2.50}$$

for every $\delta > 0$ and some constant c_δ depending on δ and Ω as well. Next, we recall a tool that is generally used in the context of problems related to the Cahn–Hilliard equations. We define

$$\operatorname{dom} \mathcal{N} := \{v^* \in V^* : \ v^*_\Omega = 0\} \quad \text{and} \quad \mathcal{N} : \operatorname{dom} \mathcal{N} \to \{v \in V : \ v_\Omega = 0\} \tag{2.51}$$

by setting for $v^* \in \operatorname{dom} \mathcal{N}$

$$\mathcal{N} v^* \in V, \quad (\mathcal{N} v^*)_\Omega = 0, \quad \text{and} \quad \int_\Omega \nabla \mathcal{N} v^* \cdot \nabla z = \langle v^*, z \rangle \quad \text{for every } z \in V \tag{2.52}$$

i.e., $\mathcal{N} v^*$ is the solution v to the generalized Neumann problem for $-\Delta$ with datum v^* that satisfies $v^*_\Omega = 0$. As Ω is bounded, smooth, and connected, it turns out that (2.52) yields a well-defined isomorphism, which satisfies

$$\langle u^*, \mathcal{N} v^* \rangle = \langle v^*, \mathcal{N} u^* \rangle = \int_\Omega (\nabla \mathcal{N} u^*) \cdot (\nabla \mathcal{N} v^*) \quad \text{for } u^*, v^* \in \operatorname{dom} \mathcal{N}. \tag{2.53}$$

Moreover, if we define $\| \cdot \|_* : V^* \to [0, +\infty)$ by the formula

$$\|v^*\|_*^2 := \|\nabla \mathcal{N}(v^* - (v^*)_\Omega)\|_H^2 + |(v^*)_\Omega)|^2 \quad \text{for } v^* \in V^* \tag{2.54}$$

it is straightforward to prove that $\| \cdot \|_*$ is a norm that makes V^* a Hilbert space. It follows that $\| \cdot \|_*$ is equivalent to the usual dual norm by the open mapping theorem and it thus can be used as a norm in V^*. It follows that

$$|\langle v^*, v \rangle| \le C \|v^*\|_* \|v\|_V \quad \text{for every } v^* \in V^* \text{ and } v \in V$$

where C depends only on Ω. Note that

$$\langle v^*, \mathcal{N} v^* \rangle = \|v^*\|_*^2 \quad \text{for every } v^* \in \operatorname{dom} \mathcal{N} \tag{2.55}$$

by (2.53)–(2.54). Finally, owing to (2.53) once more, we see that

$$2 \langle \partial_t v^*(t), \mathcal{N} v^*(t) \rangle = \frac{d}{dt} \int_\Omega |\nabla \mathcal{N} v^*(t)|^2 = \frac{d}{dt} \|v^*(t)\|_*^2 \quad \text{for a.a. } t \in (0,T) \tag{2.56}$$

for every $v^* \in H^1(0,T; V^*)$ satisfying $v^*(t)_\Omega = 0$ for every $t \in [0,T]$.

We conclude this section by stating a general rule we use as far as constants are concerned, in order to avoid a boring notation. Throughout the paper, the small-case symbol c stands for different constants which depend only on Ω, on the final time T, and on the constants and the norms of the functions involved in the assumptions of our statements. In particular, c is independent of the approximation parameter ε we introduce later on. A notation like c_δ allows the constant to depend on the positive parameter δ, in addition. Hence, the meaning of c and c_δ might change from line to line and even in the same chain of inequalities. On the contrary, we use capital letters to denote precise constants which we could refer to (see, e.g., (2.47)).

3 Uniqueness and continuous dependence

This section is devoted to the proof of Theorem 2.2. We closely follow [13, Thm. 1] and just adapt the argument used there. For convenience, we set $y := y_1 - y_2$ and similarly define y_Γ, w, ξ, ξ_Γ, g, g_Γ and y_0. As the initial data have the same mean value, by Remark 2.1 applied to y_i for $i = 1, 2$, we see that $y(t)$ has zero mean value and thus belongs to the domain of $\mathcal{N}$ for every $t \in [0, T]$. Therefore, we can write (2.25) at any time s for both solutions and test the difference by $\mathcal{N}y(s)$. Then, we integrate over $(0, t)$ with respect to s, where $t \in (0, T]$ is arbitrary. At the same time, we write (2.26) for both solutions and take $-y$ as test function. Finally, we add the obtained equalities to each other. We have

$$\begin{aligned}
&\int_0^t \langle \partial_t y(s), \mathcal{N}y(s)\rangle \, ds + \int_{Q_t} \nabla w \cdot \nabla \mathcal{N}y - \int_{Q_t} wy \\
&\quad + \frac{\tau}{2} \int_\Omega |y(t)|^2 - \frac{\tau}{2} \int_\Omega |y_0|^2 + \frac{1}{2} \int_\Gamma |y_\Gamma(t)|^2 - \frac{1}{2} \int_\Gamma |y_0|_\Gamma|^2 \\
&\quad + \int_{Q_t} |\nabla y|^2 + \int_{\Sigma_t} |\nabla_\Gamma y_\Gamma|^2 + \int_{Q_t} \xi y + \int_{\Sigma_t} \xi_\Gamma y_\Gamma \\
&= \int_{Q_t} \big(\lambda(\pi(y_2) - \pi(y_1)) + g\big)\, y + \int_{\Sigma_t} \big(\lambda_\Gamma(\pi_\Gamma(y_{\Gamma,2})) - \pi_\Gamma(y_{\Gamma,1}) + g_\Gamma\big)\, y_\Gamma \,.
\end{aligned}$$

Now, we transform the first term on the left-hand side with the help of (2.56) and cancel the next two integrals thanks to (2.52). Moreover, we neglect the last two integrals on the left-hand side since they are nonnegative for β and β_Γ are monotone. Finally, we exploit assumptions (2.4)–(2.5) and use the elementary Young inequality. We obtain

$$\begin{aligned}
&\frac{1}{2} \|y(t)\|_*^2 - \frac{1}{2} \|y_0\|_*^2 + \frac{\tau}{2} \|y(t)\|_H^2 - \frac{\tau}{2} \|y_0\|_H^2 + \frac{1}{2} \|y_\Gamma(t)\|_{H_\Gamma}^2 - \frac{1}{2} \|y_0|_\Gamma\|_{H_\Gamma}^2 \\
&\quad + \int_{Q_t} |\nabla y|^2 + \int_{\Sigma_t} |\nabla_\Gamma y_\Gamma|^2 \\
&\le c \int_{Q_t} |y|^2 + \frac{1}{4} \int_{Q_t} |g|^2 + c \int_{\Sigma_t} |y_\Gamma|^2 + \frac{1}{4} \int_{\Sigma_t} |g_\Gamma|^2.
\end{aligned}$$

At this point, we take advantage of (2.50) to infer that

$$\int_{Q_t} |y|^2 \le \delta \int_{Q_t} |\nabla y|^2 + c_\delta \int_0^t \|y(s)\|_*^2 \, ds.$$

Therefore, it suffices to choose $\delta < 1$ and apply the Gronwall lemma to obtain (2.31). The sentence of the statement regarding partial uniqueness easily follows, as we show at once. Clearly, (2.31) with the same data implies $y = 0$ and $y_\Gamma = 0$ with the notation we have introduced at the beginning, so that the difference of the equation (2.26) simply reduces to

$$\int_\Omega w(t)\, v = \int_\Omega \xi(t)\, v + \int_\Gamma \xi_\Gamma(t)\, v \quad \text{for a.a. } t \in (0, T) \text{ and every } v \in \mathcal{V}. \tag{3.1}$$

In view of (2.25), it turns out that w is a function depending only on time (i.e., for a.a. $t \in (0, T)$ $w(t)$ is constant in Ω). Taking now test functions $v \in \mathcal{D}(\Omega)$ in (3.1), we easily infer that $\xi(t) = w(t)$ for a.a. $t \in (0,T)$. Next, letting v vary in $\mathcal{V}$ we also deduce that $\xi_\Gamma(t) = 0$ for a.a. $t \in (0,T)$. Assume now β to be single-valued. Then, we have $\xi = 0$ in (3.1) as well. We immediately conclude that $w = 0$ and the proof is complete.

