Existence, uniqueness, and longtime behavior for a nonlinear Volterra integrodifferential equation

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Abstract. We consider an initial and boundary value problem for a nonlinear Volterra integrodifferential equation. This equation governs the evolution of a pair of state variables, u and ϑ, which are mutually related by a maximal monotone graph γ in $\mathbb{R}\times\mathbb{R}$. The model can be viewed, for instance, as a generalized Stefan problem within the theory of heat conduction in materials with memory. Besides, it can be used for describing some diffusion processes in fractured media. The relation defined by γ is properly interpreted and generalized in terms of a subdifferential operator associated with γ and acting from $H^1(\Omega)$ to its dual space. Then, the generalized problem is formulated as an abstract Cauchy problem for a perturbation of a nonlinear semigroup, and existence and uniqueness of a solution (u,ϑ) can be proved via a fixed point argument whatever the maximal monotone graph γ is. Moreover, the meaning of γ as a pointwise relationship is recovered almost everywhere, in the case when γ is bounded on bounded subsets of $\mathbb{R}$. Finally, under some other restrictions on γ, the longtime behavior of the solution is investigated, in a more specific context related to the generalized Stefan problem.

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1. Introduction

The present analysis regards a nonlinear Volterra integrodifferential equation which can describe the dynamic behavior of different phenomena like, e.g., phase transitions in materials with memory or diffusion in fractured media (let us refer to the papers [10, 3] at once, since they are of related interest). This equation rules the evolution of two unknown fields, u and ϑ, which must also satisfy a relation induced by a maximal monotone graph $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$. More precisely, letting Ω be a bounded and connected domain in $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary Γ, the equation we are going to study can be written in the form

$$(u + \varphi * u)_t - \Delta(k_0 \vartheta + k * \vartheta) = g \qquad \text{in } \Omega \times (0, +\infty) \tag{1.1}$$

$$u \in \gamma(\vartheta) \qquad \text{in } \Omega \times (0, +\infty) \tag{1.2}$$

where $*$ denotes the usual time convolution product on $(0,t)$, namely $(a * b)(t) := \int_0^t a(t-s)b(s)ds$, $t > 0$, Δ stands for the Laplacian with respect to the space variables, φ and k are given time–dependent memory kernels, k_0 is a positive constant, and g is a known source term.

Referring to the heat conduction theory for materials with memory (see, e.g., [6, 8, 11] and references therein), we can interpret u and ϑ as the enthalpy and the (relative) temperature, respectively. Then we introduce the constitutive assumptions

$$e(x,t) = u(x,t) + \int_{-\infty}^t \varphi(t-s)u(x,s)ds$$

$$\mathbf{q}(x,t) = -k_0 \nabla \vartheta(x,t) - \int_{-\infty}^t k(t-s) \nabla \vartheta(x,s)ds$$

for any $(x,t) \in \Omega \times \mathbb{R}$. Here e is the internal energy, while $\mathbf{q}$ is the heat flux. Assume now that the past histories of u and ϑ are known up to $t = 0$ and consider the balance equation

$$e_t + \nabla \cdot \mathbf{q} = \widetilde{g} \qquad \text{in } \Omega \times (0, +\infty)$$

where $\nabla \cdot$ is the spatial divergence operator and $\widetilde{g}$ is the heat supply. By plugging constitutive laws in this equality, we deduce equation (1.1) in which g depends both on $\widetilde{g}$ and on the past histories of u and ϑ. Relationship (1.2) characterizes the phase transition occuring in the material. In particular, if $\gamma(r) = r + H(r)$, $r \in \mathbb{R}$ and H denotes the Heaviside graph $(H(r) = 0$ if $r < 0$, $H(0) = [0, 1]$, $H(r) = 1$ if $r > 0)$, then the system (1.1–2) defines the Stefan problem for materials with memory that has been examined, for instance, in [6, 8, 11].

As far as diffusion in fractured media is concerned, we take $k \equiv 0$ and rewrite equation (1.1) in the form

$$u_t + \varphi * u_t - k_0 \Delta \vartheta = \widehat{g} \qquad \text{in } \Omega \times (0, +\infty)$$

where $\widehat{g} := g - \varphi u_0$. In the case $\gamma(r) = |r|^{\eta-1}r,\ r \in \mathbb{R}$, for some $\eta > 0$, equations (1.1–2) turn out to yield a generalization of the macro–model obtained via homogenization by Hornung and Showalter (cf. [17, eqs. (2.2a–b) and eq. (3.1)]). This model describes the evolution of fluid density u in the fracture system and, consequently, the dynamics of the fluid density z in the block system by means of the relation $z_t = \varphi' * u_t$.

Initial and boundary value problems for (1.1–2) have been already formulated and analyzed in [10] and [3] (see also [15, 19] and their references for analogous problems). Nevertheless, the case of third–type boundary condition has not been solved yet in full generality, namely for an arbitrary maximal monotone graph γ . This kind of boundary conditions is perhaps the most resonable one from the physical viewpoint, when we are dealing with phase transition problems. On the other hand, in the present case the third–type boundary condition looks more difficult to handle in the variational setting due to the presence of memory kernels, and it leads to a more general formulation (see Section 3). Here we want to prove that initial and third–type boundary value problems associated with (1.1–2) admit a unique solution for a significant class of graphs γ . Thus, we consider the following conditions

$$\partial_n(k_0\vartheta + k * \vartheta) + \alpha(\vartheta - h) = 0 \qquad \text{on } \Gamma \times (0, +\infty) \tag{1.3}$$

$$u(0) = u_0 \qquad \text{in } \Omega \tag{1.4}$$

where ∂_n denotes the outward normal partial derivative on Γ , α is a positive constant and h , u_0 are given functions. Note that (1.3) derives from some law stating that the normal component of the heat flux $\mathbf{q}$ on the boundary is proportional to the difference of internal and external temperatures.

The first part of this paper is devoted to investigate existence and uniqueness of the solution to a suitable variational formulation of (1.1–4) on a finite time interval $(0, T)$, trying to keep the maximum of generality for γ . The main difficulty lies in the following fact. If γ is an arbitrary maximal monotone graph, then we are not able to find a solution (u, ϑ) of (1.1–4) such that u is, at least, a measurable function defined almost everywhere in $Q := \Omega \times (0, T)$. Consequently, we cannot show that relation (1.2) holds almost everywhere in Q . Thus our strategy is based on the interpretation of (1.2) as a relation given by a maximal monotone operator in the dual space of $H^1(\Omega)$ associated with γ in a quite natural way (see Section 2). This argument allows us to formulate a generalized version of (1.1–2) which just requires $u(t)$ to be a suitable functional for any $t \in [0, T]$. Then we can discuss an extended version of our original variational formulation of (1.1–4), for which we can prove existence and uniqueness. The validity of (1.2) almost everywhere in Q can be recovered for the solution in the situation when the domain of γ is the whole real line (see Sections 3 and 4).

The second part of the work (see Section 5) is concerned with the asymptotic behavior of the solution to (1.1–4) as t goes to $+\infty$. In this case, our results are obtained under some restrictions on γ and for $\varphi \equiv 0$. However, the hypotheses on γ and k are general enough to cover the case of Stefan problems with heat conduction laws of memory type. In particular, k is assumed to be smooth enough and to fulfill the inequality

$$\int_0^t (k_0 v + k * v)(s)\, v(s)\, ds \geq \omega \int_0^t |v(s)|^2\, ds \qquad \forall\, v \in L^2(0,t), \quad \forall\, t \in (0,+\infty) \quad (1.5)$$

for some positive constant ω. It is worth recalling that condition (1.5) is consistent with the Second Principle of Thermodynamics (as it is claimed, for instance, in [14]). Let us point out that the longtime behavior has been already investigated for related equations with memory terms in, e.g., [2, 4, 18, 19]. In our framework, we show that, as $t \to +\infty$, $\vartheta(t)$ suitably converges to the (unique) solution ϑ_∞ of the stationary problem

$$-k_\infty \Delta \vartheta_\infty = g_\infty \qquad \text{in } \Omega \quad (1.6)$$

$$k_\infty \frac{\partial \vartheta_\infty}{\partial n} + \alpha(\vartheta_\infty - h_\infty) = 0 \qquad \text{on } \Gamma \quad (1.7)$$

where $g_\infty = \lim g(t)$ and $h_\infty = \lim h(t)$ as $t \to +\infty$, and (cf. (1.5))

$$k_\infty := k_0 + \int_0^\infty k(s)\, ds \geq \omega.$$

Moreover, we deduce that $u_t(t)$ tends to 0 in a specified weak sense. Finally, slightly stronger assumptions on γ allow us to describe the $\omega-$ limit set of the trajectories of $u(t)$. In addition, we give sufficient conditions in order that $u(t)$ has a limit as $t \to +\infty$.

2. Preliminary results

This section is devoted to discuss some preliminary issues on the convex functionals which are related to maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$. We set for convenience

$$V = H^1(\Omega) \qquad \text{and} \qquad H = L^2(\Omega)$$

and identify H with its dual space H', so that

$$V \subset H \subset V' \quad (2.1)$$

with dense and compact injections. Let $(\,\cdot\,,\,\cdot\,)$ and $|\cdot|$ be the inner product and the corresponding norm in H, and denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the duality pairing between V' and V. We fix a linear continuous symmetric coercive operator $A : V \to V'$ and define

$$((u,v)) = \langle Au, v\rangle \qquad \forall\, u, v \in V.$$

Then $(\!(\cdot\,, \cdot)\!)$ is an inner product in V which is equivalent to the standard one. In the following $\|\cdot\|$, $\|\cdot\|_*$, and $(\!(\cdot\,, \cdot)\!)_*$ stand for the associated norm in V and for the induced norm and inner product in V', respectively.

