# Global smooth solution to the standard phase-field model with memory 

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#### Abstract

This paper is devoted to study the so-called phase-field model when the classical Fourier law is replaced by the Gurtin-Pipkin constitutive assumption. The resulting system of partial differential equations is investigated in a quite general setting. A hyperbolic equation is coupled with a parabolic variational inequality, the state variables being temperature and non-conserved order parameter. By including initial and boundary conditions, the existence and uniqueness of strong solutions is shown along with regularity results ensuring the global boundedness of both the unknowns.


1. Introduction. In this paper we address the initial and homogeneous Neumann boundary value problem for two coupled integrodifferential equations governing the dynamics of solid-liquid phase transitions, i.e., the evolution of the temperature field $\vartheta$ and the phase field $\chi$ (which may stand for the local proportion of one of the two phases) in a three-dimensional body.

The corresponding system turns out to provide an extended version of the wellknown phase field model [7, 22], but with the heat flux $\mathbf{q}$ expressed by a non-Fourier law. In fact, according to [26] the heat flux is assumed to depend only on the temporal history of the temperature gradient $\nabla \vartheta$ through a constitutive relation of this type

$$
\begin{equation*}
\left.\mathbf{q}(x, t)=-\int_{-\infty}^{t} k(t-s) \nabla \vartheta(x, s) d s \quad \text { for }(x, t) \in \Omega \times\right] 0, T[ \tag{1.1}
\end{equation*}
$$

Here the bounded smooth domain $\Omega \subset \mathbb{R}^{3}$ represents the container of the solid-liquid body, $T>0$ denotes some final time, and $k:[0,+\infty[\rightarrow \mathbb{R}$ is a given function depending on the material. Thanks to laws like (1.1), memory effects are accounted for

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in the description of heat conduction phenomena. In particular, the choice (1.1) has received a great deal of attention in the last decades (see, among others, [5, 34-35]), dating back from the pioneering work of Cattaneo [10] and going to the recent fairly complete review done by Joseph and Preziosi [27, 28]. Let us just mention one feature of the Gurtin and Pipkin theory [26] (relying on (1.1)), namely, it predicts finite speed of propagation for thermal disturbances (see also [11-13]).

As a first and related consequence, the energy balance

$$
\begin{equation*}
\left.(\vartheta+\lambda(\chi))_{t}+\nabla \cdot \mathbf{q}=\bar{g} \quad \text { in } Q:=\Omega \times\right] 0, T[ \tag{1.2}
\end{equation*}
$$

supplied with (1.1), provides an equation which is no longer parabolic with respect to $\vartheta$. Its hyperbolic character will become clear in the sequel. Let us recall that $\bar{g}: Q \rightarrow \mathbb{R}$ plays as source term in (1.2) and that, according to former approaches (cf., e.g., [7-9]), $\lambda(\chi)$ is usually supposed to be linear in $\chi$. In our analysis we allow $\lambda \in C^{2}(\mathbb{R})$ to be Lipschitz continuous along with its derivative.

In the case where the order parameter $\chi$ is not conserved (an interesting classification of both kinds of models for non-isothermal phase changes can be found in [2]), (1.2) is combined with the second order differential inclusion

$$
\begin{equation*}
\mu \chi_{t}-\nu \Delta \chi+\beta(\chi) \ni \bar{\gamma}(\vartheta, \chi)+\lambda^{\prime}(\chi) \vartheta \quad \text { in } Q \tag{1.3}
\end{equation*}
$$

the relaxation parameters $\mu$ and $\nu$ being small but positive in the framework of the phase field description of the transition dynamics. While the nonlinearity $\bar{\gamma}$ is Lipschitz continuous with respect to both variables, $\beta$ denotes a maximal monotone graph from $\mathbb{R}$ to $\mathbb{R}$, possibly multivalued, and such that $0 \in \beta(0)$.

Relation (1.3) can be derived following the Ginzburg-Landau theory for the construction of the free energy functional, then applying the second law of thermodynamics (via the Clausius-Duhem inequality), and finally making a first order approximation around a critical temperature (the phase change temperature). Details of the procedure are given, for instance, in $[7,22,23,36,37]$. Thus, we point out that our $\vartheta$ does not specify the absolute temperature. Instead, it is relative to the phase change temperature to agree that $\vartheta>0$ in the liquid phase, $\vartheta<0$ in the solid phase. We know of another phase field model, proposed by Penrose and Fife [36, 37], that remains consistent with thermodynamics because no linearization around equilibrium temperatures is done. Such alternative model has been recently investigated by several authors for nonlinear (but local) heat flux laws which become singular as the temperature approaches the absolute zero value. Among the works on this subject, let us mention the papers [30, 31, 41] and the review done in [38].

However, system (1.2-3) has a quite general form which includes various possibilities. If $\mu=\nu=0, \lambda(\chi)=\chi, \beta$ coincides with the subdifferential of the indicator function of the interval $[0,1]$, and $\bar{\gamma} \equiv 0$, then (1.2-3) reduces to the weak formulation of the classical Stefan problem (see, e.g., [32, p. 196] and the references in the related "Commentaires" at p. 306). Moreover, letting $\mu>0$ and possibly retaining the
nonlinear term $\bar{\gamma}(\vartheta, \chi)$ lead to Stefan problems with phase relaxation and dissipation that have been introduced in [42, 23] and further analyzed in [20].

These two classes of Stefan and relaxed problems have been already faced also keeping constitutive laws like (1.1). Nonetheless, as far as we know (see [17, 19, 27, 28] for a number of references on hyperbolic Stefan problems), the only available existence result in this framework [16] addresses to the case $\mu>0$ and $\bar{\gamma} \equiv 0$. Otherwise, existence and uniqueness, regularity, continuous dependence properties have been deduced [39, 17, 14] for initial-boundary value problems applying to a significant variation of (1.1-3). Indeed, equation (1.2) was modified (cf. [16] and [17]) by substituting $\lambda(\chi)=\chi$ with a convolution product like (1.1), between $\chi$ and a memory kernel (which could simply consists in an approximation of the Dirac mass). Now, it has been shown [19, 18] that these alternative models are, in some rigorous sense, acceptable from the physical side since aymptotically closed to their parabolic counterparts without memories (cf. in particular [19]). However, it is not our intention to proceed along this direction in the present paper. Indeed, one of the main points for us was discovering that, provided some diffusion is assumed for the order parameter $\chi \quad(\nu>0)$, in regard of the conclusion of [16] here one can handle nonlinearities and prove existence and uniqueness of smooth solutions.

Coming back to the possible applications of (1.2-3), we must note that the proper phase-field model is obtained, neglecting constant factors, for $\mu>0, \nu>0, \lambda(\chi)=\chi$, $\beta(\chi)=\chi^{3}$, and $\bar{\gamma}(\vartheta, \chi)=\chi$, the last choices corresponding to the double-well potential in the free energy proposed by Fix [22] and Caginalp [7]. Coupling (1.2) with the Fourier law, the resulting system has been widely studied and somehow generalized, going from the results of [21] to the recent investigation [29]. Concerning the relation (1.1), we are eventually able to quote the paper [1] where questions like existence, regularity, and asymptotic behaviour as $t$ tends to $+\infty$ are treated with assumptions and methods different from ours. In fact, while we can conclude that both $\vartheta$ and $\chi$ belong to $L^{\infty}(Q)$ whenever $k$ is smooth enough and $k(0)>0$, in [1] Aizicovici and Barbu find a weaker solution for kernels $k$ of positive type. In addition, we use variational techniques and deal with the general conditions of (1.2-3), instead of their semigroup approach directed to the above particular setting. Anyway, in the case of a linear function $\lambda$, a careful and sharp analysis of weak solutions (reproducing the weak theory for linear hyperbolic problems) is developed in the twin paper [15].