4 Existence and regularity

In this section, we prove our existence and regularity results. The method we use relies on a regularization depending on the parameter $\varepsilon \in (0,1)$ that will tend to zero. Namely, we introduce the approximating problem of finding a pair $(y_\varepsilon, w_\varepsilon)$ such that a suitably corresponding quintuplet $(y_\varepsilon, y_{\Gamma,\varepsilon}, w_\varepsilon, \xi_\varepsilon, \xi_{\Gamma,\varepsilon})$ solves the system obtained by replacing the graphs β and β_Γ by the everywhere defined functions β_ε and $\beta_{\Gamma,\varepsilon}$ we make precise below and g by a suitably regularized datum. For clarity, we write both the construction of the corresponding quintuplet and the regularized problem, at once:

$$y_{\Gamma,\varepsilon}(t) := y_\varepsilon(t)_{|\Gamma} \quad \text{for a.a. } t \in (0,T), \quad \xi_\varepsilon := \beta_\varepsilon(y_\varepsilon) \quad \text{and} \quad \xi_{\Gamma,\varepsilon} := \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}) \tag{4.1}$$

$$\langle \partial_t y_\varepsilon(t), v \rangle + \int_\Omega \nabla w_\varepsilon(t) \cdot \nabla v = 0 \tag{4.2}$$

$$\begin{aligned} \int_\Omega w_\varepsilon(t) v &= \tau_\varepsilon \int_\Omega \partial_t y_\varepsilon(t)\, v + \int_\Gamma \partial_t y_{\Gamma,\varepsilon}(t)\, v + \int_\Omega \nabla y_\varepsilon(t) \cdot \nabla v + \int_\Gamma \nabla_\Gamma y_{\Gamma,\varepsilon}(t) \cdot \nabla_\Gamma v \\ &\quad + \int_\Omega \bigl(\xi_\varepsilon(t) + \lambda(t)\, \pi(y_\varepsilon(t)) - g_\varepsilon(t)\bigr) v + \int_\Gamma \bigl(\xi_{\Gamma,\varepsilon}(t) + \lambda_\Gamma(t)\, \pi_\Gamma(y_{\Gamma,\varepsilon}(t)) - g_\Gamma(t)\bigr) v \end{aligned} \tag{4.3}$$

$$y_\varepsilon(0) = y_0 \tag{4.4}$$

where (4.2) and (4.3) are required to hold for every $v \in V$ and every $v \in \mathcal{V}$, respectively, and

$$\tau_\varepsilon := \tau \quad \text{if } \tau > 0 \quad \text{and} \quad \tau_\varepsilon := \varepsilon \quad \text{if } \tau = 0. \tag{4.5}$$

Thus, we first solve problem (4.2)–(4.4) in the proper functional framework. Then, we perform a number of a priori estimates and use compactness and monotonicity techniques that ensure that the ε-solution converges to a solution to the original problem in a proper topology as ε tends to zero. Due to uniqueness, the whole family of approximating solution will converge, even though it is necessary to take convergent subsequences, in principle. The power of the estimates we can derive (thus, the topology of the convergence that follows) depends on the assumptions on the data we can account for, i.e., on the theorem we want to prove. We start with Theorem 2.3 and suppose that just (2.15)–(2.18) are fulfilled. However, the whole argument partially works for the proof of the regularity results as well. Just further a priori estimates are necessary for the latter, indeed. For the approximating solution we postulate the following regularity

$$y_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \tag{4.6}$$

$$y_{\Gamma,\varepsilon} \in H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;H^2(\Gamma)) \tag{4.7}$$

$$w_\varepsilon \in L^2(0,T;V). \tag{4.8}$$

Thus, we look for a pair $(y_\varepsilon, w_\varepsilon)$ such that the quintuplet $(y_\varepsilon, y_{\Gamma,\varepsilon}, w_\varepsilon, \xi_\varepsilon, \xi_{\Gamma,\varepsilon})$ defined by (4.1) satisfies (4.6)–(4.8) and solves (4.2)–(4.4).

Let us come to the definition of the regularized monotone operators. Inspired by [5], for each graph, we take the proper Yosida regularization (see (2.10)), namely

$$\beta_\varepsilon := \beta_\varepsilon^Y \quad \text{and} \quad \beta_{\Gamma,\varepsilon} := (\beta_\Gamma)_{\eta\varepsilon}^Y \tag{4.9}$$

where η is the the same as in (2.11). Such a choice yields (see [5, Lemma 4.4])

$$|\beta_\varepsilon(r)| \le \eta|\beta_{\Gamma,\varepsilon}(r)| + C \quad \text{for every } r \in \mathbb{R} \text{ and } \varepsilon \in (0,1) \tag{4.10}$$

where C is some positive constant and η still is the same as in (2.11). We also define for convenience

$$\widehat{\beta}_\varepsilon(r) := \int_0^r \beta_\varepsilon(s)\,ds \quad \text{and} \quad \widehat{\beta}_{\Gamma,\varepsilon}(r) := \int_0^r \beta_{\Gamma,\varepsilon}(s)\,ds \quad \text{for } r \in \mathbb{R} \tag{4.11}$$

and recall that the set of properties

$$0 \le \widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r), \;\; \widehat{\beta}_\varepsilon(r) \nearrow \widehat{\beta}(r) \text{ monotonically as } \varepsilon \searrow 0 \tag{4.12}$$

$$|\beta_\varepsilon(r)| \le |\beta^\circ(r)|, \;\; \beta_\varepsilon(r) \text{ tends to } \beta^\circ(r) \text{ monotonically as } \varepsilon \searrow 0 \tag{4.13}$$

and the analogue for $\widehat{\beta}_{\Gamma,\varepsilon}$ and $\beta_{\Gamma,\varepsilon}$ hold true (see, e.g., [2, Prop. 2.11, p. 39]). Notice that inequality (4.10) implies

$$\widehat{\beta}_\varepsilon(r) \le \eta\,\widehat{\beta}_{\Gamma,\varepsilon}(r) + C|r| \quad \text{for every } r \in \mathbb{R} \text{ and } \varepsilon \in (0,1) \tag{4.14}$$

since $\beta_\varepsilon(0) = \beta_{\Gamma,\varepsilon}(0) = 0$ by (2.3), whence β_ε and $\beta_{\Gamma,\varepsilon}$ have the same sign. For the approximating datum g_ε, we require the following regularity and convergence properties

$$g_\varepsilon \in H^1(0,T;H) \quad \text{and} \quad g_\varepsilon \to g \quad \text{strongly in } L^2(0,T;H) \quad \text{if } \tau > 0 \tag{4.15}$$

$$g_\varepsilon = g \quad \text{if } \tau = 0. \tag{4.16}$$

To start with our program, we first have to solve the approximating problem.

Theorem 4.1. *Assume* (2.3)–(2.8), (2.15)–(2.18), (4.5), (4.9) *and* (4.15)–(4.16). *Then, there is a unique pair* $(y_\varepsilon, w_\varepsilon)$ *such that the corresponding quintuplet* $(y_\varepsilon, y_{\Gamma,\varepsilon}, w_\varepsilon, \xi_\varepsilon, \xi_{\Gamma,\varepsilon})$ *satisfies the regularity given by* (4.6)–(4.8) *and solves problem* (4.2)–(4.4).

Proof. Uniqueness follows from Theorem 2.2 as a particular case. As far as existence is concerned, we can first quote [13, Thm. 3], even though π and π_Γ are multiplied by coefficients that depend on x and t. Indeed, minor changes in the proof are sufficient to adapt the argument and obtain the generalized result we need here, namely, the existence of a solution satisfying

$$y_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad y_{\Gamma,\varepsilon} \in H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \quad \text{and} \quad w_\varepsilon \in L^2(0,T;V).$$