Our aim is to interpret and to generalize the relationship $u \in \gamma(\vartheta)$ stated in (1.2). Let us introduce our ingredients, namely

$$j : \mathbb{R} \to \,]-\infty, +\infty] \quad \text{convex, proper, and lower semicontinuous} \tag{2.2}$$
$$j^* : \mathbb{R} \to \,]-\infty, +\infty] \quad \text{the conjugate function of } j \tag{2.3}$$
$$\gamma = \partial j \qquad \text{and} \qquad \beta = \gamma^{-1} = \partial j^*. \tag{2.4}$$

We associate the functionals J_H and J_V on H and on V as follows

$$J_H(v) = \int_\Omega j(v) \quad \text{if } v \in H \text{ and } j(v) \in L^1(\Omega) \tag{2.5}$$
$$J_H(v) = +\infty \quad \text{if } v \in H \text{ and } j(v) \notin L^1(\Omega) \tag{2.6}$$
$$J_V(v) = J_H(v) \quad \text{if } v \in V. \tag{2.7}$$

As is well known, J_H and J_V are convex and lower semicontinuous on H and V, respectively. Note that they are also proper since V contains all the constant functions. We now consider the corresponding subdifferentials $\partial_{V,V'} J_V : V \to 2^{V'}$ and $\partial_H J_H : H \to 2^H$ of J_V and J_H, respectively. We remind that

$$\begin{aligned} &u \in \partial_{V,V'} J_V(\vartheta) \quad \text{if and only if} \\ &\quad u \in V', \quad \vartheta \in D(J_V), \quad \text{and} \quad J_V(\vartheta) \le \langle u, \vartheta - v\rangle + J_V(v) \quad \forall\, v \in V \end{aligned} \tag{2.8}$$
$$\begin{aligned} &u \in \partial_H J_H(\vartheta) \quad \text{if and only if} \\ &\quad u \in H, \quad \vartheta \in D(J_H), \quad \text{and} \quad J_H(\vartheta) \le (u, \vartheta - v) + J_H(v) \quad \forall\, v \in H \end{aligned} \tag{2.9}$$

where $D(\,\cdot\,)$ denotes in general the effective domain for functionals and multivalued operators. We recall that $\partial_{V,V'} J_V : V \to 2^{V'}$ and $\partial_H J_H : H \to 2^H$ are maximal monotone operators. Moreover, observe that for $\vartheta, u \in H$ we have (see, e.g., [7, Ex. 2.1.3, p. 21])

$$u \in \partial_H J_H(\vartheta) \qquad \text{if and only if} \qquad u \in \partial j(\vartheta) \quad \text{a.e. in } \Omega\,. \tag{2.10}$$

On the other hand, one can easily check that the inclusion

$$\partial_H J_H(\vartheta) \subseteq H \cap \partial_{V,V'} J_V(\vartheta) \qquad \forall\, \vartheta \in V \tag{2.11}$$

holds, just as a consequence of (2.7) (compare (2.8) with (2.9)). Therefore, for $\vartheta \in V$ and $u \in H$, the condition

$$u \in \gamma(\vartheta) \quad \text{a.e. in } \Omega \tag{2.12}$$

implies

$$u \in \partial_{V,V'} J_V(\vartheta). \tag{2.13}$$

On the contrary, the equality

$$\partial_H J_H(\vartheta) = H \cap \partial_{V,V'} J_V(\vartheta) \qquad \forall\, \vartheta \in V \tag{2.14}$$

cannot be inferred simply by (2.7), i.e., it is false for more general functionals. Indeed, taking, e.g., $v_0 \in H \setminus V$, setting

$$J_H(v) = \begin{cases} 0 & \text{if } v = cv_0 \text{ for some constant } c \in \mathbb{R} \\ +\infty & \text{elsewhere on } H \end{cases}$$

and defining J_V by (2.7), then (2.14) is false for $\vartheta = 0$. Here, we want both to prove (2.14) in our case and to generalize the above relationship between (2.12) and (2.13). To this aim, it is convenient to identify some functions in $L^1(\Omega)$ with elements of V' according to the following definition.

Definition 2.1. *We say that a function $u \in L^1(\Omega)$ belongs to $V' \cap L^1(\Omega)$ if the functional*

$$v \mapsto \int_\Omega uv, \qquad v \in V \cap L^\infty(\Omega) \tag{2.15}$$

is continuous with respect to the topology of V.

If $u \in V' \cap L^1(\Omega)$, by density the functional (2.15) has a unique linear continuous extension to V, which we still term u. Hence, we have

$$\langle u, v\rangle = \int_\Omega uv \qquad \forall\, v \in V \cap L^\infty(\Omega)$$

and we regard $V' \cap L^1(\Omega)$ mainly as a subspace of V'. Note that $H \subset V' \cap L^1(\Omega)$. In particular, $V' \cap L^1(\Omega)$ is dense both in V' and in $L^1(\Omega)$.

Lemma 2.2. *Let $u \in V' \cap L^1(\Omega)$, $\psi \in L^1(\Omega)$, $\varphi : \Omega \to \mathbb{R}$ measurable, and $v \in V$ satisfy*

$$\psi \le \varphi \le uv \quad \text{a.e. in } \Omega .$$

Then

$$\varphi \in L^1(\Omega) \qquad \text{and} \qquad \int_\Omega \varphi \le \langle u, v\rangle.$$

Proof. Replace $H_0^1(\Omega)$ with V in [5, Lemma 2.1, p. 68]. The same argument works indeed, provided that $V' \cap L^1(\Omega)$ is defined exactly as above. ∎

Proposition 2.3. *Assume $\vartheta \in V$, $u \in V' \cap L^1(\Omega)$, and (2.12). Then $j(\vartheta) \in L^1(\Omega)$ and (2.13) holds.*

Proof. By (2.12), we have

$$j(\vartheta) \le u(\vartheta - z) + j(z) \qquad \forall\, z \in \mathbb{R}, \quad \text{a.e. in } \Omega$$

and we deduce

$$j(\vartheta) \le u(\vartheta - v) + j(v) \qquad \forall\, v \in D(J_V), \quad \text{a.e. in } \Omega\,.$$

Since any proper convex lower semicontinuous function is bounded from below by an affine function (cf., e.g., [5, Prop. 2.1, p. 51]), we can choose $a, b \in \mathbb{R}$ such that $j(z) \ge az + b$ for any $z \in \mathbb{R}$. Hence, defining $\psi_v := a\vartheta + b - j(v)$ for $v \in D(J_V)$, it turns out that

$$\psi_v \le j(\vartheta) - j(v) \le u(\vartheta - v) \quad \text{a.e. in } \Omega\,.$$

As $\psi_v \in L^1(\Omega)$, we can apply the previous lemma and conclude that

$$j(\vartheta) - j(v) \in L^1(\Omega) \qquad \text{and} \qquad J_V(\vartheta) - J_V(v) \le \langle u, \vartheta - v\rangle \quad \forall\, v \in D(J_V).$$

This implies $j(\vartheta) \in L^1(\Omega)$ and (2.13). ∎

The next proposition shows that (2.14) holds for the particular functionals under consideration (cf. (2.5–7)). Its proof is based on the following lemma, which slightly improves [12, Lemma 2.3].

Lemma 2.4. *Let $v \in D(J_H)$. Then there exists a sequence $\{v_n\}$ in $D(J_V)$ such that*

$$\begin{aligned} &v_n \to v \quad \text{in } H \quad \text{and} \quad J_V(v_n) \to J_H(v) \qquad \text{as } n \to \infty \\ &J_V(v_n) \le J_H(v) \qquad \forall\, n. \end{aligned}$$

Proof. Let $v_n \in V$ be the solution to the variational equation

$$(v_n, w) + \frac{1}{n}\int_\Omega \nabla v_n \cdot \nabla w = (v, w) \qquad \forall\, w \in V. \tag{2.16}$$

Then, taking $w = v_n$ yields

$$\frac{1}{2}\,|v_n|^2 + \frac{1}{n}\int_\Omega |\nabla v_n|^2 \le \frac{1}{2}\,|v|^2 \qquad \forall\, n$$

whence the weak convergence of v_n to v is derived. Since we also infer $|v_n| \to |v|$, it follows that

$$v_n \to v \qquad \text{in } H\,. \tag{2.17}$$

Now, we prove that

$$J_V(v_n) = J_H(v_n) \le J_H(v) \qquad \forall\, n. \tag{2.18}$$

Let us assume first that $\gamma = \partial j$ is a Lipschitz continuous function. We have that

$$j(v_n) - j(v) \le \gamma(v_n)(v_n - v) \qquad \text{a.e. in } \Omega\,.$$

Therefore, accounting for (2.16) with $w = \gamma(v_n)$, we obtain

$$J_H(v_n) - J_H(v) \le \int_\Omega \gamma(v_n)(v_n - v) = \langle v_n - v, \gamma(v_n)\rangle$$
$$= -\frac{1}{n}\int_\Omega \nabla v_n \cdot \nabla\gamma(v_n) = -\frac{1}{n}\int_\Omega \gamma'(v_n)|\nabla v_n|^2 \le 0.$$

In the general case, by arguing first on the Yosida approximation of γ and using [7, Prop. 2.11, p. 39], one still gets (2.18). Then, we can conclude by combining (2.18) with the lower semicontinuity of J_H. We have indeed

$$J_H(v) \le \liminf_{n\to\infty} J_H(v_n) \le \limsup_{n\to\infty} J_H(v_n) \le J_H(v). \ \blacksquare$$