Assuming that the history of $\vartheta$ is known up to the time $t=0$ and introducing the notation

$$
(a * b)(t):=\int_{0}^{t} a(s) b(t-s) d s, \quad t \in[0, T]
$$

for the convolution product with respect to time (being understood that the functions $a$ and $b$ may also depend on space variables), from (1.1-2) we can infer the equation

$$
\begin{equation*}
(\vartheta+\lambda(\chi))_{t}-\Delta(k * \vartheta)=g \quad \text { in } Q \tag{1.4}
\end{equation*}
$$

whose right hand side $g$ collects $\bar{g}$ and a term involving the values of temperature for $t<0$ (see [18] for a precise derivation accounting for regularity of data). As boundary conditions to couple with (1.4) and (1.3) we take

$$
\begin{equation*}
\partial_{\mathrm{n}}(k * \vartheta)=\partial_{\mathrm{n}} \chi=0 \quad \text { on } \Sigma:=\Gamma \times(0, T) \tag{1.5}
\end{equation*}
$$

where $\Gamma=\partial \Omega$ and $\partial_{\mathrm{n}}$ stands for the outer normal derivative on $\Gamma$. Observe that (1.5) prescribes that the system is isolated from the exterior, the conditions meaning that there is no flow across the boundary (in particular, we can rewrite the former as $\mathbf{q} \cdot \mathbf{n}=0$, with $\mathbf{n}$ outward normal vector, owing to (1.1)). Finally, we have to fix initial values for temperature and order parameter

$$
\vartheta(\cdot, 0)=\vartheta_{0}, \quad \chi(\cdot, 0)=\chi_{0} \quad \text { in } \Omega
$$

(giving some $\vartheta_{0}, \chi_{0}: \Omega \rightarrow \mathbb{R}$ ) or, equivalently, we can do that for $\chi$ and the auxiliary unknown

$$
\begin{equation*}
\eta:=\vartheta+\lambda(\chi) \tag{1.6}
\end{equation*}
$$

which corresponds to the enthalpy. It is worth remarking that the latter choice happens to be more natural in view of (1.4). Therefore, we can ask that

$$
\begin{equation*}
\eta(\cdot, 0)=\eta_{0}, \quad \chi(\cdot, 0)=\chi_{0} \quad \text { in } \Omega \tag{1.7}
\end{equation*}
$$

where $\eta_{0}$ is related to $\vartheta_{0}$ and $\chi_{0}$ by the obvious position $\eta_{0}=\vartheta_{0}+\lambda\left(\chi_{0}\right)$.
This paper is then concerned with the problem (1.3-7) in the three-dimensional case, even though a number of our results still holds in any dimension with the same proofs, if one assumes that $\lambda$ is linear. We prove the existence of a unique pair ( $\vartheta, \chi)$ solving (1.3-7). Such solution is strong in the sense that equations are satisfied at least almost everywhere. The existence proof relies on an approximation - a priori estimates - passage to the limit argument, based on a Faedo-Galerkin scheme combined with monotonicity and compactness methods. Reasonable assumptions on $g, k, \eta_{0}, \chi_{0}$ in addition to the well-known three-dimensional embedding $H^{2}(\Omega) \subset L^{\infty}(\Omega)$ enable us to deduce that $\chi$ is bounded in $Q$. By further requiring that $\lambda^{\prime \prime \prime}$ is continuous and $\beta$ is a function, we can even show that $\vartheta \in C^{0}(\bar{Q})$ as well as $\chi$.

The first step of all this machinery consists in transforming the problem by finding the right variables useful for computations. As the identity $k * \vartheta=k(0)(1 * \vartheta)+k^{\prime} * 1 * \vartheta$ holds, one possibility could be to rewrite (1.4) in terms of $1 * \vartheta$. This is precisely the way followed in $[16,17]$ and it already allows to see the hyperbolic character of equation (1.4). Here it seems to be more convenient to adopt the strategy of taking a time integral function of the enthalpy instead of the temperature. More precisely, we set

$$
\begin{equation*}
w:=1 * \eta=1 * \vartheta+1 * \lambda(\chi) \tag{1.8}
\end{equation*}
$$

and reformulate the problem in terms of $w$ and $\chi$. Denoting by $\gamma$ the Lipschitz continuous function such that $\gamma\left(w_{t}, \chi\right)=\bar{\gamma}\left(w_{t}-\lambda(\chi), \chi\right)$, it is straightforward to verify that (1.4) and (1.3) become

$$
\begin{align*}
& w_{t t}-k(0) \Delta w=g+k^{\prime} * \Delta w-k * \Delta \lambda(\chi)  \tag{1.9}\\
& \mu \chi_{t}-\nu \Delta \chi+\beta(\chi) \ni \gamma\left(w_{t}, \chi\right)+\lambda^{\prime}(\chi)\left(w_{t}-\lambda(\chi)\right) \tag{1.10}
\end{align*}
$$

in $Q$. Thanks to (1.5-7) and to an inversion formula for Volterra integral equations (cf., e.g., [25, Chap. 2]), it follows that (1.9) and (1.10) are complemented by the boundary and initial conditions

$$
\begin{align*}
& \partial_{\mathrm{n}} w=\partial_{\mathrm{n}} \chi=0 \quad \text { on } \Sigma  \tag{1.11}\\
& w(\cdot, 0)=0, \quad w_{t}(\cdot, 0)=\eta_{0}, \quad \chi(\cdot, 0)=\chi_{0} \quad \text { in } \Omega \tag{1.12}
\end{align*}
$$

Referring to the sequel for a rigorous justification, let us point out that our procedure changes (1.4) into another equation where no time derivative of $\chi$ appears. Since the Laplacian of $\lambda(\chi)$ is somehow integrated in time, by this formulation it is easier to obtain the basic estimates.

Here is the plan of the paper. The next section is devoted to state the main results. In Section 3 we present the equivalent formulation of our problem relying on position (1.8) and we give the corresponding versions of the main theorems. Sections 4 is concerned with the proof of uniqueness, while the existence of the solution is shown in Section 6 passing to the limit in the approximating problems introduced in Section 5. Finally, our regularity results are discussed in Sections 7-9.
2. Main results. We first introduce the assumptions on the data and then give a rigorous formulation of the initial and boundary value problem we are dealing with. After that, the main results are stated.

Let $\Omega \subset \mathbb{R}^{3}$ be a bounded, open, and connected set of class $C^{2,1}$ with boundary $\Gamma:=\partial \Omega$ and let $T>0$. Set

$$
Q:=\Omega \times] 0, T[, \quad \Sigma:=\Gamma \times] 0, T[
$$

and consider

$$
\begin{align*}
& k \in W^{2,1}(0, T), \quad k(0)>0  \tag{2.1}\\
& \lambda \in C^{2}(\mathbb{R}), \quad \lambda^{\prime}, \lambda^{\prime \prime} \in L^{\infty}(\mathbb{R})  \tag{2.2}\\
& \mu, \nu \in] 0, \infty[  \tag{2.3}\\
& \phi: \mathbb{R} \rightarrow[0,+\infty] \quad \text { convex and lower-semicontinuous, } \quad \phi(0)=0  \tag{2.4}\\
& \beta=\partial \phi \subset \mathbb{R} \times \mathbb{R} \quad \text { and } \quad \beta(0) \ni 0  \tag{2.5}\\
& \bar{\gamma} \in C^{1}\left(\mathbb{R}^{2}\right) \quad \text { with bounded partial derivatives }  \tag{2.6}\\
& g \in W^{1,1}\left(0, T ; L^{2}(\Omega)\right)+L^{1}\left(0, T ; H^{1}(\Omega)\right)  \tag{2.7}\\
& \vartheta_{0} \in H^{1}(\Omega)  \tag{2.8}\\
& \chi_{0} \in H^{1}(\Omega) \quad \text { and } \quad \phi\left(\chi_{0}\right) \in L^{1}(\Omega) . \tag{2.9}
\end{align*}
$$

Denoting by $D(\beta)$ the effective domain of $\beta$, we introduce
Problem (P1). Find $\vartheta, \chi, \xi: Q \rightarrow \mathbb{R}$ satisfying

$$
\begin{align*}
& \vartheta \in L^{\infty}\left(0, T ; H^{1}(\Omega)\right)  \tag{2.10}\\
& 1 * \vartheta \in L^{\infty}\left(0, T ; H^{2}(\Omega)\right)  \tag{2.11}\\
& \vartheta_{t} \in L^{2}\left(0, T ; L^{2}(\Omega)\right)+L^{1}\left(0, T ; H^{1}(\Omega)\right)  \tag{2.12}\\
& \chi \in H^{1}\left(0, T ; L^{2}(\Omega)\right) \cap C^{0}\left([0, T] ; H^{1}(\Omega)\right) \cap L^{2}\left(0, T ; H^{2}(\Omega)\right)  \tag{2.13}\\
& \xi \in L^{2}(Q)  \tag{2.14}\\
& \chi \in D(\beta) \quad \text { and } \quad \xi \in \beta(\chi) \quad \text { a.e. in } Q  \tag{2.15}\\
& \partial_{t}(\vartheta+\lambda(\chi))-\Delta(k * \vartheta)=g \quad \text { a.e. in } Q  \tag{2.16}\\
& \mu \chi_{t}-\nu \Delta \chi+\xi=\bar{\gamma}(\vartheta, \chi)+\lambda^{\prime}(\chi) \vartheta \quad \text { a.e. in } Q  \tag{2.17}\\
& k * \partial_{\mathrm{n}} \vartheta=\partial_{\mathrm{n}} \chi=0 \quad \text { on } \Sigma  \tag{2.18}\\
& \vartheta(0)=\vartheta_{0} \quad \text { in } \Omega  \tag{2.19}\\
& \chi(0)=\chi_{0} \quad \text { in } \Omega . \tag{2.20}
\end{align*}
$$