For completeness, we note that the cited result regards everywhere defined monotone operators β and β_Γ satisfying suitable growth conditions (that include the sublinear case of the operators β_ε and $\beta_{\Gamma,\varepsilon}$) instead of compatibility conditions, and data satisfying assumptions that are implied by (2.15)–(2.18) and (4.15)–(4.16). Thus, we only have to show that the solution we have found is smoother than expected, that is, it satisfies (4.6)–(4.7). In this direction, we could quote [13, Rem. 12]; however, we detail the argument for the reader's convenience. We observe that the variational equation (4.3) implies that y_ε solves the partial differential equation

$$w_\varepsilon = \tau_\varepsilon \partial_t y_\varepsilon - \Delta y_\varepsilon + \beta_\varepsilon(y_\varepsilon) + \lambda\,\pi(y_\varepsilon) - g_\varepsilon \quad \text{in } Q \tag{4.17}$$

at least in the sense of distributions. Due to the regularity assumed for g_ε and the Lipschitz continuity of β_ε, all the terms of (4.17) but Δy_ε belong to $L^2(0,T;H)$. By comparison, we deduce that $\Delta y_\varepsilon \in L^2(0,T;H)$. On the other hand, $y_{\Gamma,\varepsilon} \in L^2(0,T;V_\Gamma)$. Thus, the elliptic regularity theory yields $y_\varepsilon \in L^2(0,T;H^{3/2}(\Omega))$, whence also $\partial_n y_{\varepsilon|_\Gamma} \in L^2(0,T;H_\Gamma)$ (see, e.g., [16, Thms. 7.4 and 7.3, pp. 187-188] or [3, Thm. 3.2, p. 1.79, and Thm. 2.27, p. 1.64]). In particular, all the terms of the integration by part formula for the Laplace operator are functions and we deduce that the variational equation (4.3) also implies

$$\partial_n y_{\varepsilon|_\Gamma} + \partial_t y_{\Gamma,\varepsilon} - \Delta_\Gamma y_{\Gamma,\varepsilon} + \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}) + \lambda_\Gamma \pi_\Gamma(y_{\Gamma,\varepsilon}) = g_\Gamma \quad \text{on } \Sigma \tag{4.18}$$

at least in a generalized sense, in principle. Arguing as before, we see that $\Delta y_{\Gamma,\varepsilon} \in L^2(0,T;H_\Gamma)$, whence $y_{\Gamma,\varepsilon} \in L^2(0,T;H^2(\Gamma))$ by the boundary version of the elliptic regularity theory. Coming back to y_ε, we infer that $y_\varepsilon \in L^2(0,T;H^2(\Omega))$. □

At this point, we start estimating. We are widely inspired by the techniques used in [13]. However, since our assumptions and statements are different, it is necessary to detail the argument. We remind the reader that our assumptions on the data reduce to (2.15)–(2.18) and (4.15)–(4.16). Moreover we recall the definition (2.18) of m_0 and observe that Remark 2.1 obviously applies to the approximating problem as well, i.e.

$$(\partial_t y_\varepsilon(t))_\Omega = 0 \quad \text{for a.a. } t \in (0,T) \quad \text{and} \quad y_\varepsilon(t)_\Omega = m_0 \quad \text{for every } t \in [0,T]. \tag{4.19}$$

Furthermore, we recall the useful inequalities

$$\beta_\varepsilon(r)(r-m_0) \geq \delta_0|\beta_\varepsilon(r)| - C_0 \quad \text{and} \quad \beta_{\Gamma,\varepsilon}(r)(r-m_0) \geq \delta_0|\beta_{\Gamma,\varepsilon}(r)| - C_0$$
$$\text{for some } \delta_0, C_0 > 0 \text{ and every } r \in \mathbb{R} \text{ and } \varepsilon \in (0,1) \tag{4.20}$$

which hold whenever $\widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0$ and m_0 lies in the interior of the domains of β and β_Γ (and δ_0 and C_0 depend on the position of m_0), thus under our assumptions (see (2.3), (2.18) and the inclusion in (2.11)). For a detailed proof, see [13, p. 908]; see also [17, Appendix, Prop. A.1].

First a priori estimate. We write (4.2) at the time s and take $v = \mathcal{N}(y_\varepsilon(s) - m_0)$, which is meaningful by (4.19) (see (2.51)). Then, we integrate over $(0,t)$ with respect to s, where t is arbitrary in $(0,T]$. At the same time, we analogously behave with (4.3) by choosing $-(y_\varepsilon - m_0)$ as a test function. Then, we sum the obtained equalities to each other and use (2.56) and (2.52) in order to transform the first integral we get and to cancel the next two ones. Finally, we add two additional terms to both sides for convenience.

We obtain

$$\begin{aligned}
&\frac{1}{2}\,\|y_\varepsilon(t)-m_0\|_*^2+\frac{\tau_\varepsilon}{2}\int_\Omega |y_\varepsilon(t)-m_0|^2+\frac{1}{2}\int_\Gamma |y_{\Gamma,\varepsilon}(t)-m_0|^2\\
&\quad+\int_{Q_t}|\nabla y_\varepsilon|^2+\int_{\Sigma_t}|\nabla_\Gamma y_{\Gamma,\varepsilon}|^2\\
&\quad+\int_{Q_t}\bigl(\beta_\varepsilon(y_\varepsilon)-\beta_\varepsilon(m_0)\bigr)(y_\varepsilon-m_0)+\int_{\Sigma_t}\bigl(\beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}(t))-\beta_{\Gamma,\varepsilon}(m_0)\bigr)(y_{\Gamma,\varepsilon}-m_0)\\
&=\frac{1}{2}\,\|y_0-m_0\|_*^2+\frac{\tau_\varepsilon}{2}\int_\Omega |y_0-m_0|^2+\frac{1}{2}\int_\Gamma |y_0|_\Gamma-m_0|^2-\beta_{\Gamma,\varepsilon}(m_0)\int_{\Sigma_t}(y_{\Gamma,\varepsilon}-m_0)\\
&\quad+\int_{Q_t}\bigl(g_\varepsilon-\lambda\,\pi(y_\varepsilon)\bigr)(y_\varepsilon-m_0)+\int_{\Sigma_t}\bigl(g_\Gamma-\lambda_\Gamma\pi_\Gamma(y_{\Gamma,\varepsilon})\bigr)(y_\varepsilon-m_0). \qquad (4.21)
\end{aligned}$$

The last two integrals on the left-hand side are nonnegative by monotonicity and one term on the right-hand side can be dealt with this way

$$-\,\beta_{\Gamma,\varepsilon}(m_0)\int_{\Sigma_t}(y_{\Gamma,\varepsilon}-m_0)\le c\,|\beta_{\Gamma,\varepsilon}(m_0)|^2+\int_{\Sigma_t}|y_{\Gamma,\varepsilon}-m_0|^2\le c+\int_{\Sigma_t}|y_{\Gamma,\varepsilon}-m_0|^2$$

thanks to the analogue of (4.13) for $\beta_{\Gamma,\varepsilon}$ and to assumption (2.18) on m_0. On the other hand, we can take (2.4)–(2.5) and (2.15) into account and apply (2.50) to $y_\varepsilon-m_0$, in order to estimate the sum of the last two terms on the right-hand side of (4.21) as follows

$$\begin{aligned}
&\int_{Q_t}\bigl(g_\varepsilon-\lambda\,\pi(y_\varepsilon)\bigr)(y_\varepsilon-m_0)+\int_{\Sigma_t}\bigl(g_\Gamma-\lambda_\Gamma\pi_\Gamma(y_{\Gamma,\varepsilon})\bigr)(y_\varepsilon-m_0)\\
&\le c+c\int_{Q_t}|y_\varepsilon-m_0|^2+c\int_{\Sigma_t}|y_{\Gamma,\varepsilon}-m_0|^2\\
&\le c+\frac{1}{2}\int_{Q_t}|\nabla y_\varepsilon|^2+c\int_0^t\|y_\varepsilon(s)-m_0\|_*^2\,ds+c\int_{\Sigma_t}|y_{\Gamma,\varepsilon}-m_0|^2.
\end{aligned}$$

At this point, we combine (4.21) with the above inequalities, rearrage a little and apply the Gronwall lemma. Then, we easily eliminate m_0 in the estimate we obtain. By using (2.50) once more, we recover the L^2 norm of y_ε through the norm of its gradient and eventually conclude that

$$\|y_\varepsilon\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)}+\tau_\varepsilon^{1/2}\|y_\varepsilon\|_{L^\infty(0,T;H)}+\|y_{\Gamma,\varepsilon}\|_{L^\infty(0,T;H_\Gamma)\cap L^2(0,T;V_\Gamma)}\le c\,. \qquad (4.22)$$

Consequence. In view of (2.4)–(2.5), we immediately deduce that

$$\|\lambda\,\pi(y_\varepsilon)\|_{L^2(0,T;H)}+\|\lambda_\Gamma\pi_\Gamma(y_{\Gamma,\varepsilon})\|_{L^2(0,T;H_\Gamma)}\le c\,. \qquad (4.23)$$

Moreover, by virtue of (2.7) if $\tau=0$, we have

$$|\nabla(\lambda\,\pi(y_\varepsilon))|=|\pi(y_\varepsilon)\,\nabla\lambda+\lambda\,\pi'(y_\varepsilon)\nabla y_\varepsilon|\le c\,(|\nabla\lambda|\,|y_\varepsilon|+|\nabla y_\varepsilon|)\,.$$