Proposition 2.5. *Assume $\vartheta \in V$, $u \in H$, and (2.13). Then (2.12) holds. In particular, (2.14) is fulfilled.*

Proof. We recall (2.8) and, since $u \in H$, we rewrite (2.13) in the form

$$J_H(\vartheta) \le (u, \vartheta - v) + J_H(v) \quad \forall\, v \in V. \tag{2.19}$$

Now, we apply Lemma 2.4 and deduce that (2.19) holds for any $v \in H$. Therefore, it follows that $u \in \partial_H J_H(\vartheta)$. But this implies $u \in \partial j(\vartheta)$ a.e. in Ω because of (2.10), i.e., (2.12) is satisfied. Then, on account of (2.11), we achieve the validity of (2.14) as well. $\blacksquare$

Next, let us consider the conjugate functionals and their subdifferentials. We first introduce $J_V^* : V' \to]-\infty, +\infty]$, which is specified by

$$J_V^*(w) = \sup_{v\in V}\bigl(\langle w, v\rangle - J_V(v)\bigr), \qquad w \in V'. \tag{2.20}$$

The subdifferential $\partial_{V',V} J_V^*$ maps V' into 2^V according to the definition

$$\begin{aligned} &\vartheta \in \partial_{V',V} J_V^*(u) \quad \text{if and only if} \\ &\quad \vartheta \in V, \quad u \in D(J_V^*), \quad \text{and} \quad J_V^*(u) \le \langle u - w, \vartheta\rangle + J_V^*(w) \quad \forall\, w \in V' \end{aligned} \tag{2.21}$$

and we remind that (2.8) and (2.21) are equivalent. Instead, the conjugate J_H^* of J_H is defined on H by the formula

$$J_H^*(w) = \sup_{v\in H}\bigl((w, v) - J_H(v)\bigr), \qquad w \in H \tag{2.22}$$

and it is well known that

$$J_H^*(w) = \begin{cases} \int_\Omega j^*(w) & \text{if } j^*(w) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.23}$$

Finally, we introduce one more functional on V'. For $w \in V'$ we set

$$J_{*,V'}(w) = \begin{cases} \int_\Omega j^*(w) & \text{if } w \in V' \cap L^1(\Omega) \text{ and } j^*(w) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.24}$$

Although the functional $J_{*,V'}$ is convex and proper in the general case, it need not be lower semicontinuous on V'. Hence, no interesting relationship appears between the functionals $J_{*,V'}$ and J_V^*. However, we can prove

Proposition 2.6. *The conjugate $(J_{*,V'})^*$ of $J_{*,V'}$ coincides with J_V.*

Proof. Let $z \in V$. Then, by definition, we derive

$$(J_{*,V'})^*(z) = \sup_{v \in D(J_{*,V'})} \Bigl(\langle v, z \rangle - \int_\Omega j^*(v) \Bigr).$$

On the other hand, as J_V is the restriction of J_H to V and the functional J_H is convex, lower semicontinuous, and proper on H, accounting for (2.23) we have

$$J_V(z) = J_H(z) = (J_H^*)^*(z) = \sup_{v \in D(J_H^*)} \Bigl(\langle v, z \rangle - \int_\Omega j^*(v) \Bigr).$$

Hence, it follows immediately that $J_V(z) \le (J_{*,V'})^*(z)$ and we have to prove the reverse inequality. We do it showing that, for every $v \in D(J_{*,V'})$, there exists a sequence $\{v_n\}$ in $D(J_H^*)$ such that

$$\langle v, z \rangle - \int_\Omega j^*(v) \le \liminf_{n \to \infty} \Bigl(\langle v_n, z \rangle - \int_\Omega j^*(v_n) \Bigr). \tag{2.25}$$

The assumption $v \in D(J_{*,V'})$ means $v \in V' \cap L^1(\Omega)$ and $j^*(v) \in L^1(\Omega)$. Then, we can construct v_n as in (2.16), i.e., we take $v_n \in V$ such that

$$(v_n, w) + \frac{1}{n} \int_\Omega \nabla v_n \cdot \nabla w = \langle v, w \rangle \qquad \forall\, w \in V. \tag{2.26}$$

First of all, we prove the inequality

$$\int_\Omega j^*(v_n) \le \int_\Omega j^*(v) \tag{2.27}$$

by recalling (2.4) and using the Yosida regularization β_ε of β and the corresponding regularized functionals j_ε^* defined by

$$j_\varepsilon^*(y) = \min_{y' \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon} (y' - y)^2 + j^*(y') \right\}, \qquad y \in \mathbb{R}.$$

Then $\beta_\varepsilon = \partial j_\varepsilon^*$ and $j_\varepsilon^* \le j^*$. Thus, we have that

$$j_\varepsilon^*(v_n) - j_\varepsilon^*(v) \le (v_n - v)\,\beta_\varepsilon(v_n) \qquad \text{a.e. in } \Omega\,.$$

Thanks to Lemma 2.2 with $\psi = \varphi = j_\varepsilon^*(v_n) - j_\varepsilon^*(v)$, $u = v_n - v$, and $w = \beta_\varepsilon(v_n)$, we can integrate the above inequality over Ω . This yields

$$\int_\Omega j_\varepsilon^*(v_n) - \int_\Omega j_\varepsilon^*(v) \le \langle v_n - v, \beta_\varepsilon(v_n)\rangle.$$

On the other hand, (2.26) gives

$$\langle v_n - v, \beta_\varepsilon(v_n)\rangle = -\frac{1}{n}\int_\Omega \nabla v_n \cdot \nabla \beta_\varepsilon(v_n) \le 0.$$

Hence, we conclude that

$$\int_\Omega j_\varepsilon^*(v_n) \le \int_\Omega j_\varepsilon^*(v)$$

and (2.27) follows by the monotone convergence theorem, since j_ε^* converges to j^* pointwise. Inequality (2.27) implies immediately (2.25), i.e., the conclusion of the proof. Indeed, denoting by $(\,\cdot\,,\,\cdot\,)_1$ the standard inner product in $H^1(\Omega)$ and by R the associated Riesz operator from V into V' , one can write (2.26) as

$$\left(1 - \frac{1}{n}\right)\langle v_n, w\rangle + \frac{1}{n}(v_n, w)_1 = \langle v, w\rangle \qquad \forall\, w \in V.$$

The choices $w = R^{-1}v_n$ and $w = v_n$ easily lead to the estimates

$$\left(1 - \frac{1}{n}\right)\|v_n\|_* \le \|v\|_* \qquad \text{and} \qquad \frac{1}{n}\,\|v_n\| \le \|v\|_* \qquad \forall\, n$$

and the (actually strong) convergence $v_n \rightharpoonup v$ in V' follows. ∎

We now look for the lower semicontinuity of $J_{*,V'}$. Indeed, combining this with the previous result, we could prove that $J_{*,V'}$ coincides with the conjugate J_V^* of J_V and deduce the equivalence between

$$u \in \partial_{V,V'} J_V(\vartheta) \qquad (\text{or} \quad \vartheta \in \partial_{V',V} J_V^*(u))$$

and

$$u \in D(J_V^*) \quad \text{and} \quad u \in \gamma(\vartheta) \quad \text{a.e. in } \Omega$$

under the only condition that $\vartheta \in V$ and $u \in V'$. We need the assumption

$$D(j) = \mathbb{R} \tag{2.28}$$

i.e., $j(r) < +\infty$ for any $r \in \mathbb{R}$. It is easy to check (one may also see [7, Rem. 2.3, p. 43]) that (2.28) is equivalent to either $D(\gamma) = \mathbb{R}$ or β is surjective or

$$\lim_{|r| \to +\infty} \frac{j^*(r)}{|r|} = +\infty. \tag{2.29}$$

We point out that the form (2.29) of (2.28) is mainly used in the sequel.

Proposition 2.7. *Assume (2.28). Then $J_{*,V'}$ is lower semicontinuous on V'.*

Proof. One can replace $H^{-1}(\Omega) \cap L^1(\Omega)$ with $V' \cap L^1(\Omega)$ in the first part of the proof of [5, Prop. 2.10, pp. 67–68] and conclude. ∎

Proposition 2.8. *Assume (2.28). Then*

$$J_{*,V'} = J_V^*. \tag{2.30}$$

Proof. Indeed, Propositions 2.7 and 2.6 yield, respectively

$$J_{*,V'} = (J_{*,V'})^{**} \qquad \text{and} \qquad (J_{*,V'})^{**} = J_V^*$$

owing to well–known properties of conjugate functions. ∎

Proposition 2.9. *Assume that (2.28) holds and let $\vartheta \in V$ and $u \in D(J_V^*)$. Then (2.12) and (2.13) are equivalent.*

Proof. Instead of giving a direct proof, we adapt the Brézis argument reported in [5, pp. 69–71], although with a different notation. We recall that A denotes the Riesz isomorphism from V onto V' and introduce two operators, $\partial_{V'} J_V^*$ and F, which map V' into $2^{V'}$. The first one is the subdifferential of J_V^* in V', i.e.,

$$\begin{aligned} &\zeta \in \partial_{V'} J_V^*(u) \quad \text{if and only if} \\ &\quad \zeta \in V', \quad u \in D(J_V^*), \quad \text{and} \quad J_V^*(u) \le ((u - w, \zeta))_* + J_V^*(w) \quad \forall\, w \in V'. \end{aligned} \tag{2.31}$$