The first result reads
Theorem 2.1. Let the assumptions (2.1-9) hold. Then Problem (2.10-20) admits a unique solution $(\vartheta, \chi, \xi)$. Moreover, $\vartheta$ and $\chi$ fulfill

$$
\begin{align*}
& \vartheta \in C^{0}\left([0, T] ; H^{1}(\Omega)\right), \quad 1 * \vartheta \in C^{0}\left([0, T] ; H^{2}(\Omega)\right),  \tag{2.21}\\
& \phi(\chi) \in L^{\infty}\left(0, T ; L^{1}(\Omega)\right) . \tag{2.22}
\end{align*}
$$

Suppose now

$$
\begin{array}{lll}
\chi_{0} \in H^{2}(\Omega) & \text { and } \quad \partial_{\mathrm{n}} \chi_{0}=0 & \text { on } \Gamma \\
\chi_{0} \in D(\beta) & \text { a.e. in } \Omega & \text { and } \tag{2.24}
\end{array} \beta^{0}\left(\chi_{0}\right) \in L^{2}(\Omega)
$$

where, for $y \in D(\beta), \beta^{0}(y)$ is the element of $\beta(y)$ having minimum modulus. In this case we can prove a regularity result for $\chi$, namely

Theorem 2.2. Assume (2.23-24) in addition to (2.1-9) and let $(\vartheta, \chi, \xi)$ be the unique solution to Problem (2.10-20). Then $\chi$ satisfies

$$
\begin{equation*}
\chi \in W^{1, \infty}\left(0, T ; L^{2}(\Omega)\right) \cap H^{1}\left(0, T ; H^{1}(\Omega)\right) \cap L^{\infty}\left(0, T ; H^{2}(\Omega)\right) . \tag{2.25}
\end{equation*}
$$

Remark 2.3. Clearly, Problem (P1) can be stated in any spatial dimension $N$. As it appears from the subsequent proofs, the above results still hold provided $N \leq 4$. Moreover, it is worth noting that no restriction on $N$ is needed if one assumes $\lambda$ to be linear, as we will emphasize in the sequel.

Next, working in the three-dimensional case, we point out that the validity of the regularity results we are going to state extends to $\Omega \subset \mathbb{R}^{N}$ with $N \leq 3$, of course.

Remark 2.4. Thanks to the injections $H^{1}\left(0, T ; H^{1}(\Omega)\right) \subset C^{0,1 / 2}\left([0, T] ; H^{1}(\Omega)\right)$ and $H^{s}(\Omega) \subset C^{0}(\bar{\Omega})$ for $s>3 / 2,(2.25)$ and [33, Prop. 1.1.4, p. 13] entail

$$
\begin{equation*}
\chi \in C^{0}(\bar{Q}) \tag{2.26}
\end{equation*}
$$

More precisely, $\chi \in C^{0, \alpha_{1}}\left([0, T] ; C^{0, \alpha_{2}}(\bar{\Omega})\right)$ for some $0<\alpha_{i}<1, i=1,2$.

By strengthening the assumptions, we obtain further regularity for $\vartheta$ and $\chi$ which yields continuity up to the boundary for $\vartheta$ as well. Take

$$
\begin{align*}
& \lambda \in C^{3}(\mathbb{R}), \quad \lambda^{\prime}, \lambda^{\prime \prime} \in L^{\infty}(\mathbb{R})  \tag{2.27}\\
& \beta \in C^{1}(\mathbb{R}) \quad \text { non decreasing }  \tag{2.28}\\
& g \in W^{2,1}\left(0, T ; L^{2}(\Omega)\right)+W^{1,1}\left(0, T ; H^{1}(\Omega)\right), \quad g(0) \in H^{1}(\Omega)  \tag{2.29}\\
& \vartheta_{0} \in H^{2}(\Omega), \quad \partial_{\mathrm{n}} \vartheta_{0}=0 \quad \text { on } \Gamma  \tag{2.30}\\
& \chi_{0} \in H^{3}(\Omega), \quad \partial_{\mathrm{n}} \chi_{0}=0 \quad \text { on } \Gamma . \tag{2.31}
\end{align*}
$$

Observing that (2.28) and (2.31) entail (2.24) and (2.23), the stronger hypotheses lead to

Theorem 2.5. Assume (2.27-31) besides (2.1-9) and let $(\vartheta, \chi, \xi)$ be the unique solution to Problem (2.10-20). Then

$$
\begin{align*}
& \vartheta \in W^{2,1}\left(0, T ; L^{2}(\Omega)\right) \cap C^{1}\left([0, T] ; H^{1}(\Omega)\right) \cap C^{0}\left([0, T] ; H^{2}(\Omega)\right)  \tag{2.32}\\
& \chi \in H^{2}\left(0, T ; L^{2}(\Omega)\right) \cap C^{1}\left([0, T] ; H^{1}(\Omega)\right) \cap H^{1}\left(0, T ; H^{2}(\Omega)\right) . \tag{2.33}
\end{align*}
$$

In particular one has

$$
\begin{equation*}
\vartheta, \chi \in C^{0, \alpha}(\bar{Q}) \quad \text { for some } 0<\alpha \leq 1 . \tag{2.34}
\end{equation*}
$$

3. An equivalent formulation. As we stressed in the Introduction, a convenient formulation of Problem (P1), that is (2.10-20), is based on the use of the integrated enthalpy as variable. In order to specify the setting, put

$$
\begin{equation*}
\gamma\left(y_{1}, y_{2}\right)=\bar{\gamma}\left(y_{1}-\lambda\left(y_{2}\right), y_{2}\right) \quad \text { for } \quad\left(y_{1}, y_{2}\right) \in \mathbb{R}^{2} \tag{3.1}
\end{equation*}
$$

and let $\eta_{0}$ be related to $\vartheta_{0}$ and $\chi_{0}$ by

$$
\begin{equation*}
\eta_{0}=\vartheta_{0}+\lambda\left(\chi_{0}\right) \tag{3.2}
\end{equation*}
$$

Under the assumptions (2.1-9) (see, especially, (2.2), (2.6), and (2.8-9)) it results that

$$
\begin{equation*}
\gamma \in C^{1}\left(\mathbb{R}^{2}\right) \quad \text { with partial derivatives } \gamma_{1}, \gamma_{2} \in L^{\infty}\left(\mathbb{R}^{2}\right) \tag{3.3}
\end{equation*}
$$

and that

$$
\begin{equation*}
\eta_{0} \in H^{1}(\Omega) \tag{3.4}
\end{equation*}
$$

Then, if $(\vartheta, \chi, \xi)$ is a solution to ( P 1 ) and $w$ is defined by (1.8), thanks to the identity $k * \vartheta=k(0)(1 * \vartheta)+k^{\prime} * 1 * \vartheta$, one easily checks that the triplet $(w, \chi, \xi)$ solves

Problem (P2). Find $w, \chi, \xi: Q \rightarrow \mathbb{R}$ satisfying

$$
\begin{align*}
& w \in W^{1, \infty}\left(0, T ; H^{1}(\Omega)\right) \cap L^{\infty}\left(0, T ; H^{2}(\Omega)\right)  \tag{3.5}\\
& w_{t t} \in L^{\infty}\left(0, T ; L^{2}(\Omega)\right)+L^{1}\left(0, T ; H^{1}(\Omega)\right)  \tag{3.6}\\
& \chi \in H^{1}\left(0, T ; L^{2}(\Omega)\right) \cap C^{0}\left([0, T] ; H^{1}(\Omega)\right) \cap L^{2}\left(0, T ; H^{2}(\Omega)\right)  \tag{3.7}\\
& \xi \in L^{2}(Q)  \tag{3.8}\\
& \chi \in D(\beta) \quad \text { and } \quad \xi \in \beta(\chi) \quad \text { a.e. in } Q  \tag{3.9}\\
& w_{t t}-k(0) \Delta w=g+k^{\prime} * \Delta w-k * \Delta \lambda(\chi) \quad \text { a.e. in } Q  \tag{3.10}\\
& \mu \chi_{t}-\nu \Delta \chi+\xi=\gamma\left(w_{t}, \chi\right)+\lambda^{\prime}(\chi)\left(w_{t}-\lambda(\chi)\right) \quad \text { a.e. in } Q  \tag{3.11}\\
& \partial_{\mathrm{n}} w=\partial_{\mathrm{n}} \chi=0 \quad \text { on } \Sigma  \tag{3.12}\\
& w(0)=0 \quad \text { and } \quad w_{t}(0)=\eta_{0} \quad \text { in } \Omega  \tag{3.13}\\
& \chi(0)=\chi_{0} \quad \text { in } \Omega . \tag{3.14}
\end{align*}
$$

Conversely, if ( $w, \chi, \xi$ ) is a solution to (P2), setting

$$
\begin{equation*}
\vartheta:=w_{t}-\lambda(\chi) \quad \text { a.e. in } Q \tag{3.15}
\end{equation*}
$$

straightforward arguments show that $(\vartheta, \chi, \xi)$ solves (P1). To sum up, we have
Proposition 3.1. Let (2.1-9) hold. Then Problem (P1) admits a unique solution $(\vartheta, \chi, \xi)$ if and only if Problem (P2) admits a unique solution $(w, \chi, \xi)$.