Hence, by also accounting for the Hölder inequality and the continuous embedding $V\subset L^6(\Omega)$, we deduce that

$$\|\nabla(\lambda\,\pi(y_\varepsilon))\|_{L^2(0,T;H)}\le c\|\nabla\lambda\|_{L^\infty(0,T;L^3(\Omega))}\|y_\varepsilon\|_{L^2(0,T;L^6(\Omega))}+c\|y_\varepsilon\|_{L^2(0,T;V)}\le c$$

and conclude that

$$\|\lambda\,\pi(y_\varepsilon)\|_{L^2(0,T;V)} \le c \quad \text{if } \tau = 0. \tag{4.24}$$

Second a priori estimate. We recall (4.19) and test (4.2) and (4.3) written at the time s by $\mathcal{N}(\partial_t y_\varepsilon(s))$ and $-\partial_t y_\varepsilon(s)$, respectively. We sum the obtained equalities and integrate over $(0,t)$. Then, recalling (2.52) and (2.55) once more, we have that

$$\begin{aligned}
&\int_0^t \|\partial_t y_\varepsilon(s)\|_*^2\,ds + \tau_\varepsilon \int_{Q_t} |\partial_t y_\varepsilon|^2 + \int_{\Sigma_t} |\partial_t y_{\Gamma,\varepsilon}|^2 \\
&\quad + \frac12 \int_\Omega |\nabla y_\varepsilon(t)|^2 + \frac12 \int_\Gamma |\nabla_\Gamma y_{\Gamma,\varepsilon}(t)|^2 + \int_\Omega \widehat{\beta}_\varepsilon(y_\varepsilon(t)) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}(t)) \\
&= \frac12\,\|y_0\|_*^2 + \frac{\tau_\varepsilon}{2} \int_\Omega |y_0|^2 + \frac12 \int_\Gamma |y_0{}_{|\Gamma}|^2 + \int_\Omega \widehat{\beta}_\varepsilon(y_0) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(y_0{}_{|\Gamma}) \\
&\quad + \int_{Q_t} \bigl(g_\varepsilon - \lambda\,\pi(y_\varepsilon)\bigr)\partial_t y_\varepsilon + \int_{\Sigma_t} \bigl(g_\Gamma - \lambda_\Gamma \pi_\Gamma(y_{\Gamma,\varepsilon})\bigr)\partial_t y_{\Gamma,\varepsilon}.
\end{aligned} \tag{4.25}$$

Note that all the terms on the left-hand side are nonnegative (cf. (4.12)) and recall that (2.17) holds. Then, the upper inequalities in (4.12), holding for $\widehat{\beta}_{\Gamma,\varepsilon}$ and β_Γ as well, allow us to infer

$$\int_\Omega \widehat{\beta}_\varepsilon(y_0) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(y_0{}_{|\Gamma}) \le \int_\Omega \widehat{\beta}(y_0) + \int_\Gamma \widehat{\beta}_\Gamma(y_0{}_{|\Gamma}) \le c.$$

Thus, just the last two integrals on the right-hand side need some treatement. By (2.15), (4.23) and the Young inequality (2.46), we immediately have

$$\int_{\Sigma_t} \bigl(g_\Gamma - \lambda_\Gamma \pi_\Gamma(y_{\Gamma,\varepsilon})\bigr)\partial_t y_{\Gamma,\varepsilon} \le c + \frac12\,\|\partial_t y_{\Gamma,\varepsilon}\|_{L^2(0,T;H_\Gamma)}^2.$$

On the contrary, the analogous integral over Q_t is more delicate and we distinguish the cases $\tau > 0$ and $\tau = 0$. In the former, we write

$$\int_{Q_t} \bigl(g_\varepsilon - \lambda\,\pi(y_\varepsilon)\bigr)\partial_t y_\varepsilon \le \|g_\varepsilon - \lambda\,\pi(y_\varepsilon)\|_{L^2(0,T;H)}^2 + \frac{\tau}{2}\,\|\partial_t y_\varepsilon\|_{L^2(0,T;H)}^2 \le c + \frac{\tau}{2}\,\|\partial_t y_\varepsilon\|_{L^2(0,T;H)}^2.$$

Hence, the last term can be absorbed by the left-hand side of (4.25). If $\tau = 0$, we have $g_\varepsilon = g$ and can account for (2.16). In view of (4.22), (4.24) and the interpolation inequality (2.50), we obtain

$$\begin{aligned}
&\int_{Q_t} g_\varepsilon \partial_t y_\varepsilon = \int_\Omega g(t) y_\varepsilon(t) - \int_\Omega g(0) y_0 - \int_{Q_t} \partial_t g\, y_\varepsilon \\
&\le \|g\|_{L^\infty(0,T;H)}^2 + \|y_\varepsilon(t)\|_H^2 + c + \|\partial_t g\|_{L^2(0,T;H)}^2 + \|y_\varepsilon\|_{L^2(0,T;H)}^2 \\
&\le \frac14 \int_\Omega |\nabla y_\varepsilon(t)|^2 + c\|y_\varepsilon(t)\|_*^2 + c \le \frac14 \int_\Omega |\nabla y_\varepsilon(t)|^2 + c
\end{aligned}$$

and

$$-\int_{Q_t} \lambda\,\pi(y_\varepsilon)\partial_t y_\varepsilon \le \frac12 \int_0^t \|\partial_t y_\varepsilon(s)\|_*^2\,ds + c\|\lambda\,\pi(y_\varepsilon)\|_{L^2(0,T;V)}^2 \le \frac12 \int_0^t \|\partial_t y_\varepsilon(s)\|_*^2\,ds + c\,.$$

Thus, the left-hand side of (4.25) dominates also in this case. Therefore, we conclude that

$$\begin{aligned} &\|y_\varepsilon\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)} + \tau_\varepsilon^{1/2}\|y_\varepsilon\|_{H^1(0,T;H)} + \|y_{\Gamma,\varepsilon}\|_{H^1(0,T;H_\Gamma)\cap L^\infty(0,T;V_\Gamma)} \\ &\quad + \|\widehat{\beta}_\varepsilon(y_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|\widehat{\beta}_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})\|_{L^\infty(0,T;L^1(\Gamma))} \le c\,. \end{aligned} \tag{4.26}$$

Third a priori estimate. From (4.26), we first deduce an estimate for ∇w_ε by using (4.2) with $v = w_\varepsilon(t) - (w_\varepsilon(t))_\Omega$ and the Poincaré inequality (2.49). We have for a.a. $t \in (0,T)$

$$\begin{aligned} &\int_\Omega |\nabla w_\varepsilon(t)|^2 = \int_\Omega |\nabla\big(w_\varepsilon(t) - (w_\varepsilon(t))_\Omega\big)|^2 = \langle \partial_t y_\varepsilon(t), w_\varepsilon(t) - (w_\varepsilon(t))_\Omega \rangle \\ &\le C\,\|\partial_t y_\varepsilon(t)\|_*\,\|w_\varepsilon(t) - (w_\varepsilon(t))_\Omega\|_V \le \frac{1}{2}\int_\Omega |\nabla w_\varepsilon(t)|^2 + c\,\|\partial_t y_\varepsilon(t)\|_*^2\,. \end{aligned}$$

Hence, (4.26) immediately yields

$$\|\nabla w_\varepsilon\|_{L^2(0,T;H)} \le c\,. \tag{4.27}$$

In order to recover the full V-norm, we have to estimate the mean value. Inspired by [13, p. 908], we test equations (4.2) and (4.3) as we did for our first estimate, i.e., by $\mathcal{N}(y_\varepsilon - m_0)$ and $-(y_\varepsilon - m_0)$, respectively, but we do not integrate with respect to time. Also in the present case, two terms cancel thanks to (2.52). Thus, for a.a. $t \in (0,T)$ (and we avoid writing t in the sequel, for brevity) we have

$$\begin{aligned} &\int_\Omega |\nabla y_\varepsilon|^2 + \int_\Gamma |\nabla_\Gamma y_{\Gamma,\varepsilon}|^2 + \int_\Omega \xi_\varepsilon(y_\varepsilon - m_0) + \int_\Gamma \xi_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon} - m_0) \\ &= F_\varepsilon := -\langle \partial_t y_\varepsilon, \mathcal{N}(y_\varepsilon - m_0)\rangle + \int_\Omega \big(g_\varepsilon - \lambda\,\pi(y_\varepsilon) - \tau_\varepsilon \partial_t y_\varepsilon\big)(y_\varepsilon - m_0) \\ &\quad + \int_\Gamma \big(g_\Gamma - \lambda_\Gamma \pi_\Gamma(y_{\Gamma,\varepsilon}) - \partial_t y_{\Gamma,\varepsilon}\big)(y_{\Gamma,\varepsilon} - m_0). \end{aligned} \tag{4.28}$$