Comparing (2.31) and (2.21), we see that

$$\partial_{V'} J_V^*(u) = A(\partial_{V',V} J_V^*(u)) \qquad \forall\, u \in V'. \tag{2.32}$$

The second operator is defined in this way

$$\begin{aligned} &\zeta \in F(u) \quad \text{if and only if} \\ &\quad u \in V' \cap L^1(\Omega), \quad \zeta \in V', \quad \text{and} \quad u \in \gamma(A^{-1}\zeta) \quad \text{a.e. in } \Omega. \end{aligned} \tag{2.33}$$

Since we are assuming (2.28), we can apply Proposition 2.8 and our assumptions on ϑ and u become $\vartheta \in V$ and $u \in D(J_{*,V'})$. In particular, $u \in V' \cap L^1(\Omega)$, so that (2.12) is equivalent to $A\vartheta \in F(u)$. On the other hand, (2.13) is equivalent to $\vartheta \in \partial_{V',V} J_V^*(u)$, i.e., to $A\vartheta \in \partial_{V'} J_V^*(u)$ by (2.32). Hence, the equivalence between (2.12) and (2.13) follows if we show that

$$\partial_{V'} J_V^*(u) = F(u) \qquad \forall\, u \in V'$$

which means that the graphs $\partial_{V'} J_V^*$ and F are identical. To prove this assertion, we observe that Proposition 2.3 can be reformulated as

$$F(u) \subseteq \partial_{V'} J_V^*(u) \qquad \forall\, u \in V' \tag{2.34}$$

since $F(u) = \partial_{V'} J_V^*(u) = \emptyset$ if $u \notin D(J_{*,V'})$. Therefore, it suffices to show that F is maximal monotone in $V' \times V'$. Let $\zeta_i \in F(u_i)$ for $i = 1, 2$. Setting

$$u = u_1 - u_2, \qquad v = A^{-1}\zeta_1 - A^{-1}\zeta_2, \qquad \text{and} \qquad \varphi = uv$$

we have $u \in V' \cap L^1(\Omega)$ and $\varphi \geq 0$ a.e. in Ω, because γ is monotone. Applying Lemma 2.2 with $\psi = 0$, we deduce that $\varphi \in L^1(\Omega)$ and

$$((u_1 - u_2, \zeta_1 - \zeta_2)) = \langle u, v \rangle \geq \int_\Omega \varphi \geq 0.$$

This proves monotonicity for F, and maximal monotonicity follows provided we show that the inclusion

$$F(u) + u \ni f_0 \tag{2.35}$$

is solvable in V' for any $f_0 \in V'$. In view of (2.33) and (2.34), problem (2.35) is equivalent to

$$(\vartheta, u) \in V \times (V' \cap L^1(\Omega)), \quad j^*(u) \in L^1(\Omega), \quad u + A\vartheta = f_0, \quad \text{and} \quad u \in \gamma(\vartheta) \quad \text{a.e. in } \Omega.$$

Actually, we can derive the existence of a solution by arguing as in [5, pp. 69–71] with just one modification, due to the fact that V' is not a space of distributions on Ω. The equation $f_\lambda + Jv_\lambda = f_0$ of [5] reads

$$u_\lambda + A\vartheta_\lambda = f_0 \tag{2.36}$$

in our notation. For the reader's convenience, we recall that $\lambda > 0$ is subject to tend to 0 and that $\vartheta_\lambda = \gamma_\lambda(u_\lambda)$, where γ_λ stands for the Yosida approximation of γ. Now, from (2.36) we should deduce that

$$u \in V' \cap L^1(\Omega) \qquad \text{and} \qquad u + A\vartheta = f_0$$

for the weak limits u and ϑ. This can be done as follows. As $u_\lambda \in H$, we can write down the equality

$$(u_\lambda, z) + \langle A\vartheta_\lambda, z \rangle = \langle f_0, z \rangle \qquad \forall\, z \in V.$$

Since $u_\lambda \rightharpoonup u$ in $L^1(\Omega)$ and $\vartheta_\lambda \rightharpoonup \vartheta$ in V, we infer

$$\int_\Omega uz = \langle f_0 - A\vartheta, z \rangle \qquad \forall\, z \in V \cap L^\infty(\Omega).$$

Noting that the above right hand side is continuous on V, it turns out that the left hand side is continuous on $V \cap L^\infty(\Omega)$ with respect to the topology of V, i.e., $u \in V' \cap L^1(\Omega)$ according to Definition 2.1. Moreover, we have

$$\langle u, z \rangle = \langle f_0 - A\vartheta, z \rangle \qquad \forall\, z \in V \cap L^\infty(\Omega)$$

and we conclude by density. ∎

3. Generalized formulation and main result

We consider here Problem (1.1–4) of the Introduction and state it in a generalized form. The notation introduced in the previous section on spaces, graphs, and functionals is still in force. However, for the reader's convenience, let us recall that

$$j : \mathbb{R} \to \,]-\infty, +\infty] \quad \text{is convex, proper, and l.s.c.,} \quad \text{and} \quad \gamma = \partial j. \tag{3.1}$$

Concerning the structure of the equations in Problem (1.1–4), we take $k_0 = 1$ without any loss of generality, and assume

$$\alpha \in (0, +\infty) \tag{3.2}$$

$$\varphi, k \in W^{1,1}(0,T). \tag{3.3}$$

Moreover, we choose the following inner product in V

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v + \alpha \int_\Gamma uv, \qquad u, v \in V \tag{3.4}$$

while we keep the standard one in H. Let us introduce the linear and continuous operators $A, B : V \to V'$,

$$\langle Au, v \rangle = \int_\Omega \nabla u \cdot \nabla v + \alpha \int_\Gamma uv \qquad \forall\, u, v \in V \tag{3.5}$$

$$\langle Bu, v \rangle = \int_\Omega \nabla u \cdot \nabla v \qquad \forall\, u, v \in V. \tag{3.6}$$

According to the framework of the previous section, A is the Riesz map associated with the inner product (3.4). Instead, B is not coercive. Setting now

$$\langle f(t), v \rangle = \int_\Omega g(t)v + \alpha \int_\Gamma h(t)v \qquad \forall\, v \in V \tag{3.7}$$

multiplying equation (1.1) by $v \in V$, integrating the resulting equation over Ω, and using (1.3), we formally get

$$(u + \varphi * u)' + A\vartheta + k * B\vartheta = f \qquad \text{in } V', \quad \text{a.e. in } (0,T). \tag{3.8}$$

Moreover, on account of the previous section, we can interpret (1.2) as

$$u(t) \in \partial_{V,V'} J_V(\vartheta(t)) \qquad \text{a.e. in } (0,T). \tag{3.9}$$

This formal argument suggests a quite natural generalized formulation of the problem we are dealing with, and our main result reads as follows.

Theorem 3.1. *Assume* (3.1–6) *and*

$$f \in L^2(0,T;V') \tag{3.10}$$

$$u_0 \in D(J_V^*). \tag{3.11}$$

Then, there exists a unique pair (u,ϑ) *satisfying*

$$u \in H^1(0,T;V') \qquad \text{and} \qquad \vartheta \in L^2(0,T;V) \tag{3.12}$$

which solves (3.8–9) and fulfills the Cauchy condition

$$u(0) = u_0. \tag{3.13}$$

Morover, we also have

$$u(t) \in D(J_V^*) \quad \forall\, t \in [0,T] \qquad \text{and} \qquad J_V^*(u) \in W^{1,1}(0,T). \tag{3.14}$$

The proof is given in the next section. Here, we just prove that (u,ϑ) solves (3.8–9) in a stronger sense, provided (2.28) is fulfilled. We have indeed

Corollary 3.2. *In the framework of Theorem 3.1, assume (2.28) in addition. Then the solution* (u,ϑ) *satisfies*

$$u(t) \in V' \cap L^1(\Omega) \quad \text{and} \quad j^*(u(t)) \in L^1(\Omega) \qquad \forall\, t \in [0,T] \tag{3.15}$$

$$j^*(u) \in L^\infty(0,T;L^1(\Omega)) \tag{3.16}$$

$$u : [0,T] \to L^1(\Omega) \quad \text{is weakly continuous} \tag{3.17}$$

$$u(t) \in \gamma(\vartheta(t)) \quad \text{a.e. in } \Omega, \quad \text{for a.a. } t \in (0,T). \tag{3.18}$$

Proof. Conditions (3.15–16) and property (3.18) come out directly from Propositions 2.8 and 2.9, (2.24), and (3.14), since (2.28) is equivalent to (2.29). In this particular case, the continuity of $J_V^*(u)$ given by (3.14) ensures that

$$\int_\Omega j^*(u(t)) \le C \qquad \forall\, t \in [0,T]$$

for some constant C. From this bound, we deduce property (3.17) as well. Assume indeed $t_n \to t$. Arguing as in [5, pp. 67–68] and applying [13, Cor. 11, p. 294], we infer that the sequence $\{u(t_n)\}$ is weakly (relatively) compact in $L^1(\Omega)$. Therefore, at least for a subsequence, we have that $u(t_n) \rightharpoonup u_*$ in $L^1(\Omega)$ for some $u_* \in L^1(\Omega)$. On the other hand, $u(t_n) \to u(t)$ in V' because of (3.12), whence

$$\langle u(t), v\rangle = \lim_{n\to\infty} \langle u(t_n), v\rangle = \lim_{n\to\infty} \int_\Omega u(t_n) v = \int_\Omega u_* v$$

for any $v \in V \cap L^\infty(\Omega)$. According to Definition 2.1, we deduce $u_* = u(t)$ and $u(t_n) \rightharpoonup u(t)$ in $L^1(\Omega)$.