In view of Proposition 3.1, Theorem 2.1 turns out to be equivalent to
Theorem 3.2. Let (2.1-9) hold. Then Problem (3.5-14) admits a unique solution $(w, \chi, \xi)$. Moreover,

$$
\begin{equation*}
w \in C^{1}\left([0, T] ; H^{1}(\Omega)\right) \cap C^{0}\left([0, T] ; H^{2}(\Omega)\right) \tag{3.16}
\end{equation*}
$$

and (2.22) is satisfied.

Also, it is evident that proving Theorem 2.2 is equivalent to proving
Theorem 3.3. Assume (2.23-24) in addition to (2.1-9). Then the unique solution ( $w, \chi, \xi$ ) to Problem (3.5-14) fulfills (2.25).

Finally, in the regularity setting specified by (2.27-31) and entailing (cf. (3.2))

$$
\begin{equation*}
\eta_{0} \in H^{2}(\Omega), \quad \partial_{\mathrm{n}} \eta_{0}=0 \quad \text { on } \quad \Gamma \tag{3.17}
\end{equation*}
$$

we can establish the equivalent version of Theorem 2.5.
Theorem 3.4. Assume (2.27-31) besides (2.1-9). Then the unique solution ( $w, \chi, \xi$ ) to Problem (3.5-14) satisfies (2.33) and

$$
\begin{equation*}
w \in W^{3,1}\left(0, T ; L^{2}(\Omega)\right) \cap C^{2}\left([0, T] ; H^{1}(\Omega)\right) \cap C^{1}\left([0, T] ; H^{2}(\Omega)\right) \tag{3.18}
\end{equation*}
$$

In particular one has

$$
\begin{equation*}
w, w_{t}, \chi \in C^{0}(\bar{Q}) \tag{3.19}
\end{equation*}
$$

For the sake of convenience, let us fix here, once and for all, some notation and recall standard tools. We set

$$
\begin{aligned}
& \left.Q_{t}:=\Omega \times\right] 0, t[, \quad t \in] 0, T[ \\
& H:=L^{2}(\Omega), \quad V:=H^{1}(\Omega), \quad \text { and } \quad W:=H^{2}(\Omega) .
\end{aligned}
$$

Further, we denote by the same symbol the norm of a space of scalar functions and the norm of the space of corresponding vector-valued functions. For instance, $\|\cdot\|_{V}$ means the norm of both $V$ and $V^{3}$. The same is done for scalar products. Moreover, we use the formulas (cf., e.g., [25])

$$
\begin{equation*}
a * b=a(0)(1 * b)+a_{t} * 1 * b \quad \text { and } \quad(a * b)_{t}=a(0) b+a_{t} * b \tag{3.20}
\end{equation*}
$$

which hold whenever they make sense, the well-known Young theorem

$$
\begin{equation*}
\|a * b\|_{L^{r}(0, T ; X)} \leq\|a\|_{L^{p}(0, T)}\|b\|_{L^{q}(0, T ; X)} \tag{3.21}
\end{equation*}
$$

$X$ being a real Banach space, $1 \leq p, q, r \leq \infty, 1 / r=(1 / p)+(1 / q)-1$, and the elementary inequality

$$
\begin{equation*}
2 a b \leq \sigma a^{2}+\frac{1}{\sigma} b^{2} \quad \forall a, b \geq 0 \quad \forall \sigma>0 \tag{3.22}
\end{equation*}
$$

Warning 3.5. In the proofs we give in the sequel the same symbol $c$ is employed for different constants, even in the same formula, in regard of simplicity. These constants may generally depend on $\Omega, T, \mu, \nu, k(0),\left\|k^{\prime}\right\|_{W^{1,1}(0, T)},|\lambda(0)|,\left\|\lambda^{\prime}\right\|_{L^{\infty}(\mathbb{R})}$, $\left\|\lambda^{\prime \prime}\right\|_{L^{\infty}(\mathbb{R})},|\gamma(0,0)|,\left\|\gamma_{1}\right\|_{L^{\infty}\left(\mathbb{R}^{2}\right)}$, and $\left\|\gamma_{2}\right\|_{L^{\infty}\left(\mathbb{R}^{2}\right)}$. Further dependences are specified in each section, separately.
4. Uniqueness. Let $\left(w_{1}, \chi_{1}, \xi_{1}\right)$ and $\left(w_{2}, \chi_{2}, \xi_{2}\right)$ be two solutions to Problem (3.5-14). We prove that they necessarily coincide. To do that, we derive some a priori estimates for the functions

$$
\begin{equation*}
w:=w_{1}-w_{2}, \quad \chi:=\chi_{1}-\chi_{2}, \quad \xi:=\xi_{1}-\xi_{2} \tag{4.1}
\end{equation*}
$$

The constants $c$ can depend on the just mentioned quantities and on the norms of the two solutions in the spaces listed in (3.5-8), as well.

In view of (3.10-11), one can easily check that $(w, \chi, \xi)$ satisfies

$$
\begin{align*}
& w_{t t}-k(0) \Delta w=k^{\prime} * \Delta w-k * \Delta\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)  \tag{4.2}\\
& \mu \chi_{t}-\nu \Delta \chi+\xi=\gamma\left(\partial_{t} w_{1}, \chi_{1}\right)-\gamma\left(\partial_{t} w_{2}, \chi_{2}\right)  \tag{4.3}\\
& \quad+\lambda^{\prime}\left(\chi_{1}\right)\left(w_{t}-\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)\right)+\left(\lambda^{\prime}\left(\chi_{1}\right)-\lambda^{\prime}\left(\chi_{2}\right)\right)\left(\partial_{t} w_{2}-\lambda\left(\chi_{2}\right)\right)
\end{align*}
$$

a.e. in $Q$. Moreover, the following boundary and initial conditions hold

$$
\begin{align*}
& \partial_{\mathrm{n}} w=\partial_{\mathrm{n}} \chi=0 \quad \text { on } \Sigma  \tag{4.4}\\
& w(0)=w_{t}(0)=\chi(0)=0 \quad \text { in } \Omega . \tag{4.5}
\end{align*}
$$

Multiplying (4.2) by $w_{t}$, integrating over $Q_{t}$, where $t \in(0, T)$ is arbitrary, and using (4.4-5), we obtain

$$
\begin{equation*}
\frac{1}{2}\left\|w_{t}(t)\right\|_{H}^{2}+\frac{k(0)}{2}\|\nabla w(t)\|_{H}^{2}=I_{1}(t)+I_{2}(t) \tag{4.6}
\end{equation*}
$$

with

$$
\begin{align*}
& I_{1}(t):=-\int_{0}^{t}\left(\left(k^{\prime} * \nabla w\right)(s), \nabla w_{t}(s)\right)_{H} d s  \tag{4.7}\\
& I_{2}(t):=\int_{0}^{t}\left(\left(k * \nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)\right)(s), \nabla w_{t}(s)\right)_{H} d s \tag{4.8}
\end{align*}
$$

Recalling (2.1) and (3.20), we can integrate by parts in time in (4.7) and (4.8). This gives

$$
\begin{align*}
I_{1}(t)= & -\int_{\Omega}\left(k^{\prime} * \nabla w\right)(t) \cdot \nabla w(t)+\iint_{Q_{t}}\left(k^{\prime}(0) \nabla w+k^{\prime \prime} * \nabla w\right) \cdot \nabla w  \tag{4.9}\\
I_{2}(t)= & \int_{\Omega}\left(k * \nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)\right)(t) \cdot \nabla w(t)  \tag{4.10}\\
& -\iint_{Q_{t}}\left(k(0) \nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)+k^{\prime} * \nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)\right) \cdot \nabla w .
\end{align*}
$$

Using well-known inequalities as (3.21) and (3.22), from (4.9) we are able to infer (cf. also (2.1))

$$
\begin{equation*}
\left|I_{1}(t)\right| \leq \frac{k(0)}{8}\|\nabla w(t)\|_{H}^{2}+c \int_{0}^{t}\|\nabla w(s)\|_{H}^{2} d s \tag{4.11}
\end{equation*}
$$