Now, we account for (4.20) and deduce that

$$\int_\Omega \xi_\varepsilon(y_\varepsilon - m_0) + \int_\Gamma \xi_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon} - m_0) \ge \delta_0 \int_\Omega |\beta_\varepsilon(y_\varepsilon)| + \delta_0 \int_\Gamma |\beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})| - c\,. \tag{4.29}$$

On the other hand, recalling (2.54) and that y_ε and $y_{\Gamma,\varepsilon}$ are bounded in $L^\infty(0,T;V)$ and in $L^\infty(0,T;V_\Gamma)$, respectively (see (4.26)), we deduce for a.a. $t \in (0,T)$

$$\begin{aligned} |F_\varepsilon| &\le c\|\partial_t y_\varepsilon\|_*\,\|y_\varepsilon - m_0\|_* + \big(\|g_\varepsilon\|_H + \|\lambda\,\pi(y_\varepsilon)\|_H + \tau_\varepsilon\|\partial_t y_\varepsilon\|_H\big)\|y_\varepsilon - m_0\|_H \\ &\qquad + \big(\|g_\Gamma\|_{H_\Gamma} + \|\lambda_\Gamma \pi_\Gamma(y_{\Gamma,\varepsilon})\|_{H_\Gamma} + \|\partial_t y_{\Gamma,\varepsilon}\|_{H_\Gamma}\big)\|y_{\Gamma,\varepsilon} - m_0\|_{H_\Gamma} \\ &\le c\|\partial_t y_\varepsilon\|_* + c\big(\|g_\varepsilon\|_H + \|\lambda\,\pi(y_\varepsilon)\|_H + \tau_\varepsilon\|\partial_t y_\varepsilon\|_H\big) \\ &\quad + c\big(\|g_\Gamma\|_{H_\Gamma} + \|\lambda_\Gamma \pi_\Gamma(y_{\Gamma,\varepsilon})\|_{H_\Gamma} + \|\partial_t y_{\Gamma,\varepsilon}\|_{H_\Gamma}\big). \end{aligned}$$

Hence, F_ε is bounded in $L^2(0,T)$ thanks to (4.26), (4.23) and assumptions (2.15), (4.15) on the data. Therefore, even the integrals on the right-hand side of (4.29) are estimated in $L^2(0,T)$. By choosing $v = 1$ in (4.3), we immediately deduce that the same holds for the space integral of w_ε, whence

$$\big\|(w_\varepsilon)_\Omega\big\|_{L^2(0,T)} \le c \tag{4.30}$$

i.e., the mean value of w_ε is estimated in $L^2(0,T)$. By combining this and (4.27) and using the Poincaré inequality (2.49), we eventually infer that

$$\|w_\varepsilon\|_{L^2(0,T;V)} \le c\,. \tag{4.31}$$

Fourth a priori estimate. Our aim is to find a bound for ξ_ε in $L^2(Q)$. To this end, we take $v = \xi_\varepsilon$ in (4.3) and integrate over Ω. We have

$$\begin{aligned}&\int_\Omega \beta_\varepsilon'(y_\varepsilon)|\nabla y_\varepsilon|^2 + \int_\Gamma \beta_\varepsilon'(y_{\Gamma,\varepsilon})|\nabla_\Gamma y_{\Gamma,\varepsilon}|^2 + \int_\Omega |\xi_\varepsilon|^2 + \int_\Gamma \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})\beta_\varepsilon(y_{\Gamma,\varepsilon})\\ &= \int_\Omega \bigl(g - \lambda\,\pi(y_\varepsilon) + w_\varepsilon - \tau\partial_t y_\varepsilon\bigr)\xi_\varepsilon + \int_\Gamma \bigl(g_\Gamma - \lambda_\Gamma\pi_\Gamma(y_{\Gamma,\varepsilon}) - \partial_t y_{\Gamma,\varepsilon}\bigr)\beta_\varepsilon(y_{\Gamma,\varepsilon}).\end{aligned} \tag{4.32}$$

The first three terms on the left-hand side are nonnegative. For the last one, we recall the compatibility condition (4.10): note that the functions β_ε and $\beta_{\Gamma,\varepsilon}$ have the same sign since they are non-decreasing and null at 0 (due to (2.3)). Then, we deduce that

$$\int_\Gamma \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})\beta_\varepsilon(y_{\Gamma,\varepsilon}) \ge \frac{1}{\eta}\int_\Gamma \bigl(|\beta_\varepsilon(y_{\Gamma,\varepsilon})|^2 - C|\beta_\varepsilon(y_{\Gamma,\varepsilon})|\bigr) \ge \frac{1}{2\eta}\int_\Gamma |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^2 - c\,.$$

Let us come to the right-hand side of (4.32). The first two terms are bounded thanks to (4.12), (4.14) and (2.17). Furthermore, using the Young inequality (2.46) we can estimate the last integrals as follows

$$\begin{aligned}&\int_\Omega \bigl(g - \lambda\,\pi(y_\varepsilon) + w_\varepsilon - \tau\partial_t y_\varepsilon\bigr)\xi_\varepsilon \le \frac{1}{2}\int_\Omega |\xi_\varepsilon|^2 + \frac{1}{2}\|g - \lambda\,\pi(y_\varepsilon) + w_\varepsilon - \tau\partial_t y_\varepsilon\|_H^2\\ &\int_\Gamma \bigl(g_\Gamma - \lambda_\Gamma\pi_\Gamma(y_{\Gamma,\varepsilon}) - \partial_t y_{\Gamma,\varepsilon}\bigr)\beta_\varepsilon(y_{\Gamma,\varepsilon}) \le \frac{1}{4\eta}\int_\Gamma |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^2 + \eta\,\|g_\Gamma - \lambda_\Gamma\pi_\Gamma(y_{\Gamma,\varepsilon}) - \partial_t y_{\Gamma,\varepsilon}\|_{H_\Gamma}^2\end{aligned}$$

and remark that, thanks to (4.23), (4.26) and (4.31), the last terms in the above inequalites are uniformly bounded in $L^1(0,T)$. Hence, by combining and integrating over $(0,T)$, we find out that

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} \le c. \tag{4.33}$$

Consequence. By partially repeating the argument used in the proof of Theorem 4.1 and noting that each deduction has a corresponding estimate, we derive the following chain of bounds

$$\|\Delta y_\varepsilon\|_{L^2(0,T;H)} \le c, \quad \|y_\varepsilon\|_{L^2(0,T;H^{3/2}(\Omega))} \le c, \quad \|\partial_n y_{\varepsilon|_\Gamma}\|_{L^2(0,T;H_\Gamma)} \le c.$$

By comparison in (4.18) with the help of (4.26), we conclude that

$$\|-\Delta_\Gamma y_{\Gamma,\varepsilon} + \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})\|_{L^2(0,T;H_\Gamma)} \le c\,. \tag{4.34}$$

Fifth a priori estimate. By (4.34), we can simply write

$$-\Delta_\Gamma y_{\Gamma,\varepsilon} + \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}) = F_\varepsilon\,, \quad \text{with} \quad \|F_\varepsilon\|_{L^2(0,T;H_\Gamma)} \le c$$

and multiply such an equation by $\beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})$. We immediately obtain

$$\int_\Sigma \beta'_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})|\nabla_\Gamma y_{\Gamma,\varepsilon}|^2 + \int_\Sigma |\beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})|^2 = \int_\Sigma F_\varepsilon\, \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}) \le \frac{1}{2}\int_\Sigma |\beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})|^2 + c$$

and infer that

$$\|\beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})\|_{L^2(0,T;H_\Gamma)} \le c\,. \tag{4.35}$$

Consequence. By entering the proof of Theorem 4.1 once more and arguing as above, we deduce the following chain of bounds

$$\|\Delta_\Gamma y_{\Gamma,\varepsilon}\|_{L^2(0,T;H_\Gamma)} \le c, \quad \|y_{\Gamma,\varepsilon}\|_{L^2(0,T;H^2(\Gamma))} \le c, \quad \|y_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \le c\,.$$

Therefore, we conclude that

$$\|y_{\Gamma,\varepsilon}\|_{L^2(0,T;H^2(\Gamma))} \le c \quad \text{and} \quad \|y_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \le c \tag{4.36}$$

whence also

$$\|y_{\Gamma,\varepsilon}\|_{L^2(0,T;L^\infty(\Gamma))} \le c \quad \text{and} \quad \|y_\varepsilon\|_{L^2(0,T;L^\infty(\Omega))} \le c \tag{4.37}$$

thanks to the continuous embeddings (2.47)–(2.48).