Remark 3.3. Note that, under the validity of (2.29), condition (3.11) reduces to

$$u_0 \in V' \cap L^1(\Omega) \qquad \text{and} \qquad j^*(u_0) \in L^1(\Omega).$$

Remark 3.4. We emphasize that (cf. (3.9) and Proposition 2.5) inclusion (3.18) still holds even though (2.28) does not, provided that u satisfies the further regularity condition $u(t) \in H$ for a.a. $t \in (0,T)$. This is precisely the case examined by Damlamian and Kenmochi in [12] for the problem without memory kernels. There, the authors are able to achieve that $u \in L^\infty(0,T;H)$ (see [12, Thm. 1.13]). However, even when memory terms are present, there are cases for which we can still conclude $u \in L^\infty(0,T;H)$. This happens, for instance, for the problem addressed in [3]. In fact, the paper [3] deals with other boundary conditions (slightly different from (1.3)) and the respective equation (3.8) has a simpler form, namely A and B are the same linear, continuous, and coercive operator from V to V'. Finally, let us notice that, if (2.28) holds, then the regularity property $u \in L^\infty(0,T;H)$ is simply ensured by (3.16) whenever γ is linearly bounded (see [10]).

4. Existence and uniqueness

Our aim is showing Theorem 3.1. In fact, we prove a more general result obtained by allowing the kernels φ and k in (3.8) to take values in V' instead of in $\mathbb{R}$, that is, we replace (3.3) by

$$\varphi, k \in W^{1,1}(0,T;\mathcal{L}(V')) \tag{4.1}$$

where $\mathcal{L}(V')$ denotes the space of continuous linear operators from V' into itself. Note that (4.1) allows, in particular, to consider real kernels depending smoothly on the space variables. Our proof takes advantage of the following lemma, whose proof is reported, for the reader's convenience, at the end of this section. However, one can also refer to [16, Sect. I.2.3, pp. 42–45] for the scalar case or [20, Sect. 0.3, pp. 12–15] for a quite general setting.

Lemma 4.1. *Let X be a real Banach space and $M \in L^1(0,T;\mathcal{L}(X))$. Then, there exists a unique $\overline{M} \in L^1(0,T;\mathcal{L}(X))$, named conjugate kernel of M, such that*

$$\overline{M} + M * \overline{M} = M \qquad \text{a.e. in } (0,T). \tag{4.2}$$

Consequently, for any $z \in L^1(0,T;X)$, there exists a unique $w \in L^1(0,T;X)$ such that

$$w + M * w = z \qquad \text{a.e. in } (0,T) \tag{4.3}$$

and the following representation holds

$$w = z - \overline{M} * z \qquad \textit{a.e. in } (0,T)\,. \tag{4.4}$$

In order to prove the generalized version we have mentioned, we observe that equation (3.8) can be written in the form

$$A\vartheta + M * A\vartheta = f - u' - (\varphi * u)' \qquad \text{in } V'\,, \quad \text{a.e. in } (0,T) \tag{4.5}$$

where M is defined by

$$M(t) := k(t)BA^{-1} \qquad \forall\, t \in [0,T]\,. \tag{4.6}$$

Hence, (4.4) yields

$$A\vartheta = \bigl(f - u' - (\varphi * u)'\bigr) - \overline{M} * \bigl(f - u' - (\varphi * u)'\bigr) \quad \text{in } V'\,, \quad \text{a.e. in } (0,T)\,. \tag{4.7}$$

Next, from (3.5–6), (4.1), and (4.2), we deduce that

$$M, \overline{M} \in W^{1,1}(0,T;\mathcal{L}(V')). \tag{4.8}$$

Then, taking (4.1) into account and using the formulas

$$(\varphi * u)' = \varphi(0)u + \varphi' * u \qquad \text{and} \qquad \overline{M} * u' = \overline{M}\,' * u + \overline{M}(0)u - \overline{M}u_0$$

we transform (4.7) into

$$\begin{aligned} u' + A\vartheta = {} & f - \varphi(0)u - \varphi' * u \\ & - \overline{M} * f + \overline{M}\,' * u + \overline{M}(0)u - \overline{M}u_0 + \overline{M} * (\varphi(0)u) + \overline{M} * \varphi' * u \end{aligned}$$

or, equivalently,

$$u' + A\vartheta = F + M_0 u + \Phi * u \qquad \text{in } V'\,, \quad \text{a.e. in } (0,T) \tag{4.9}$$

where we have set

$$F := f - \overline{M} * f - \overline{M}u_0 \tag{4.10}$$
$$M_0 := -\varphi(0) + \overline{M}(0) \tag{4.11}$$
$$\Phi := -\varphi' + \overline{M}\,' + \overline{M}\varphi(0) + \overline{M} * \varphi'. \tag{4.12}$$

On account of (3.10–11), (4.1), and (4.8), we have

$$F \in L^2(0,T;V'), \qquad M_0 \in \mathcal{L}(V'), \qquad \text{and} \qquad \Phi \in L^1(0,T;\mathcal{L}(V')). \tag{4.13}$$

We now state a perfecly equivalent formulation of our problem in V'. To this aim, we consider the subdifferential of J_V^* in V', i.e., the map $\partial_{V'} J_V^*$ already defined in (2.31) and related to $\partial_{V,V'} J_V$ by

$$\partial_{V'} J_V^*(u) = A(\partial_{V',V} J_V^*(u)) \qquad \forall\, u \in V' \tag{4.14}$$

(as already observed in the proof of Proposition 2.9). Making use of this operator, (3.8–9) become

$$u' + \xi = F + M_0 u + \Phi * u \qquad \text{and} \qquad \xi \in \partial_{V'} J_V^*(u) \qquad \text{a.e. in } (0,T) \tag{4.15}$$

$$A\vartheta = \xi \qquad \text{a.e. in } (0,T) \tag{4.16}$$

and our problem turns out to be equivalent to finding $u \in H^1(0,T;V')$ which satisfies the following inclusion and Cauchy condition

$$u' + \partial_{V'} J_V^*(u) \ni F + M_0 u + \Phi * u \qquad \text{in } V', \quad \text{a.e. in } (0,T) \tag{4.17}$$

$$u(0) = u_0. \tag{4.18}$$

Indeed, the function $\vartheta \in L^2(0,T;V)$ is obtained solving (4.16), where $\xi \in L^2(0,T;V')$ is uniquely selected in the subdifferential in order to fulfill the first equation in (4.15).

At this point, we prove that (4.17–18) has a unique solution $u \in H^1(0,T;V')$, which satisfies (3.14) as well. We apply the generalized contraction mapping principle in the Banach space $C^0([0,T];V')$ and, for any given $w \in C^0([0,T];V')$, we consider the Cauchy problem

$$u' + \partial_{V'} J_V^*(u) \ni F + M_0 w + \Phi * w \qquad \text{in } V', \quad \text{a.e. in } (0,T) \tag{4.19}$$

$$u(0) = u_0. \tag{4.20}$$

Since J_V^* is bounded form below by an affine function on V', we can assume that $\min J_V^* = 0$ without loss of generality. Indeed, it suffices to modify J_V^* along with F and M_0 in an obvious way. Therefore, we are allowed to use [7, Thm. 3.6, pp. 72–73] and deduce that there is a unique solution u to (4.19–20). Moreover, u satisfies (3.14) in addition to further regularity properties. Hence, we can define the operator

$$\mathcal{S} : C^0([0,T];V') \to C^0([0,T];V')$$

by the formula $\mathcal{S}(w) := u$ and check that $\mathcal{S}^m$ is a (strict) contraction for a suitable m. Take $w_i \in C^0([0,T];V')$, $i = 1,2$, and set $u_i = \mathcal{S}(w_i)$. Writing (4.19–20) for the pairs (u_i, w_i) in the form (4.15), i.e.,

$$u_i' + \xi_i = F + M_0 w_i + \Phi * w_i \qquad \text{with} \qquad \xi_i \in \partial_{V'} J_V^*(u_i)$$

taking the difference, and applying a well–known inequality (see, e.g., [21, Cor. 8.2, pp. 221]), we obtain the estimate

$$\|u_1(t) - u_2(t)\|_* \leq \int_0^t \left\| M_0(w_1 - w_2)(s) + \big(\Phi * (w_1 - w_2)\big)(s) \right\|_* \, ds$$

for any $t \in [0,T]$. Next, using the Young inequality for convolutions, we infer

$$\|\mathcal{S}(w_1) - \mathcal{S}(w_1)\|_{C^0([0,t];V')} \leq C \int_0^t \|w_1 - w_2\|_{C^0([0,s];V')} \, ds$$

for any $t \in [0,T]$, where

$$C = \|M_0\|_{\mathcal{L}(V')} + \|\Phi\|_{L^1(0,T;\mathcal{L}(V'))} \, .$$

By iteration, we conclude that

$$\|\mathcal{S}^m(w_1) - \mathcal{S}^m(w_1)\|_{C^0([0,t];V')} \leq \frac{(Ct)^m}{m!} \|w_1 - w_2\|_{C^0([0,t];V')}$$

for any $t \in [0,T]$ and any $m \geq 0$. Such an estimate implies that $\mathcal{S}^m$ is a contraction mapping for m large enough. This ensures that the transformed problem has a unique solution u which fulfills (3.14). Therefore, our generalized version of Theorem 3.1 is proved.