To deal with the first integral in the right hand side of (4.10), note that, owing to (2.2), one has

$$
\left|\nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)\right| \leq c\left(|\nabla \chi|+\left|\nabla \chi_{2}\right||\chi|\right)
$$

Then, on account of (2.1), the Hölder inequality and (3.21) give

$$
\begin{align*}
& \int_{\Omega}\left(k * \nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)\right)(t) \cdot \nabla w(t)  \tag{4.12}\\
& \leq c\|\nabla w(t)\|_{H} \int_{0}^{t}\left\{\|\nabla \chi(s)\|_{H}+\left\|\nabla \chi_{2}(s)\right\|_{L^{4}(\Omega)}\|\chi(s)\|_{L^{4}(\Omega)}\right\} d s
\end{align*}
$$

As $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^{3}$, there holds

$$
\begin{equation*}
\|v\|_{L^{4}(\Omega)} \leq C\|v\|_{V} \quad \forall v \in V \tag{4.13}
\end{equation*}
$$

for some constant $C$ depending only on $\Omega$. Consequently, from (4.12) we deduce that

$$
\begin{align*}
& \int_{\Omega}\left(k * \nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)\right)(t) \cdot \nabla w(t)  \tag{4.14}\\
& \leq c\|\nabla w(t)\|_{H}\left(1+\left\|\chi_{2}\right\|_{L^{2}(0, T ; W)}\right)\left(\int_{0}^{t}\|\chi(s)\|_{V}^{2} d s\right)^{1 / 2} \\
& \leq \frac{k(0)}{8}\|\nabla w(t)\|_{H}^{2}+c \int_{0}^{t}\|\chi(s)\|_{V}^{2} d s
\end{align*}
$$

Arguing similarly, one can treat the second integral in the right hand side of (4.10) and get the estimate

$$
\begin{align*}
& -\iint_{Q_{t}}\left(k(0) \nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)+k^{\prime} * \nabla\left(\lambda\left(\chi_{1}\right)-\lambda\left(\chi_{2}\right)\right)\right) \cdot \nabla w \\
& \leq c \int_{0}^{t}\left(\|\nabla w(s)\|_{H}^{2}+\|\chi(s)\|_{V}^{2}\right) d s \tag{4.15}
\end{align*}
$$

Combining (4.6), (4.11), and (4.14-15), we obtain

$$
\frac{1}{2}\left\|w_{t}(t)\right\|_{H}^{2}+\frac{k(0)}{4}\|\nabla w(t)\|_{H}^{2} \leq c \int_{0}^{t}\left(\|\nabla w(s)\|_{H}^{2}+\|\chi(s)\|_{V}^{2}\right) d s
$$

and the Gronwall lemma yields

$$
\begin{equation*}
\left\|w_{t}(t)\right\|_{H}^{2}+\|\nabla w(t)\|_{H}^{2} \leq c \int_{0}^{t}\|\chi(s)\|_{V}^{2} d s \quad \forall t \in[0, T] \tag{4.16}
\end{equation*}
$$

At this point, multiply equation (4.3) by $\chi$. Then an integration over $Q_{t}$ leads to

$$
\begin{equation*}
\frac{\mu}{2}\|\chi(t)\|_{H}^{2}+\nu \int_{0}^{t}\|\nabla \chi(s)\|_{H}^{2} d s+\int_{0}^{t}(\xi(s), \chi(s))_{H} d s=\sum_{j=3}^{6} I_{j}(t) \tag{4.17}
\end{equation*}
$$

where

$$
\begin{align*}
& I_{3}(t):=\int_{0}^{t}\left(\gamma\left(\partial_{t} w_{1}(s), \chi_{1}(s)\right)-\gamma\left(\partial_{t} w_{2}(s), \chi_{2}(s)\right), \chi(s)\right)_{H} d s  \tag{4.18}\\
& I_{4}(t):=\int_{0}^{t}\left(\lambda^{\prime}\left(\chi_{1}(s)\right)\left(w_{t}(s)-\left(\lambda\left(\chi_{1}(s)\right)-\lambda\left(\chi_{2}(s)\right)\right)\right), \chi(s)\right)_{H} d s  \tag{4.19}\\
& I_{5}(t):=\int_{0}^{t}\left(\left(\lambda^{\prime}\left(\chi_{1}(s)\right)-\lambda^{\prime}\left(\chi_{2}(s)\right)\right) \partial_{t} w_{2}(s), \chi(s)\right)_{H} d s  \tag{4.20}\\
& I_{6}(t):=-\int_{0}^{t}\left(\left(\lambda^{\prime}\left(\chi_{1}(s)\right)-\lambda^{\prime}\left(\chi_{2}(s)\right)\right) \lambda\left(\chi_{2}(s)\right), \chi(s)\right)_{H} d s \tag{4.21}
\end{align*}
$$

Thanks to (3.3) and (2.2), it is straightforward to conclude that

$$
\begin{equation*}
\left|I_{3}(t)\right|+\left|I_{4}(t)\right| \leq c \int_{0}^{t}\left(\left\|w_{t}(s)\right\|_{H}^{2}+\|\chi(s)\|_{H}^{2}\right) d s \tag{4.22}
\end{equation*}
$$

As far as $I_{5}$ is concerned, using also inequality (4.13), we have

$$
\begin{gathered}
\left|I_{5}(t)\right| \leq c \int_{0}^{t}\left\|\left(\chi \partial_{t} w_{2}\right)(s)\right\|_{H}\|\chi(s)\|_{H} d s \\
\leq c \int_{0}^{t}\|\chi(s)\|_{L^{4}(\Omega)}\left\|\partial_{t} w_{2}(s)\right\|_{L^{4}(\Omega)}\|\chi(s)\|_{H} d s \\
\leq c \int_{0}^{t}\|\chi(s)\|_{V}\left\|\partial_{t} w_{2}(s)\right\|_{V}\|\chi(s)\|_{H} d s .
\end{gathered}
$$

Since $w_{2}$ fulfills (3.5), it turns out that

$$
\begin{equation*}
\left|I_{5}(t)\right| \leq \frac{\nu}{4} \int_{0}^{t}\|\nabla \chi(s)\|_{H}^{2} d s+c \int_{0}^{t}\|\chi(s)\|_{H}^{2} d s \tag{4.23}
\end{equation*}
$$

Besides, holding (3.7) for $\chi_{2}$, recalling (2.2), and arguing as above, one easily infers

$$
\begin{equation*}
\left|I_{6}(t)\right| \leq \frac{\nu}{4} \int_{0}^{t}\|\nabla \chi(s)\|_{H}^{2} d s+c \int_{0}^{t}\|\chi(s)\|_{H}^{2} d s \tag{4.24}
\end{equation*}
$$

Collecting (4.22-24) and accounting for the monotonicity of $\beta$, from (4.17) it follows that

$$
\begin{equation*}
\frac{\mu}{2}\|\chi(t)\|_{H}^{2}+\frac{\nu}{2} \int_{0}^{t}\|\nabla \chi(s)\|_{H}^{2} d s \leq c \int_{0}^{t}\left(\left\|w_{t}(s)\right\|_{H}^{2}+\|\chi(s)\|_{H}^{2}\right) d s \tag{4.25}
\end{equation*}
$$

for any $t \in[0, T]$.
Next, by estimating the right hand side of (4.16) with the help of (4.25), then summing the obtained inequality to (4.25), and finally applying the Gronwall lemma,
we achieve that $w=0$ and $\chi=0$ a.e. in $Q$. A comparison in (4.3) gives $\xi=0$ a.e. in $Q$ and the proof is complete.

Remark 4.1. Note that the restriction to the dimension is due to the use of (4.13) in estimating the integrals $I_{2}, I_{5}$, and $I_{6}$. Therefore, it is clear that our proof holds unchanged up to dimension 4 since (4.13) remains true if $\Omega \subset \mathbb{R}^{4}$.

Remark 4.2. Referring to the above remark, in the case $\lambda$ linear, i.e. if $\lambda^{\prime}$ is a constant $\lambda_{0}$, (4.13) is no longer needed in the above estimates and our proof works in any dimension. Indeed, $I_{5}$ and $I_{6}$ vanish and $I_{2}$ reduces to

$$
I_{2}(t)=\lambda_{0} \int_{\Omega}(k * \nabla \chi)(t) \cdot \nabla w(t)+\lambda_{0} \iint_{Q_{t}}\left(k(0) \nabla \chi+k^{\prime} * \nabla \chi\right) \cdot \nabla w
$$

so that estimates (4.14-15) can be easily recovered without using (4.13).
5. Approximation. The existence of a solution to Problem (2.10-20) and the regularity results stated in Section 3 are shown via an approximation-a priori estimatepassage to the limit argument. In particular, we apply the Faedo-Galerkin method combined with a regularization procedure. Thus we consider a sequence $\left\{V_{n}\right\}$ of finite dimensional subspaces filling up $V$ and approximate the data. At the same time, we regularize the maximal monotone graph $\beta$ with Lipschitz continuous functions $\beta_{\varepsilon}$. Then we define and solve the approximating problems.