Conclusion of the proof of Theorem 2.3. Recalling all the estimates we have obtained, we see that the following convergence holds true

$$\begin{aligned}
y_\varepsilon \to y &\quad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) && (4.38)\\
\partial_t y_\varepsilon \to \partial_t y &\quad \text{weakly in } L^2(0,T;H) \ \text{ if } \tau > 0 && (4.39)\\
y_{\Gamma,\varepsilon} \to y_\Gamma &\quad \text{weakly star in } H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;H^2(\Gamma)) && (4.40)\\
w_\varepsilon \to w &\quad \text{weakly in } L^2(0,T;V) && (4.41)\\
\xi_\varepsilon \to \xi &\quad \text{weakly in } L^2(0,T;H) && (4.42)\\
\xi_{\Gamma,\varepsilon} \to \xi_\Gamma &\quad \text{weakly in } L^2(0,T;H_\Gamma) && (4.43)
\end{aligned}$$

at least for a subsequence, in principle. Clearly, y satisfies the Cauchy condition (2.27), y_Γ is the trace of y, and the quintuplet $(y, y_\Gamma, w, \xi, \xi_\Gamma)$ satisfies (2.28) and a variational equation like (2.29), where the terms related to π and π_Γ are not yet identified. Moreover, the relationships contained in (2.23)–(2.24) have to be proved. Thanks to, e.g., [21, Sect. 8, Cor. 4], it is not difficult to infer that

$$\begin{aligned}
y_\varepsilon \to y &\quad \text{strongly in } C^0([0,T];H) && (4.44)\\
y_{\Gamma,\varepsilon} \to y_\Gamma &\quad \text{strongly in } C^0([0,T];H). && (4.45)
\end{aligned}$$

Then, recalling (2.4), we deduce that $\pi(y_\varepsilon)$ and $\pi_\Gamma(y_{\Gamma,\varepsilon})$ converge to $\pi(y)$ and to $\pi_\Gamma(y_\Gamma)$ in $C^0([0,T];H)$ and in $C^0([0,T];H_\Gamma)$, respectively. Moreover, by applying well-known results on maximal monotone operators (see, e.g., [1, Lemma 1.3, p. 42]), we infer that $\xi \in \beta(y)$ a.e. in Q and $\xi_\Gamma \in \beta_\Gamma(y_\Gamma)$ a.e. on Σ. This completes the proof of Theorem 2.3. $\square$

The rest of the section is devoted to the proof of our regularity results. We start with Theorem 2.4 and thus suppose that its assumptions are satisfied. Then we can take

$g_\varepsilon = g$ in both cases. As observed at the beginning, just further a priori estimates on the solution to the approximating problem are necessary. In order to confine the length of the paper, we proceed formally, by assuming that the solution to the approximating problem is as smooth as needed. We prepare a lemma.

Lemma 4.2. *We have*

$$\|\partial_t y_\varepsilon(0)\|_* + \tau_\varepsilon^{1/2} \|\partial_t y_\varepsilon(0)\|_H + \|\partial_t y_{\Gamma,\varepsilon}(0)\|_{H_\Gamma} \le c\,. \tag{4.46}$$

Proof. The values $\partial_t y_\varepsilon(0)$ and $\partial_t y_{\Gamma,\varepsilon}(0)$ can be obtained by taking $t = 0$ in (4.2)–(4.3). Hence, they satisfy

$$\langle \partial_t y_\varepsilon(0), v\rangle + \int_\Omega \nabla w_\varepsilon(0) \cdot \nabla v = 0 \tag{4.47}$$

$$\int_\Omega w_\varepsilon(0) v = \tau_\varepsilon \int_\Omega \partial_t y_\varepsilon(0)\, v + \int_\Gamma \partial_t y_{\Gamma,\varepsilon}(0)\, v + \int_\Omega \nabla y_0 \cdot \nabla v + \int_\Gamma \nabla_\Gamma (y_0{}_{|_\Gamma}) \cdot \nabla_\Gamma v$$
$$+ \int_\Omega \bigl(\beta_\varepsilon(y_0) + \lambda(0)\, \pi(y_0) - g_\varepsilon(0)\bigr) v + \int_\Gamma \bigl(\beta_{\Gamma,\varepsilon}(y_0{}_{|_\Gamma}) + \lambda_\Gamma(0)\, \pi_\Gamma(y_0{}_{|_\Gamma}) - g_\Gamma(0)\bigr) v \tag{4.48}$$

for every $v \in V$ and every $v \in \mathcal{V}$, respectively. We choose $v = \mathcal{N}(\partial_t y_\varepsilon(0))$ and $v = -\partial_t y_\varepsilon(0)$ in (4.47) and in (4.48), respectively, sum the obtained equalities to each other and exploit (2.52) and (2.55), as usual. By observing that $\partial_t y_{\Gamma,\varepsilon}(0) = (\partial_t y_\varepsilon(0))_{|_\Gamma}$ (since y_ε is smooth), integrating by parts both in Ω (with the help of our assumption (2.37) on y_0) and on Γ, and rearranging a little, we have

$$\begin{aligned}
&\|\partial_t y_\varepsilon(0)\|_*^2 + \tau_\varepsilon \|\partial_t y_\varepsilon(0)\|_H^2 + \|\partial_t y_{\Gamma,\varepsilon}(0)\|_{H_\Gamma}^2 \\
&= - \int_\Omega \bigl(-\Delta y_0 + \beta_\varepsilon(y_0) + \lambda(0)\, \pi(y_0) - g_\varepsilon(0)\bigr) \partial_t y_\varepsilon(0) \\
&\quad - \int_\Gamma \bigl(-\Delta_\Gamma y_0{}_{|_\Gamma} + \beta_{\Gamma,\varepsilon}(y_0{}_{|_\Gamma}) + \lambda_\Gamma(0)\, \pi_\Gamma(y_0{}_{|_\Gamma}) - g_\Gamma(0)\bigr) \partial_t y_{\Gamma,\varepsilon}(0).
\end{aligned}$$

The last integral can be easily handled by using (2.37), (2.39), (4.13), assumptions (2.4)–(2.5) on π_Γ and λ_Γ, and (2.36). Then, we deduce that

$$- \int_\Gamma \bigl(-\Delta_\Gamma y_0{}_{|_\Gamma} + \beta_{\Gamma,\varepsilon}(y_0{}_{|_\Gamma}) + \lambda_\Gamma(0)\, \pi_\Gamma(y_0{}_{|_\Gamma}) - g_\Gamma(0)\bigr) \partial_t y_{\Gamma,\varepsilon}(0) \le \frac{1}{2}\, \|\partial_t y_{\Gamma,\varepsilon}(0)\|_H^2 + c\,.$$

We can deal with the integral over Ω in a similar way if $\tau > 0$ (e.g., (2.38) replaces (2.39) in the argument). In this case, we obtain

$$- \int_\Omega \bigl(-\Delta y_0 + \beta_\varepsilon(y_0) + \lambda(0)\, \pi(y_0) - g_\varepsilon(0)\bigr) \partial_t y_\varepsilon(0) \le \frac{\tau_\varepsilon}{2}\, \|\partial_t y_\varepsilon(0)\|_H^2 + c\,.$$

On the contrary, if $\tau = 0$, the treatment of the integral is more delicate and requires the help of (2.40), (2.7) and (2.50). We have

$$\begin{aligned}
&- \int_\Omega \bigl(-\Delta y_0 + \beta_\varepsilon(y_0) + \lambda(0)\, \pi(y_0) - g_\varepsilon(0)\bigr) \partial_t y_\varepsilon(0) \\
&\le \frac{1}{2}\, \|\partial_t y_\varepsilon(0)\|_*^2 + c\, \|-\Delta y_0 + \beta_\varepsilon(y_0) + \lambda(0)\, \pi(y_0) - g_\varepsilon(0)\|_V^2 \\
&\le \frac{1}{2}\, \|\partial_t y_\varepsilon(0)\|_*^2 + c\, \|-\Delta y_0 + \beta_\varepsilon(y_0) - g_\varepsilon(0)\|_V^2 + c\, \|\lambda(0)\, \pi(y_0)\|_V^2 \le \frac{1}{2}\, \|\partial_t y_\varepsilon(0)\|_*^2 + c\,.
\end{aligned}$$