Remark 4.2. It is worth noting that, once the equivalent problem (4.17–18) has been set, the existence and uniqueness result can be also obtained by applying [15, Thm. 3]. Indeed, it suffices to check that the right hand side of (4.17) obeys [15, (2.13)]. Besides, one can realize that similar arguments allow to infer a continuous dependence estimate on F and u_0 (see, e.g., [3, Thm. 2.4]).

Remark 4.3. In Remark 3.4 we incidentally observed that one can associate some boundary conditions with (1.1) for which the related operators A and B (cf. (3.5–6) and (3.8)) coincide with a linear, continuous, and coercive operator from V to V' . For instance, homogeneous Dirichlet boundary conditions yield an example characterized by the choice $V = H_0^1(\Omega)$. However, in this situation the operator $M(t)$ defined by (4.6) always reduces to $k(t)I$, where I stands for the identity operator in $\mathcal{L}(V')$, and then $M(t)$ can be viewed as an element of $\mathcal{L}(H)$ as well. Recalling then (3.9) and Proposition 2.5, it should not be difficult for the reader to see how an estimate for u in $L^\infty(0,T;H)$ can be obtained and the related problem can be set in H . On the contrary, in our case, marked by (1.3), BA^{-1} does not map H into itself, but it is just an element of $\mathcal{L}(V')$.

Proof of Lemma 4.1. Put $\mathcal{M} = L^1(0,T;\mathcal{L}(X))$ and note that an element $\overline{M} \in \mathcal{M}$ solves (4.2) if and only if it is a fixed point for the operator $\mathcal{T}_M : \mathcal{M} \to \mathcal{M}$ defined by

$$\mathcal{T}_M K := M - M * K.$$

Thus, we can conclude immediately by the Banach contraction theorem, once we prove that $\mathcal{T}_M$ is a contraction mapping in $\mathcal{M}$. To this aim, we choose a particular norm in $\mathcal{M}$. For $\lambda \in \mathbb{R}$, we set

$$\|K\|_\lambda = \int_0^T e^{-\lambda t} \|K(t)\| \, dt, \qquad K \in \mathcal{M} \tag{4.21}$$

where $\|\cdot\|$ here denotes the norm in $\mathcal{L}(X)$. Clearly, for any λ, (4.21) is a norm in $\mathcal{M}$ which is equivalent to the usual one. Therefore, for any $K_1, K_2 \in \mathcal{M}$, letting $K = K_1 - K_2$ and using the Young inequality for convolution products, we have

$$\begin{aligned} \|\mathcal{T}_M K_1 - \mathcal{T}_M K_2\|_\lambda = \|M * K\|_\lambda &\le \int_0^T e^{-\lambda t} \int_0^t \|M(t-s)\| \, \|K(s)\| \, ds \, dt \\ = \int_0^T \int_0^t \left\|e^{-\lambda(t-s)} M(t-s)\right\| \, \left\|e^{-\lambda s} K(s)\right\| \, ds \, dt &\le \|M\|_\lambda \, \|K\|_\lambda \, . \end{aligned}$$

Hence, it suffices to observe that the assumption $M \in \mathcal{M}$ implies that $\|M\|_\lambda \to 0$ as $\lambda \to +\infty$ and to choose λ such that $\|M\|_\lambda < 1$.

The second part of the lemma is straightforward. Indeed, from (4.2) it is clear that (4.4) provides a solution w to (4.3) which belongs to $L^1(0,T;X)$, and we only have to show uniqueness. This can be easily seen by taking $\mathcal{M} = L^1(0,T;X)$, $\mathcal{T}_M w := z - M * w$, and arguing as above. ∎

5. Asymptotic behavior

In this section, we study the longtime behavior of the solution (u, ϑ) given by Corollary 3.2, whose assumptions are required to hold for any $T > 0$. However, we must ask for further requirements which are listed below. We recal definitions (3.4–6) and still take $k_0 = 1$ for the sake of simplicity.

About memory kernels, we assume

$$\varphi = 0, \qquad k \in W^{1,1}(0, +\infty) \tag{5.1}$$

$$\int_0^t (v + k * v)(s) \, v(s) \, ds \ge \omega \int_0^t |v(s)|^2 \, ds \tag{5.2}$$

for some constant $\omega > 0$, any $v \in L^2(0,t)$, and any $t \in (0, +\infty)$. We assume $\omega \le 1$ without any loss of generality. Moreover, for later convenience we define the function $\iota : [0, +\infty) \to \mathbb{R}$ by the formula

$$\iota(t) := \int_t^\infty k(s) \, ds \tag{5.3}$$

and suppose that

$$\iota \in L^2(0, +\infty). \tag{5.4}$$

As far as γ is concerned, we recall and reinforce the assumptions of the previous sections, namely

$$j:\mathbb{R}\to\mathbb{R} \quad \text{convex} \tag{5.5}$$
$$\gamma=\partial j \qquad \text{and} \qquad \beta=\gamma^{-1}=\partial j^* \tag{5.6}$$
$$\beta \quad \text{is a Lipschitz continuous function with Lipschitz constant} \quad C_\beta. \tag{5.7}$$

Note that (5.5) includes both (2.2) and (2.28). Moreover, it implies that j is a continuous function on $\mathbb{R}$ and that

$$\lim_{|r|\to+\infty}\frac{j^*(r)}{|r|}=+\infty. \tag{5.8}$$

Furthermore, owing to (5.7), we see that also j^* is a continuous function on $\mathbb{R}$ and that both $j(r)$ and $j^*(r)$ tend to $+\infty$ as $|r|\to+\infty$.

We now state a precise formulation of problem (1.6–7). On account of notation (5.3), we have that $k_\infty=1+\iota(0)$ and the stationary problem can be written in the form

$$A\vartheta_\infty+\iota(0)\,B\vartheta_\infty=f_\infty \tag{5.9}$$

where $f_\infty\in V'$ is related to g_∞ and h_∞ by

$$\langle f_\infty,v\rangle=\int_\Omega g_\infty v+\alpha\int_\Gamma h_\infty v \qquad \forall\, v\in V. \tag{5.10}$$

However, we do not require this particular structure for f_∞. Indeed, the data must satisfy the conditions listed below, where the new function ϑ_0 is defined as well.

$$f_\infty\in V', \qquad f-f_\infty\in L^2(0,\infty;H)+H^1(0,+\infty;V'), \tag{5.11}$$
$$u_0\in D(J_V^*), \qquad \vartheta_0:=\beta(u_0)\in V. \tag{5.12}$$

We point out that the definition of ϑ_0 makes sense for $u_0\in L^1(\Omega)$. Moreover, we note that taking $v=1$ in (5.2), dividing by t, and letting $t\to+\infty$, we get $1+\iota(0)\ge\omega$. As $\omega\le 1$, this easily yields

$$\langle Av,v\rangle+\iota(0)\langle Bv,v\rangle\ge\omega\,\|v\|^2 \qquad \forall\, v\in V. \tag{5.13}$$

Therefore, (5.9) has a unique solution $\vartheta_\infty\in V$ and we assume that

$$j((1+\delta)\vartheta_\infty)\in L^1(\Omega) \qquad \text{for some } \delta>0\,. \tag{5.14}$$

Remark 5.1. In the concrete cases f and f_∞ are defined by (3.7) and (5.10), respectively. Therefore, sufficient conditions for the validity of (5.11) are

$$g_\infty\in L^2(\Omega), \qquad g-g_\infty\in L^2(\Omega\times(0,+\infty))$$
$$h_\infty\in L^2(\Gamma), \qquad h-h_\infty\in L^2(\Gamma\times(0,+\infty)), \qquad \partial_t h\in L^2(\Gamma\times(0,+\infty)).$$

Remark 5.2. Assuming that (5.1) holds and extending k to $(-\infty,0)$ by 0, it is not difficult to check that (5.2) is satisfied whenever

$$1+\operatorname{Re}\widehat{k}\geq\omega \qquad \text{on } \mathbb{R}$$

where $\widehat{k}$ denotes the Fourier transform of k. Moreover, (5.4) is fulfilled if, e.g., $|k(t)|$ is bounded by c/t^2 for some $c>0$ and any t large enough. In particular, any decreasing exponential is an admissible kernel.

Remark 5.3. We note that (5.14) is equivalent to

$$j(\vartheta_\infty)\in L^1(\Omega) \tag{5.15}$$

if $j(r)$ grows like the power $|r|^p$ at infinity for some p. Moreover, in this case, (5.15) holds automatically if $V\subset L^p(\Omega)$. On the contrary, restriction (5.14) is strictly stronger than (5.15) if $j(r)$ grows, e.g., exponentially. However, if $\Omega\subset\mathbb{R}^2$, (5.14) is satisfied for any $\delta>0$ if $j(r)\leq c\exp(|r|^\sigma)$ for any $r\in\mathbb{R}$ and some $\sigma\in(0,2)$ and $c>0$. Indeed, in this case, for any $\delta,\varepsilon>0$, the inequality $j((1+\delta)r)\leq c_{\varepsilon,\delta}\exp\bigl(\varepsilon r^2\bigr)$ holds for any $r\in\mathbb{R}$ and some $c_{\varepsilon,\delta}>0$, and a result due to Trudinger (see, e.g., [1, Thm. 8.25, pp. 242–245]) says that $\exp\bigl(cv^2/\|v\|^2\bigr)\in L^1(\Omega)$ for some c depending only on Ω and any $v\in V\setminus\{0\}$. Finally, in any dimension, (5.14) is satisfied if ϑ_∞ is bounded, and this is true provided that f_∞ is smooth enough. ∎