In view of a number of a priori estimates, we need a special choice of the subspaces. Consider the orthonormal basis $\left\{v_{i}\right\}_{i \geq 1}$ of $V$ formed by the normalized eigenfunctions of the Laplace operator with homogeneous Neumann boundary condition, that is,

$$
\begin{align*}
-\Delta v_{i}=\lambda_{i} v_{i} & \text { in } \Omega  \tag{5.1}\\
\partial_{\mathrm{n}} v_{i}=0 & \text { on } \Gamma \tag{5.2}
\end{align*}
$$

where $\left\{\lambda_{i}\right\}_{i \geq 1}$ is the sequence of the corresponding eigenvalues. Note that, owing to the regularity of $\Gamma$,

$$
\begin{equation*}
v_{i} \in W \quad \forall i \geq 1 \tag{5.3}
\end{equation*}
$$

Then, for any integer $n \geq 1$, we denote by $V_{n}$ the $n$-dimensional subspace of $V$ spanned by $\left\{v_{1}, \ldots, v_{n}\right\}$.

As approximations of the data $\eta_{0}$ and $\chi_{0}$ we take

$$
\begin{equation*}
\eta_{0 n} \in V_{n} \quad \text { and } \quad \chi_{0 n} \in V_{n} \tag{5.4}
\end{equation*}
$$

Since we need approximating solutions regular enough in time, we also regularize $g$ by

$$
\begin{equation*}
g_{n} \in C^{2}([0, T] ; V) \tag{5.5}
\end{equation*}
$$

Further requirements on the approximating data and their convergence properties are specified whenever is needed.

Moreover, for $\varepsilon>0$ we set

$$
\left\{\begin{array}{l}
\phi_{\varepsilon}(y):=\min _{z \in \mathbb{R}}\left\{\frac{1}{2 \varepsilon}|z-y|^{2}+\phi(z)\right\} \quad \text { for } y \in \mathbb{R}  \tag{5.6}\\
\beta_{\varepsilon}:=\frac{1}{\varepsilon}\left(I-(I+\varepsilon \beta)^{-1}\right)
\end{array}\right.
$$

$I$ being the identity. As is well known (see, e.g., [6, p. 39]), thanks to (2.4-5) $\phi_{\varepsilon}$ is a nonnegative, convex, and continuously differentiable function, $\beta_{\varepsilon}$ is the Yosida approximation of $\beta$ (thus Lipschitz continuous), and the following equalities hold

$$
\begin{equation*}
\beta_{\varepsilon}=\phi_{\varepsilon}^{\prime}=\partial \phi_{\varepsilon} \tag{5.7}
\end{equation*}
$$

We also remind that (cf., e.g., [6, Prop. 2.6, p. 28])

$$
\begin{equation*}
\left|\beta_{\varepsilon}(y)\right| \leq\left|\beta^{0}(y)\right| \quad \forall \varepsilon>0 \quad \text { and } \quad \lim _{\varepsilon \rightarrow 0} \beta_{\varepsilon}(y)=\beta^{0}(y) \tag{5.8}
\end{equation*}
$$

for all $y \in \mathbb{R}$ and for $y \in D(\beta)$, respectively, where $D(\beta)$ is the effective domain of $\beta$ and $\beta^{0}(y)$ is the element of $\beta(y)$ having minimum modulus. Finally, we have that

$$
\begin{equation*}
0 \leq \phi_{\varepsilon}(y) \leq \phi(y) \quad \forall \varepsilon>0 \quad \text { and } \quad \lim _{\varepsilon \rightarrow 0} \phi_{\varepsilon}(y)=\phi(y) \tag{5.9}
\end{equation*}
$$

for any $y \in \mathbb{R}$, the limit being a consequence of [6, Prop. 2.11, p. 39], while the inequality is obvious.

However, the regularization of $\phi$ and $\beta$ is not necessary when the graph $\beta$ itself is a Lipschitz continuous function. In this particular case, (5.8-9) are plainly fulfilled by taking $\phi_{\varepsilon}=\phi$ and $\beta_{\varepsilon}=\beta$.

We are now ready to introduce the approximating problem. Even though it depends on both $n$ and $\varepsilon$, we do not plug the subscript $\varepsilon$ to its solution.

Problem (P2) $)_{\varepsilon, n}$. Find $\left.\left.t_{n} \in\right] 0, T\right]$ and $w_{n}, \chi_{n}: \Omega \times\left[0, t_{n}\right] \rightarrow \mathbb{R}$ such that

$$
\begin{align*}
& w_{n} \in C^{2}\left(\left[0, t_{n}\right] ; V_{n}\right)  \tag{5.10}\\
& \chi_{n} \in C^{1}\left(\left[0, t_{n}\right] ; V_{n}\right)  \tag{5.11}\\
& \left(\partial_{t}^{2} w_{n}(t), v\right)_{H}+\int_{\Omega} k(0) \nabla w_{n}(t) \cdot \nabla v  \tag{5.12}\\
& =-\int_{\Omega} \nabla\left(k^{\prime} * w_{n}-k * \lambda\left(\chi_{n}\right)\right)(t) \cdot \nabla v+\left(g_{n}(t), v\right)_{H} \\
& \forall v \in V_{n} \quad \forall t \in\left[0, t_{n}\right] \\
& \mu\left(\partial_{t} \chi_{n}(t), v\right)_{H}+\nu \int_{\Omega} \nabla \chi_{n}(t) \cdot \nabla v+\left(\beta_{\varepsilon}\left(\chi_{n}(t)\right), v\right)_{H}  \tag{5.13}\\
& =\left(\gamma\left(\partial_{t} w_{n}(t), \chi_{n}(t)\right), v\right)_{H}+\left(\lambda^{\prime}\left(\chi_{n}(t)\right)\left(\partial_{t} w_{n}(t)-\lambda\left(\chi_{n}(t)\right)\right), v\right)_{H} \\
& \quad \forall v \in V_{n} \quad \forall t \in\left[0, t_{n}\right] \\
& w_{n}(0)=0 \quad \text { and } \quad \partial_{t} w_{n}(0)=\eta_{0 n}  \tag{5.14}\\
& \chi_{n}(0)=\chi_{0 n} . \tag{5.15}
\end{align*}
$$

Of course, any solution to Problem (P2) $)_{\varepsilon, n}$ has the form

$$
w_{n}(t)=\sum_{i=1}^{n} a_{i n}(t) v_{i} \quad \text { and } \quad \chi_{n}(t)=\sum_{i=1}^{n} b_{i n}(t) v_{i}
$$

for some $a_{i n} \in C^{2}\left(\left[0, t_{n}\right]\right)$ and $b_{i n} \in C^{1}\left(\left[0, t_{n}\right]\right)$ and Problem (5.10-15) consists in a Cauchy problem for a system of nonlinear ordinary integrodifferential equations. Therefore, taking advantage of (2.1-3), (5.4-6) and (3.3), standard arguments show that Problem (5.10-15) admits a local solution $\left(w_{n}, \chi_{n}\right)$ defined on some interval $\left[0, t_{n}\right]$ with $t_{n}$ small enough.
6. Existence. Our aim is now proving the existence of the solution to Problem (3.5-14) stated in Theorem 3.2. Hence, in addition to (5.4), we require that

$$
\begin{equation*}
\eta_{0 n} \rightarrow \eta_{0} \quad \text { and } \quad \chi_{0 n} \rightarrow \chi_{0} \quad \text { in } V \tag{6.1}
\end{equation*}
$$

as $n \rightarrow \infty$. Moreover, after splitting $g$ as

$$
\begin{equation*}
g=g^{1}+g^{2} \quad \text { with } \quad g^{1} \in W^{1,1}(0, T ; H) \quad \text { and } \quad g^{2} \in L^{1}(0, T ; V) \tag{6.2}
\end{equation*}
$$

we choose $g_{n}^{1}, g_{n}^{2} \in C^{2}([0, T] ; V)$ such that

$$
\begin{array}{ll}
g_{n}^{1} \rightarrow g^{1} & \text { in } \quad W^{1,1}(0, T ; H) \\
g_{n}^{2} \rightarrow g^{2} & \text { in } L^{1}(0, T ; V) \tag{6.4}
\end{array}
$$

and set

$$
g_{n}=g_{n}^{1}+g_{n}^{2}
$$

Warning 6.1. We are going to deduce a priori estimates and pass to the limit, first as $n \rightarrow \infty$, then as $\varepsilon \rightarrow 0$. For this reason, we are always allowed to assume $\varepsilon$ small enough and $n$ large enough, depending on $\varepsilon$. This restriction on the parameters will not be pointed out in the sequel. Moreover, remark that the second limit can be avoided whenever $\beta$ is a Lipschitz continuous function, since we have decided to take simply $\beta_{\varepsilon}=\beta$ in this case. Concerning the generic constant $c$, in addition to the dependences specified in Warning 3.5, henceforth we let $c$ depend also on (cf. (2.7), (2.9), and (3.4)) $\left\|g^{1}\right\|_{W^{1,1}(0, T ; H)},\left\|g^{2}\right\|_{L^{1}(0, T ; V)},\left\|\eta_{0}\right\|_{V},\left\|\chi_{0}\right\|_{V}$, and $\left\|\phi\left(\chi_{0}\right)\right\|_{L^{1}(\Omega)}$, but not on $n$ and $\varepsilon$.