In both cases, we can combine and conclude that (4.46) holds true. $\square$

Sixth a priori estimate. We differentiate equations (4.2) and (4.3) with respect to time and obtain for a.a. $t \in (0,T)$ (but we avoid writing the time t everywhere, for brevity)

$$\langle \partial_t^2 y_\varepsilon, v\rangle + \int_\Omega \nabla \partial_t w_\varepsilon \cdot \nabla v = 0 \tag{4.49}$$

$$\begin{aligned}\int_\Omega \partial_t w_\varepsilon v &= \tau_\varepsilon \int_\Omega \partial_t^2 y_\varepsilon \, v + \int_\Gamma \partial_t^2 y_{\Gamma,\varepsilon} \, v + \int_\Omega \nabla \partial_t y_\varepsilon \cdot \nabla v + \int_\Gamma \nabla_\Gamma \partial_t y_{\Gamma,\varepsilon} \cdot \nabla_\Gamma v \\ &\quad + \int_\Omega \bigl(\beta_\varepsilon'(y_\varepsilon)\partial_t y_\varepsilon + (\partial_t \lambda)\, \pi(y_\varepsilon) + \lambda\, \pi'(y_\varepsilon)\partial_t y_\varepsilon - \partial_t g_\varepsilon\bigr)\, v \\ &\quad + \int_\Gamma \bigl(\beta_{\Gamma,\varepsilon}'(y_{\Gamma,\varepsilon})\partial_t y_{\Gamma,\varepsilon} + (\partial_t \lambda_\Gamma)\pi_\Gamma(y_{\Gamma,\varepsilon}) + \lambda_\Gamma \pi_\Gamma'(y_{\Gamma,\varepsilon})\partial_t y_{\Gamma,\varepsilon} - \partial_t g_\Gamma\bigr)\, v\end{aligned} \tag{4.50}$$

where (4.2) and (4.3) are required to hold for every $v \in V$ and every $v \in \mathcal{V}$, respectively. Now, we note that (4.19) implies $\partial_t y_\varepsilon(t) \in D(\mathcal{N})$ for a.a. $t \in (0,T)$ (cf. (2.51)). So, we test the above equations by $\mathcal{N}(\partial_t y_\varepsilon)$ and $-\partial_t y_\varepsilon$, integrate over $(0,t)$, sum up, and account for (2.52) and (2.55), as usual. We obtain

$$\begin{aligned}&\frac{1}{2}\, \|\partial_t y_\varepsilon(t)\|_*^2 + \frac{\tau_\varepsilon}{2} \int_\Omega |\partial_t y_\varepsilon(t)|^2 + \frac{1}{2} \int_\Gamma |\partial_t y_{\Gamma,\varepsilon}(t)|^2 \\ &\quad + \int_{Q_t} |\nabla \partial_t y_\varepsilon|^2 + \int_{\Sigma_t} |\nabla_\Gamma \partial_t y_{\Gamma,\varepsilon}|^2 + \int_{Q_t} \beta_\varepsilon'(y_\varepsilon)|\partial_t y_\varepsilon|^2 + \int_{\Sigma_t} \beta_{\Gamma,\varepsilon}'(y_{\Gamma,\varepsilon})|\partial_t y_{\Gamma,\varepsilon}|^2 \\ &= \frac{1}{2}\, \|\partial_t y_\varepsilon(0)\|_*^2 + \frac{\tau_\varepsilon}{2} \int_\Omega |\partial_t y_\varepsilon(0)|^2 + \frac{1}{2} \int_\Gamma |\partial_t y_{\Gamma,\varepsilon}(0)|^2 \\ &\quad + \int_{Q_t} \bigl(\partial_t g_\varepsilon - (\partial_t \lambda)\pi(y_\varepsilon) - \lambda\, \pi'(y_\varepsilon)\partial_t y_\varepsilon\bigr)\partial_t y_\varepsilon \\ &\quad + \int_{\Sigma_t} \bigl(\partial_t g_\Gamma - (\partial_t \lambda_\Gamma)\pi_\Gamma(y_{\Gamma,\varepsilon}) - \lambda_\Gamma \pi_\Gamma'(y_{\Gamma,\varepsilon})\partial_t y_{\Gamma,\varepsilon}\bigr)\partial_t y_{\Gamma,\varepsilon}\,.\end{aligned} \tag{4.51}$$

All the integrals on the left-hand side are nonnegative and the first three terms on the right-hand side are bounded thanks to Lemma 4.2. The next integral is estimated as follows

$$\begin{aligned}&\int_{Q_t} \bigl(\partial_t g_\varepsilon - (\partial_t \lambda)\pi(y_\varepsilon) - \lambda\, \pi'(y_\varepsilon)\partial_t y_\varepsilon\bigr)\partial_t y_\varepsilon \\ &\leq c \int_{Q_t} |\partial_t y_\varepsilon|^2 + \|\partial_t g_\varepsilon\|_{L^2(0,T;H)}^2 + \|(\partial_t \lambda)\pi(y_\varepsilon)\|_{L^2(0,T;H)}^2 \\ &\leq c \int_{Q_t} |\partial_t y_\varepsilon|^2 + c + c\|\partial_t \lambda\|_{L^\infty(0,T;H)}^2 \|y_\varepsilon\|_{L^2(0,T;L^\infty(\Omega))}^2 \\ &\leq \frac{1}{2} \int_{Q_t} |\nabla \partial_t y_\varepsilon|^2 + c \int_0^t \|\partial_t y_\varepsilon(s)\|_*^2 \, ds + c\end{aligned}$$

thanks to (2.35), the boundedness of π' (cf. (2.4)), (2.36), the interpolation inequality (2.50) and (4.37). As the last integral can be treated in a similar and even simpler way, from (4.51) we conclude that

$$\begin{aligned}&\|\partial_t y_\varepsilon\|_{L^\infty(0,T;V^*)} + \tau_\varepsilon^{1/2} \|\partial_t y_\varepsilon\|_{L^\infty(0,T;H)} + \|\partial_t y_{\Gamma,\varepsilon}\|_{L^\infty(0,T;H_\Gamma)} \\ &\quad + \|\partial_t y_\varepsilon\|_{L^2(0,T;V)} + \|\partial_t y_{\Gamma,\varepsilon}\|_{L^2(0,T;V_\Gamma)} \leq c\,.\end{aligned} \tag{4.52}$$

Conclusion of the proof of Theorem 2.4. Due to the existence proof, we already know that $(y_\varepsilon, y_{\Gamma,\varepsilon}, w_\varepsilon, \xi_\varepsilon, \xi_{\Gamma,\varepsilon})$ converges to the solution to problem (2.19)–(2.27). Thus, the last estimate of ours just improves the regularity of the limit (as well as the topology of the convergence), and (2.32)–(2.33) are partially proved. In order to achieve the H^2 regularity requirements of the statement, one can argue as in the proof of Theorem 2.3, going from the Third a priori estimate to the Conclusion of the proof. In the argument used there, time was just a parameter, indeed, since everything was based on the theory of elliptic regularity and traces in Ω. In the present case one immediately checks that L^∞ instead of L^2 bounds hold with respect to time. Hence, we get, in this order,

$$\|w_\varepsilon\|_{L^\infty(0,T;V)} \le c \tag{4.53}$$

$$\|\xi_\varepsilon\|_{L^\infty(0,T;H)} \le c \tag{4.54}$$

$$\|\xi_{\Gamma,\varepsilon}\|_{L^\infty(0,T;H)} \le c \tag{4.55}$$

$$\|y_\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} + \|y_{\Gamma,\varepsilon}\|_{L^\infty(0,T;H^2(\Gamma))} \le c \tag{4.56}$$

i.e., the last part of (2.32)–(2.33) as well as (2.41). This completes the proof. □

Proof of Theorem 2.6. We recall that $\tau > 0$. Hence, $\partial_t y_\varepsilon$ is bounded in $L^\infty(0,T;H)$. On the other hand, equation (4.2) implies that $\partial_t y_\varepsilon - \Delta w_\varepsilon = 0$ in Q and $\partial_n w_\varepsilon = 0$ on the boundary, whence

$$\|\Delta w_\varepsilon\|_{L^\infty(0,T;H)} \le c \quad \text{and} \quad \|w_\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \le c\,. \tag{4.57}$$