Henceforth, we show that the above assumptions are sufficient to describe the longtime behavior of ϑ and to verify that $\vartheta(t)\rightharpoonup\vartheta_\infty$ in V. As a consequence, it turns out that $f(t)-u'(t)$ tends to f_∞ weakly in V' as $t\to+\infty$. In particular, $u'(t)$ tends to 0 whenever $f(t)-f_\infty$ does. On the contrary, in order to handle the limit of $u(t)$ as t goes to infinity, we need a further restriction on j^*. More precisely, we require a compactness condition on the sets

$$E_c:=\bigl\{w\in V'\cap L^1(\Omega):\ \textstyle\int_\Omega j^*(w)\leq c\bigr\}, \qquad c\in\mathbb{R}. \tag{5.16}$$

Theorem 5.4. *Assume (5.1–7), (5.11–12), (5.14), where $\vartheta_\infty\in V$ is the unique solution to (5.9), and let (u,ϑ) be the solution to problem (3.8–9) given by Theorem 3.1. Then we have the following stability results*

$$\vartheta\in L^\infty(0,\infty;V),\quad \vartheta-\vartheta_\infty\in L^2(0,\infty;V),\quad \vartheta'\in L^2(0,\infty;H) \tag{5.17}$$

$$u\in L^\infty(0,\infty;L^1(\Omega)),\quad j^*(u)\in L^\infty(0,\infty;L^1(\Omega)),\quad u'\in L^2(0,\infty;V'). \tag{5.18}$$

Moreover, ϑ is a weakly continuous $V-$ valued function and

$$\vartheta(t)\to\vartheta_\infty \quad \text{weakly in } V \text{ and strongly in } H, \quad \text{as } t\to+\infty. \tag{5.19}$$

Corollary 5.5. *The difference $f-u'$ is a weakly continuous $V'-$ valued function and*

$$f(t)-u'(t)\to f_\infty \qquad \text{weakly in } V', \quad \text{as } t\to+\infty. \tag{5.20}$$

Theorem 5.6. *In the framework of Theorem 5.4, assume in addition that the set* E_c *in (5.16) is strongly relatively compact in* V' *for any* $c \in \mathbb{R}$. *Moreover, let* $\{t_n\}$ *be a sequence of times such that* $t_n \to +\infty$ *as* $n \to \infty$. *Then we have*

$$u(t_n) \to u_\infty \qquad \text{weakly in } L^1(\Omega) \text{ and strongly in } V' \tag{5.21}$$

at least for a subsequence, for some u_∞ *satisfying*

$$u_\infty \in V' \cap L^1(\Omega) \qquad \text{and} \qquad j^*(u_\infty) \in L^1(\Omega). \tag{5.22}$$

Moreover, any cluster point u_∞ *in the above conditions fulfills*

$$u_\infty \in \gamma(\vartheta_\infty) \quad \text{a.e. in } \Omega\,. \tag{5.23}$$

In particular, if γ *is single valued, the entire function* $u(t)$ *converges to* $\gamma(\vartheta_\infty)$ *weakly in* $L^1(\Omega)$ *and strongly in* V' *as* $t \to +\infty$.

Remark 5.7. If $N = 1$ then $V' \cap L^1(\Omega)$ clearly coincides with $L^1(\Omega)$ and is compactly embedded in V'. Therefore, the assumption of Theorem 5.6 is automatically satisfied. If $N \geq 2$, all the sets (5.16) are strongly relatively compact in V' if j^* satisfies a growth condition from below. More precisely, if $N \geq 3$, $p := 2N/(N-2)$, and $q < p$, then V is compactly and densely embedded in $L^q(\Omega)$. By transposition, $L^{q'}(\Omega)$ is compactly embedded in V', where q' is the conjugate exponent of q. Thererore, relative compactness in V' for all sets (5.16) is ensured if

$$j^*(r) \geq c_1 |r|^{q'} - c_2 \qquad \forall\, r \in \mathbb{R} \tag{5.24}$$

for some constants $c_1, c_2 > 0$ and $q' > p'$. Note that this condition is compatible with the Lipschitz continuity of $\beta = \partial j^*$ since we can assume $q' \leq 2$. Moreover, this type of assumption agrees with Remark 5.3 since (5.24) is equivalent to

$$j(r) \leq c_3 \left(|r|^q + 1\right) \qquad \forall\, r \in \mathbb{R} \tag{5.25}$$

for some $c_3 > 0$ and $q < p$. If $N = 2$, we have to do usual modifications and assume $q < \infty$, or $q' > 1$, in (5.25) and (5.24), respectively. Finally, note that everything works well if γ behaves linearly at infinity, as in the standard Stefan problem.

Proof of Theorem 5.4. Let us write (5.9) in the form

$$A\vartheta_\infty + B(k * \vartheta_\infty) = f_\infty - \iota B \vartheta_\infty \tag{5.26}$$

and take the difference between (3.8) and (5.26). We obtain

$$u' + A\widehat{\vartheta} + B(k * \widehat{\vartheta}) = \widehat{f} - \iota B \vartheta_\infty \tag{5.27}$$

where

$$\widehat{\vartheta} = \vartheta - \vartheta_\infty \qquad \text{and} \qquad \widehat{f} = f - f_\infty. \tag{5.28}$$

We split the proof into several steps.

First a priori estimate. Testing equation (5.27) by $\widehat{\vartheta}$, and integrating over $(0,t)$, where $t>0$ is arbitrary, we get

$$\begin{aligned} &\int_0^t \langle u'(s), \widehat{\vartheta}(s)\rangle\, ds + \int_0^t \Big(\langle A\widehat{\vartheta}(s), \widehat{\vartheta}(s)\rangle + \langle B(k*\widehat{\vartheta})(s), \widehat{\vartheta}(s)\rangle \Big)\, ds \\ &= \int_0^t \langle \widehat{f}(s), \widehat{\vartheta}(s)\rangle\, ds - \int_0^t \iota(s)\langle B\vartheta_\infty, \widehat{\vartheta}(s)\rangle\, ds \end{aligned} \tag{5.29}$$

and we estimate each term, separately. The first one is equal to

$$\int_0^t \langle u'(s), \vartheta(s)\rangle\, ds - \langle u(t)-u_0, \vartheta_\infty\rangle = J_V^*(u(t)) - J_V^*(u_0) - \langle u(t)-u_0, \vartheta_\infty\rangle \tag{5.30}$$

thanks to (3.12) and [7, Lemme 3.3 p. 73]. On the other hand, we have that $u(t) \in V' \cap L^1(\Omega)$ for any $t>0$ (cf. (3.15)). Then, owing to the definition of conjugate function and to (5.14), we get

$$u(t)\vartheta_\infty \le \frac{1}{1+\delta}\Big(j^*(u(t)) + j((1+\delta)\vartheta_\infty)\Big) \qquad \text{a.e. in } \Omega\,.$$

Since the right hand side of this inequality belongs to $L^1(\Omega)$, we are in a position to apply Lemma 2.2 and deduce

$$\langle u(t), \vartheta_\infty\rangle \le \frac{1}{1+\delta} J_V^*(u(t)) + J_V((1+\delta)\vartheta_\infty).$$

Therefore, (5.30) can be bounded from below as follows

$$\begin{aligned} &J_V^*(u(t)) - J_V^*(u_0) - \langle u(t)-u_0, \vartheta_\infty\rangle \\ &\ge \frac{\delta}{1+\delta} J_V^*(u(t)) - J_V((1+\delta)\vartheta_\infty) - J_V^*(u_0) + \langle u_0, \vartheta_\infty\rangle. \end{aligned}$$

The second integral in (5.29) is estimated with the help of (5.2) and (3.4–5). Namely, we have that

$$\begin{aligned} &\int_0^t \int_\Omega \Big(|\nabla\widehat{\vartheta}|^2 + (k*\nabla\widehat{\vartheta})\cdot\nabla\widehat{\vartheta}\Big) + \alpha\int_0^t\int_\Gamma |\widehat{\vartheta}|^2 \\ &\ge \omega\int_0^t\int_\Omega |\nabla\widehat{\vartheta}|^2 + \alpha\int_0^t\int_\Gamma |\widehat{\vartheta}|^2 \ge \omega\int_0^t \|\widehat{\vartheta}(s)\|^2\, ds \end{aligned}$$

because $\omega \le 1$. Concerning the right hand side of (5.29), it turns out that

$$\int_0^t \langle \widehat{f}(s), \widehat{\vartheta}(s)\rangle\, ds \le \frac{\omega}{2}\int_0^t \|\widehat{\vartheta}(s)\|^2\, ds + \frac{1}{2\omega}\int_0^t \|\widehat{f}(s)\|_*^2\, ds$$

as well as

$$-\int_0^t \iota(s)\langle B\vartheta_\infty, \widehat{\vartheta}(s)\rangle\, ds \le \frac{\omega}{4}\int_0^t \|\widehat{\vartheta}(s)\|^2\, ds + \frac{1}{\omega}\|\vartheta_\infty\|^2 \int_0^t \iota^2(s)\, ds.$$

Collecting all these inequalities, we obtain from (5.29)

$$\begin{aligned} &\frac{\delta}{1+\delta} J_V^*(u(t)) + \frac{\omega}{4}\int_0^t \|\widehat{\vartheta}(s)\|^2\, ds \\ &\le J_V((1+\delta)\vartheta_\infty) + J_V^*(u_0) - \langle u_0, \vartheta_\infty\rangle \\ &\qquad + \frac{1}{2\omega}\int_0^t \|\widehat{f}(s)\|_*^2\, ds + \frac{1}{\omega}\|\vartheta_\infty\|^2 \int_0^t \iota^2(s)\, ds \end{aligned} \tag{5.31}$$

for any $t > 0$. Thus, accounting also for (5.4) and (5.8), we have proved the first two properties listed in (5.18) and the second one in (5.17).