Since $\phi_{\varepsilon}$ has a quadratic growth, (6.1) implies that $\phi_{\varepsilon}\left(\chi_{0 n}\right)$ converges to $\phi_{\varepsilon}\left(\chi_{0}\right)$ strongly in $L^{1}(\Omega)$ as $n \rightarrow \infty$. Hence, on account of (5.9), without loss of generality we can assume

$$
\begin{equation*}
\int_{\Omega} \phi_{\varepsilon}\left(\chi_{0 n}\right) \leq 1+\int_{\Omega} \phi_{\varepsilon}\left(\chi_{0}\right) \leq 1+\int_{\Omega} \phi\left(\chi_{0}\right) \tag{6.5}
\end{equation*}
$$

for any $n$ and any $\varepsilon$.
First a priori estimate. Take $v=\partial_{t} w_{n}(t)$ in equation (5.12) and integrate in time over $[0, t]$. Then we have

$$
\begin{equation*}
\frac{1}{2}\left\|\partial_{t} w_{n}(t)\right\|_{H}^{2}+\frac{k(0)}{2}\left\|\nabla w_{n}(t)\right\|_{H}^{2}=\sum_{j=7}^{9} I_{j}(t)+\frac{1}{2}\left\|\eta_{0 n}\right\|_{H}^{2} \tag{6.6}
\end{equation*}
$$

where

$$
\begin{align*}
I_{7}(t) & :=-\iint_{Q_{t}}\left(k^{\prime} * \nabla w_{n}\right) \cdot \nabla \partial_{t} w_{n}  \tag{6.7}\\
I_{8}(t) & :=\iint_{Q_{t}}\left(k * \nabla \lambda\left(\chi_{n}\right)\right) \cdot \nabla \partial_{t} w_{n}  \tag{6.8}\\
I_{9}(t) & :=\iint_{Q_{t}} g_{n} \partial_{t} w_{n} \tag{6.9}
\end{align*}
$$

for any $t \in\left[0, t_{n}\right]$.
Next, consider equation (5.13), choose $v=\partial_{t} \chi_{n}(t)$, and integrate over $[0, t]$. We obtain

$$
\begin{align*}
& \mu \int_{0}^{t}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}^{2} d s+\frac{\nu}{2}\left\|\nabla \chi_{n}(t)\right\|_{H}^{2}+\int_{0}^{t}\left(\beta_{\varepsilon}\left(\chi_{n}(s)\right), \partial_{t} \chi_{n}(s)\right)_{H} d s  \tag{6.10}\\
& =I_{10}(t)+I_{11}(t)+\frac{\nu}{2}\left\|\nabla \chi_{0 n}\right\|_{H}^{2}
\end{align*}
$$

where

$$
\begin{align*}
I_{10}(t) & :=\iint_{Q_{t}} \gamma\left(\partial_{t} w_{n}, \chi_{n}\right) \partial_{t} \chi_{n}  \tag{6.11}\\
I_{11}(t) & :=\iint_{Q_{t}} \lambda^{\prime}\left(\chi_{n}\right)\left(\partial_{t} w_{n}-\lambda\left(\chi_{n}\right)\right) \partial_{t} \chi_{n} \tag{6.12}
\end{align*}
$$

In view of (2.1), an integration by parts in time in (6.7) and (6.8) gives

$$
\begin{align*}
I_{7}(t)= & -\int_{\Omega}\left(k^{\prime} * \nabla w_{n}\right)(t) \cdot \nabla w_{n}(t)+\iint_{Q_{t}}\left(k^{\prime}(0) \nabla w_{n}+k^{\prime \prime} * \nabla w_{n}\right) \cdot \nabla w_{n}  \tag{6.13}\\
I_{8}(t)= & \int_{\Omega}\left(k * \nabla \lambda\left(\chi_{n}\right)\right)(t) \cdot \nabla w_{n}(t)  \tag{6.14}\\
& -\iint_{Q_{t}}\left(k(0) \nabla \lambda\left(\chi_{n}\right)+k^{\prime} * \nabla \lambda\left(\chi_{n}\right)\right) \cdot \nabla w_{n}
\end{align*}
$$

Using (3.20), (3.21), and (2.2), from (6.13-14) we infer

$$
\begin{align*}
& \left|I_{7}(t)\right| \leq \frac{k(0)}{8}\left\|\nabla w_{n}(t)\right\|_{H}^{2}+c \int_{0}^{t}\left\|\nabla w_{n}(s)\right\|_{H}^{2} d s  \tag{6.15}\\
& \left|I_{8}(t)\right| \leq \frac{k(0)}{8}\left\|\nabla w_{n}(t)\right\|_{H}^{2}+c \int_{0}^{t}\left\{\left\|\nabla w_{n}(s)\right\|_{H}^{2}+\left\|\nabla \chi_{n}(s)\right\|_{H}^{2}\right\} d s \tag{6.16}
\end{align*}
$$

Regarding (6.9), one easily derives

$$
\begin{equation*}
\left|I_{9}(t)\right| \leq \int_{0}^{t}\left\|g_{n}(s)\right\|_{H}\left\|\partial_{t} w_{n}(s)\right\|_{H} d s \tag{6.17}
\end{equation*}
$$

On the other hand, recalling (5.7) and (5.15) we have

$$
\begin{equation*}
\int_{0}^{t}\left(\beta_{\varepsilon}\left(\chi_{n}(s)\right), \partial_{t} \chi_{n}(s)\right)_{H} d s=\int_{\Omega} \phi_{\varepsilon}\left(\chi_{n}(t)\right)-\int_{\Omega} \phi_{\varepsilon}\left(\chi_{0 n}\right) \tag{6.18}
\end{equation*}
$$

By (3.3) we see that

$$
\begin{equation*}
\left|\gamma\left(\partial_{t} w_{n}, \chi_{n}\right)\right| \leq c\left(\left|\partial_{t} w_{n}\right|+\left|\chi_{n}\right|+1\right) \tag{6.19}
\end{equation*}
$$

whence, taking advantage of (2.2), the integrals in (6.11-12) can be bounded as follows

$$
\begin{align*}
& \left|I_{10}(t)\right|+\left|I_{11}(t)\right|  \tag{6.20}\\
& \quad \leq \frac{\mu}{2} \int_{0}^{t}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}^{2} d s+c\left\{1+\int_{0}^{t}\left(\left\|\partial_{t} w_{n}(s)\right\|_{H}^{2}+\left\|\chi_{n}(s)\right\|_{H}^{2}\right) d s\right\}
\end{align*}
$$

Therefore, adding (6.6) to (6.10), the estimates (6.15-18) and (6.20) entail

$$
\begin{align*}
& \frac{1}{2}\left\|\partial_{t} w_{n}(t)\right\|_{H}^{2}+\frac{k(0)}{4}\left\|\nabla w_{n}(t)\right\|_{H}^{2}  \tag{6.21}\\
& +\frac{\mu}{2} \int_{0}^{t}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}^{2} d s+\frac{\nu}{2}\left\|\nabla \chi_{n}(t)\right\|_{H}^{2}+\int_{\Omega} \phi_{\varepsilon}\left(\chi_{n}(t)\right) \\
& \leq c \int_{0}^{t}\left(\left\|\nabla w_{n}(s)\right\|_{H}^{2}+\left\|\partial_{t} w_{n}(s)\right\|_{H}^{2}+\left\|g_{n}(s)\right\|_{H}\left\|\partial_{t} w_{n}(s)\right\|_{H}\right) d s \\
& +c \int_{0}^{t}\left(\left\|\nabla \chi_{n}(s)\right\|_{H}^{2}+\left\|\chi_{n}(s)\right\|_{H}^{2}\right) d s \\
& +c\left\{\int_{\Omega} \phi_{\varepsilon}\left(\chi_{0 n}\right)+\frac{1}{2}\left\|\eta_{0 n}\right\|_{H}^{2}+\frac{\nu}{2}\left\|\nabla \chi_{0 n}\right\|_{H}^{2}+1\right\}
\end{align*}
$$