This implies the first conclusion in (2.43). In order to prove boundedness for ξ, it suffices to find an a priori estimate for the L^p norm of ξ_ε that is uniform with respect to both p and ε. Thus, in the sequel, the dependence on p of the constants is explicitely written and carefully controlled. In order to perform our estimate, we write (4.3) as

$$\begin{aligned} &\tau \int_\Omega \partial_t y_\varepsilon\, v + \int_\Gamma \partial_t y_{\Gamma,\varepsilon}\, v + \int_\Omega \nabla y_\varepsilon \cdot \nabla v + \int_\Gamma \nabla_\Gamma y_{\Gamma,\varepsilon} \cdot \nabla_\Gamma v + \int_\Omega \beta_\varepsilon(y_\varepsilon) v + \int_\Gamma \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}) v \\ &= \int_\Omega f_\varepsilon v + \int_\Gamma f_{\Gamma,\varepsilon} v \quad \text{a.e. in } (0,T) \text{ and for every } v \in \mathcal{V} \end{aligned} \tag{4.58}$$

where $f_\varepsilon := w + g - \lambda\, \pi(y_\varepsilon)$ and $f_{\Gamma,\varepsilon} := g_\Gamma - \lambda_\Gamma \pi_\Gamma(y_{\Gamma,\varepsilon})$, and observe that

$$\|f_\varepsilon\|_{L^\infty(Q)} \le c \quad \text{and} \quad \|f_{\Gamma,\varepsilon}\|_{L^\infty(\Sigma)} \le c \tag{4.59}$$

due to (2.42) and (4.56)–(4.57). Then, we test (4.58) by $|\beta_\varepsilon(y_\varepsilon)|^{p-1} \operatorname{sign} y_\varepsilon$ with an arbitrary $p > 2$, where the sign function is extended by $\operatorname{sign} 0 = 0$, and integrate over $(0,t)$. We have

$$\begin{aligned} &\tau \int_\Omega B_{p,\varepsilon}(y_\varepsilon(t)) + \int_\Gamma B_{p,\varepsilon}(y_{\Gamma,\varepsilon}(t)) \\ &\quad + (p-1) \int_\Omega \beta'_\varepsilon(y_\varepsilon) |\beta_\varepsilon(y_\varepsilon)|^{p-2} |\nabla y_\varepsilon|^2 + (p-1) \int_\Gamma \beta'_\varepsilon(y_{\Gamma,\varepsilon}) |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^{p-2} |\nabla_\Gamma y_{\Gamma,\varepsilon}|^2 \\ &\quad + \int_{Q_t} |\beta_\varepsilon(y_\varepsilon)|^p + \int_{\Sigma_t} \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon})\, |\beta_\varepsilon(y_\varepsilon)|^{p-1} \operatorname{sign} y_{\Gamma,\varepsilon} \\ &= \tau \int_\Omega B_{p,\varepsilon}(y_0) + \int_\Gamma B_{p,\varepsilon}(y_0{}_{|\Gamma}) \\ &\quad + \int_{Q_t} f_\varepsilon\, |\beta_\varepsilon(y_\varepsilon)|^{p-1} \operatorname{sign} y_\varepsilon + \int_{\Sigma_t} f_{\Gamma,\varepsilon} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^{p-1} \operatorname{sign} y_{\Gamma,\varepsilon} \end{aligned} \tag{4.60}$$

where we have set

$$B_{p,\varepsilon}(r) := \int_0^r |\beta_\varepsilon(s)|^{p-1} \operatorname{sign} s \, ds \quad \text{for } r \in \mathbb{R}. \tag{4.61}$$

We recall that β_ε and $\beta_{\Gamma,\varepsilon}$ are monotone functions that vanish at the origine. It follows that they and the identity map have the same sign, whence all the terms on the left-hand side of (4.60) are nonnegative. Moreover, the last of them can be estimated from below on account of the compatibility condition (2.11) and of the Young inequality (2.46) (with p' in place of p and $\delta > 0$ to be chosen) as follows

$$\begin{aligned}
&\int_{\Sigma_t} \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}) \, |\beta_\varepsilon(y_\varepsilon)|^{p-1} \operatorname{sign} y_{\Gamma,\varepsilon} \geq \int_{\Sigma_t} \bigl(\eta |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^p - C |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^{p-1} \bigr) \\
&\geq \eta \int_{\Sigma_t} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^p - \int_{\Sigma_t} \Bigl(\frac{\delta}{p'} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^{(p-1)p'} + \frac{\delta^{-p/p'}}{p} \, C^p \Bigr) \\
&= \eta \int_{\Sigma_t} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^p - \frac{\delta}{p'} \int_{\Sigma_t} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^p - c \, \frac{\delta^{-p/p'}}{p} \, C^p .
\end{aligned}$$

By choosing $\delta = \eta p'/2$, we conclude that

$$\int_{\Sigma_t} \beta_{\Gamma,\varepsilon}(y_{\Gamma,\varepsilon}) \, |\beta_\varepsilon(y_\varepsilon)|^{p-1} \operatorname{sign} y_{\Gamma,\varepsilon} \geq \frac{\eta}{2} \int_{\Sigma_t} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^p - c^p \, . \tag{4.62}$$

Let us come to the right-hand side and denote by M the L^∞ norm of $\beta^\circ(y_0)$ (cf. (2.42)). By (4.13), we deduce that

$$|\beta_\varepsilon(y_0)| \leq M \quad \text{a.e. in } \Omega \quad \text{and} \quad |\beta_\varepsilon(y_0{}_{|\Gamma})| \leq M \quad \text{a.e. on } \Gamma.$$

Thus, on account of (4.61) and of the L^∞ bound that follows from (4.56), we can estimate the sum of the first two integrals this way

$$\tau \int_\Omega B_{p,\varepsilon}(y_0) + \int_\Gamma B_{p,\varepsilon}(y_0{}_{|\Gamma}) \leq c M^{p-1} \leq c^p .$$

Now, we consider the volume integral. We recall (4.59), apply the Young inequality and have

$$\int_\Omega f_\varepsilon \, |\beta_\varepsilon(y_\varepsilon)|^{p-1} \operatorname{sign} y_\varepsilon \leq \int_{Q_t} \Bigl(\frac{1}{p} \, c^p + \frac{1}{p'} |\beta_\varepsilon(y_\varepsilon)|^{(p-1)p'} \Bigr) \leq c^p + \frac{1}{p'} \int_{Q_t} |\beta_\varepsilon(y_\varepsilon)|^p .$$

Arguing as for (4.62), we obtain

$$\int_{\Sigma_t} f_{\Gamma,\varepsilon} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^{p-1} \operatorname{sign} y_{\Gamma,\varepsilon} \leq \frac{\eta}{4} \int_{\Sigma_t} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^p + c^p .$$

By collecting (4.60) (where we neglect a number of nonnegative terms on the left-hand side), (4.62) and the last two estimates, and rearranging, we infer that

$$\frac{1}{p} \int_{Q_t} |\beta_\varepsilon(y_\varepsilon)|^p + \frac{\eta}{4} \int_{\Sigma_t} |\beta_\varepsilon(y_{\Gamma,\varepsilon})|^p \leq c^p$$

and easily conclude that

$$\|\beta_\varepsilon(y_\varepsilon)\|_{L^p(Q)} + \|\beta_\varepsilon(y_{\Gamma,\varepsilon})\|_{L^p(\Sigma)} \leq c \, .$$

This completes the proof. □

Remark 4.3. The above proof also provides an L^∞ bound for $\beta_\varepsilon(y_{\Gamma,\varepsilon})$. This implies that

$$\beta_\varepsilon(y_{\Gamma,\varepsilon}) \to \zeta \quad \text{weakly star in } L^\infty(\Sigma)$$

for some $\zeta \in L^\infty(\Sigma)$ and at least for a subsequence. This and the strong convergence of $y_{\Gamma,\varepsilon}$, e.g., in $L^2(\Sigma)$ yield $\zeta \in \beta(y_\Gamma)$ by maximal monotonicity. Hence, we have also proved that some selection of $\beta(y_\Gamma)$ is bounded on Σ. On the contrary, nothing can be inferred as far as $\beta_\Gamma(y_\Gamma)$ is concerned, unless the assumptions of Corollary 2.7 are supposed to hold.

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