Consequent estimates. Clearly, (5.1) and (5.31) imply $B(k * \widehat{\vartheta}) \in L^2(0,\infty;V')$. Moreover, writing

$$B(k * \widehat{\vartheta})' = k(0)B\widehat{\vartheta} + k' * (B\widehat{\vartheta})$$

we conclude that

$$B(k * \widehat{\vartheta}) \in H^1(0,+\infty;V'). \tag{5.32}$$

Next, recalling (5.1) again and owing to (5.4), we deduce that

$$\iota B\vartheta_\infty \in H^1(0,+\infty;V'). \tag{5.33}$$

Finally, a comparison in (5.27) yields the last of (5.18). In addition, the norms of all the above functions are bounded by a computable constant.

Second a priori estimate. We are going to prove

$$\vartheta - \vartheta_\infty \in L^\infty(0,\infty;V) \qquad \text{and} \qquad \vartheta' \in L^2(0,\infty;H). \tag{5.34}$$

This allows us to conclude the proof of the stability part of the statement. To this aim, after splitting $\widehat{f}$ as

$$\widehat{f} = f_1 + f_2 \qquad \text{with} \qquad f_1 \in L^2(0,\infty;H) \quad \text{and} \quad f_2 \in H^1(0,+\infty;V')$$

we write (5.27) in the form

$$u' + A\widehat{\vartheta} = f_1 + F_2 \tag{5.35}$$

where

$$F_2 = f_2 - B(k * \widehat{\vartheta}) - \iota B\vartheta_\infty.$$

By (5.32–33) it has already been achieved that $F_2 \in H^1(0,+\infty;V')$. We now test (5.35) by $\vartheta' = \widehat{\vartheta}\,'$ and integrate over $(0,t)$. Although the procedure is formal, we perform the calculation, suggesting then how to make the argument rigorous. We have

$$\begin{aligned}
&\int_0^t \langle u'(s), \vartheta'(s)\rangle\, ds + \int_0^t ((\widehat{\vartheta}(s), \widehat{\vartheta}\,'(s)\,))\, ds \\
&= \int_0^t (f_1(s), \widehat{\vartheta}\,'(s))\, ds + \int_0^t \langle F_2(s), \widehat{\vartheta}\,'(s)\rangle\, ds.
\end{aligned} \tag{5.36}$$

The first integral is estimated from below as follows

$$\int_0^t \langle u'(s), \vartheta'(s)\rangle\, ds \geq \frac{1}{C_\beta}\int_0^t |\vartheta'(s)|^2\, ds$$

thanks to (5.7) and to the fact that $u \in \beta^{-1}(\vartheta)$. The second term is nothing but

$$\frac{1}{2}\,\|\widehat{\vartheta}(t)\|^2 - \frac{1}{2}\,\|\vartheta_0 - \vartheta_\infty\|^2 .$$

The first term on the right hand side is trivially controlled by

$$\frac{1}{2C_\beta}\int_0^t |\vartheta'(s)|^2\, ds + \frac{C_\beta}{2}\int_0^t |f_1(s)|^2\, ds.$$

Finally, we handle the last term this way

$$\begin{aligned}
\int_0^t \langle F_2(s), \widehat{\vartheta}\,'(s)\rangle\, ds &= \langle F_2(t), \widehat{\vartheta}(t)\rangle - \langle F_2(0), \vartheta_0 - \vartheta_\infty\rangle - \int_0^t \langle F_2'(s), \widehat{\vartheta}(s)\rangle\, ds \\
&\leq \frac{1}{4}\,\|\widehat{\vartheta}(t)\|^2 + \|F_2\|^2_{L^\infty(0,\infty;V')} + \frac{1}{2}\,\|\vartheta_0 - \vartheta_\infty\|^2 \\
&\quad + \frac{1}{2}\,\|F_2\|^2_{L^\infty(0,\infty;V')} + \|F_2'\|_{L^2(0,\infty;V')}\,\|\widehat{\vartheta}\|_{L^2(0,\infty;V)}
\end{aligned}$$

and note that $\|\widehat{\vartheta}\|_{L^2(0,\infty;V)}$ is estimated by (5.31) and that $\|F_2\|_{L^\infty(0,\infty;V')}$ can be bounded by means of $\|F_2\|_{H^1(0,+\infty;V')}$. Therefore, combining everything, we finally derive (5.34).

This argument can be made rigorous in the following way. The purpose is that of controlling an appropriate norm of the difference quotient $\delta_h(t) := h^{-1}(\widehat{\vartheta}(t) - \widehat{\vartheta}(t-h))$ by a constant independent of h and then take $h \to 0^+$ in the obtained inequality. More precisely, for any $h > 0$, we consider the mean value of (5.35) on $(t-h,t)$ with respect to time, then we test the obtained equation with δ_h and integrate over $(0,t)$. However, this procedure is correct provided that the equation holds also in $(-h,0)$. Therefore, we extend u, $\widehat{\vartheta}$, F_2 by their values at 0 in order to preserve continuity, and the constant value we must use to extend f_1 is necessarily

$$f_1^0 := A(\vartheta_0 - \vartheta_\infty) - F_2(0).$$

Then everything goes like in the above formal argument, but for a contribution near $t = 0$, namely

$$\frac{1}{h}\langle f_1^0, \int_0^h (h-s)\delta_h(s)\,ds\rangle. \tag{5.37}$$

Thus, we assume for a moment that $f_1^0 \in H$ and that its norm $\left|f_1^0\right|$ is bounded by a known constant C. Then (5.37) is estimated as follows

$$\begin{aligned}
&\le \frac{1}{h}\left|f_1^0\right| \int_0^h (h-s)\left|\delta_h(s)\right|\,ds \\
&\le \frac{1}{4C_\beta}\int_0^t \left|\delta_h(s)\right|^2\,ds + \frac{C_\beta h}{3}\left|f_1^0\right|^2 \le \frac{1}{4C_\beta}\int_0^t \left|\delta_h(s)\right|^2\,ds + \frac{C_\beta C^2}{3}h
\end{aligned}$$

and the last addendum actually figures in the right hand side of the estimate and tends to 0 as $h \to 0^+$. On the contrary, if $f_1^0 \notin H$, the above argument cannot be performed. In this case, we approximate F_2 by $F_{2,\varepsilon}$ in order that the corresponding value $f_{1,\varepsilon}^0$ belongs to H. This leads to an approximate solution $(u_\varepsilon, \widehat{\vartheta}_\varepsilon)$ of (5.35), which converges (in some suitable sense) to $(u, \widehat{\vartheta})$ as $\varepsilon \to 0^+$ (in particular, note that $u_\varepsilon \to u$ strongly in V' for all t due to the continuous dependence property stated in [3, Thm. 2.4]). Then, we can argue as before. The constant C is replaced by some C_ε, but we can take first $h \to 0^+$ and then $\varepsilon \to 0^+$.

Conclusion. The weak continuity of ϑ and (5.19) easily follow from (5.17). We have indeed $\widehat{\vartheta} \in H^1(0, +\infty; H)$ and deduce that $\widehat{\vartheta}$ is an $H-$ valued continuous function which goes to 0 at infinity, whence the strong convergence of $\vartheta(t)$ to ϑ_∞ in H is derived. On the other hand, $\widehat{\vartheta}$ is also an essentially bounded $V-$ valued function. Therefore, using a standard argument based on the density of H in V', we deduce that $\widehat{\vartheta}$ is a weakly continuous $V-$ valued function and that $\|\widehat{\vartheta}(t)\|$ is bounded everywhere in $\mathbb{R}$. This yields immediately the weak convergence in V. ∎

Proof of Corollary 5.5. It suffices to recall equation (5.27) and apply Theorem 5.4 and (5.32–33). ∎

Proof of Theorem 5.6. Suppose that $t_n \to +\infty$. From (5.31) we have

$$J_V^*(u(t_n)) \le C \qquad \forall\, n$$

for some fixed constant C, i.e., $u(t_n) \in E_C$ for all n. As any E_c is strongly relatively compact in V', we can choose a subsequence, still termed $\{t_n\}$ for simplicity, such that $\{u(t_n)\}$ is strongly convergent in V' to some $u_\infty \in V'$. Moreover, arguing as in the last part of the proof of Corollary 3.2, we derive that (5.22) holds and that some subsequence from $\{u(t_n)\}$ weakly converges to u_∞ in $L^1(\Omega)$. This also yields (5.21). Assume now (5.21–22) for some sequence $t_n \to +\infty$. Using (3.9) and the continuity of u given by (3.12), we can choose t_n' such that

$$|t_n' - t_n| \le 1/n, \quad \|u(t_n') - u(t_n)\|_* \le 1/n, \quad \text{and} \quad u(t_n') \in \partial_{V,V'} J_V(\vartheta(t_n')) \quad \forall\, n.$$

Thus, also $u(t'_n)$ converges to u_∞ strongly in V'. On the other hand, we have that $\vartheta(t'_n) \rightharpoonup \vartheta_\infty$ in V by (5.19). Hence, we can apply [5, Lemma 1.3, p. 42] and infer

$$u_\infty \in \partial_{V,V'} J_V(\vartheta_\infty). \tag{5.38}$$

Therefore, (5.23) follows from (5.22) and (5.38), thanks to Proposition 2.9. Finally, the last sentence in the statement can be proved using standard arguments. ■

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