Observe now that (6.1) and (6.5) imply

$$
\begin{equation*}
\int_{\Omega} \phi_{\varepsilon}\left(\chi_{0 n}\right)+\left\|\eta_{0 n}\right\|_{H}^{2}+\left\|\chi_{0 n}\right\|_{V}^{2} \leq c . \tag{6.22}
\end{equation*}
$$

Moreover, since $\chi_{n}(t)=\chi_{0 n}+\left(1 * \partial_{t} \chi_{n}\right)(t)$, for any $n$ and any $t \in\left[0, t_{n}\right]$ we have

$$
\begin{equation*}
\int_{0}^{t}\left\|\chi_{n}(s)\right\|_{H}^{2} d s \leq c\left\{1+\int_{0}^{t}\left(\int_{0}^{s}\left\|\partial_{t} \chi_{n}(r)\right\|_{H}^{2} d r\right) d s\right\} \tag{6.23}
\end{equation*}
$$

Hence, taking (6.22-23) into account and using a generalized version of the Gronwall lemma (cf., e.g., [3] or [6, pp. 156-157]), from (6.21) we derive an upper bound for

$$
\left\|\partial_{t} w_{n}(t)\right\|_{H}^{2}+\left\|\nabla w_{n}(t)\right\|_{H}^{2}+\left\|\partial_{t} \chi_{n}\right\|_{L^{2}(0, t: H)}^{2}+\left\|\nabla \chi_{n}(t)\right\|_{H}^{2}+\int_{\Omega} \phi_{\varepsilon}\left(\chi_{n}(t)\right) .
$$

In particular, the local solution can be extended to a solution defined on the whole interval $[0, T]$, i.e., we can assume $t_{n}=T$ for any $n$. Thus, owing also to (5.14-15) and (6.22), we conclude that

$$
\begin{align*}
& \left\|w_{n}\right\|_{L^{\infty}(0, T ; V) \cap W^{1, \infty}(0, T ; H)} \leq c  \tag{6.24}\\
& \left\|\chi_{n}\right\|_{L^{\infty}(0, T ; V) \cap H^{1}(0, T ; H)} \leq c  \tag{6.25}\\
& \| \phi_{\varepsilon}\left(\chi_{n} \|_{L^{\infty}\left(0, T ; L^{1}(\Omega)\right)} \leq c .\right. \tag{6.26}
\end{align*}
$$

Second a priori estimate. Since $\Delta v \in V_{n}$ whenever $v \in V_{n}$, one can take $v=$ $-\Delta \chi_{n}$ in equality (5.13). Integrating by parts in space and time, one obtains

$$
\begin{align*}
& \frac{\mu}{2}\left\|\nabla \chi_{n}(t)\right\|_{H}^{2}+\nu \int_{0}^{t}\left\|\Delta \chi_{n}(s)\right\|_{H}^{2} d s+\iint_{Q_{t}} \beta_{\varepsilon}^{\prime}\left(\chi_{n}\right)\left|\nabla \chi_{n}\right|^{2}  \tag{6.27}\\
& =I_{12}(t)+I_{13}(t)+\frac{\mu}{2}\left\|\nabla \chi_{0 n}\right\|_{H}^{2}
\end{align*}
$$

where

$$
\begin{aligned}
& I_{12}(t):=-\int_{0}^{t}\left(\gamma\left(\partial_{t} w_{n}(s), \chi_{n}(s)\right), \Delta \chi_{n}(s)\right)_{H} d s \\
& I_{13}(t):=-\int_{0}^{t}\left(\lambda^{\prime}\left(\chi_{n}(s)\right)\left(\partial_{t} w_{n}(s)-\lambda\left(\chi_{n}(s)\right)\right), \Delta \chi_{n}(s)\right)_{H} d s
\end{aligned}
$$

for any $t \in[0, T]$. Using standard inequalities, we get

$$
\begin{align*}
& \left|I_{12}(t)\right|+\left|I_{13}(t)\right| \leq \frac{\nu}{2} \int_{0}^{t}\left\|\Delta \chi_{n}(s)\right\|_{H}^{2} d s  \tag{6.28}\\
& \quad+c \int_{0}^{t}\left(\left\|\gamma\left(\partial_{t} w_{n}(s), \chi_{n}(s)\right)\right\|_{H}^{2}+\left\|\lambda^{\prime}\left(\chi_{n}(s)\right)\left(\partial_{t} w_{n}(s)-\lambda\left(\chi_{n}(s)\right)\right)\right\|_{H}^{2}\right) d s
\end{align*}
$$

Then, recalling (2.2), (6.19), and (6.24-26), one finds

$$
\begin{equation*}
\left|I_{12}(t)\right|+\left|I_{13}(t)\right| \leq c+\frac{\nu}{2} \int_{0}^{t}\left\|\Delta \chi_{n}(s)\right\|_{H}^{2} d s \tag{6.29}
\end{equation*}
$$

Combining (6.27) with (6.29), the property $\beta_{\varepsilon}^{\prime} \geq 0$ and (6.22) enable us to achieve

$$
\begin{equation*}
\frac{\mu}{2}\left\|\nabla \chi_{n}(t)\right\|_{H}^{2}+\frac{\nu}{2}\left\|\Delta \chi_{n}\right\|_{L^{2}(0, t ; H)}^{2} \leq c \tag{6.30}
\end{equation*}
$$

for any $t \in[0, T]$ and any $n$. Hence, on account of (6.25), (6.30), and the associated boundary condition, well-known regularity results for elliptic problems yield

$$
\begin{equation*}
\left\|\chi_{n}\right\|_{L^{2}(0, T ; W)} \leq c \tag{6.31}
\end{equation*}
$$

Third a priori estimate. Since $\partial_{t} \Delta w_{n} \in C^{1}\left([0, T] ; V_{n}\right)$ thanks to the special structure of $V_{n}$, we can take $v=-\partial_{t} \Delta w_{n}(t)$ in (5.12). Integrating over ( $0, t$ ) and applying the Green formula, we get

$$
\begin{equation*}
\frac{1}{2}\left\|\partial_{t} \nabla w_{n}(t)\right\|_{H}^{2}+\frac{k(0)}{2}\left\|\Delta w_{n}(t)\right\|_{H}^{2}=\sum_{j=14}^{17} I_{j}(t)+\frac{1}{2}\left\|\nabla \eta_{0 n}\right\|_{H}^{2} \tag{6.32}
\end{equation*}
$$

where

$$
\begin{align*}
I_{14}(t) & :=-\int_{0}^{t}\left(\left(k^{\prime} * \Delta w_{n}\right)(s), \partial_{t} \Delta w_{n}(s)\right)_{H} d s  \tag{6.33}\\
I_{15}(t) & :=\int_{0}^{t}\left(\left(k * \Delta \lambda\left(\chi_{n}\right)(s), \partial_{t} \Delta w_{n}(s)\right)_{H} d s\right.  \tag{6.34}\\
I_{16}(t) & :=-\int_{0}^{t}\left(g_{n}^{1}(s), \partial_{t} \Delta w_{n}(s)\right)_{H} d s  \tag{6.35}\\
I_{17}(t) & :=-\int_{0}^{t}\left(g_{n}^{2}(s), \partial_{t} \Delta w_{n}(s)\right)_{H} d s \tag{6.36}
\end{align*}
$$

Consider (6.33). An integration by parts in time gives

$$
\begin{aligned}
& I_{14}(t)=-\left(\left(k^{\prime} * \Delta w_{n}\right)(t), \Delta w_{n}(t)\right)_{H} \\
& +\int_{0}^{t}\left(k^{\prime}(0) \Delta w_{n}(s)+\left(k^{\prime \prime} * \Delta w_{n}\right)(s), \Delta w_{n}(s)\right)_{H} d s
\end{aligned}
$$

Therefore, using (2.1) and (3.21-22) we deduce that

$$
\begin{equation*}
\left|I_{14}(t)\right| \leq \frac{k(0)}{8}\left\|\Delta w_{n}(t)\right\|_{H}^{2}+c \int_{0}^{t}\left\|\Delta w_{n}(s)\right\|_{H}^{2} d s \tag{6.37}
\end{equation*}
$$

In view of $(2.1-2), I_{15}(t)$ can be rewritten as

$$
\begin{align*}
& I_{15}(t)=\left(\left(k * \Delta \lambda\left(\chi_{n}\right)\right)(t), \Delta w_{n}(t)\right)_{H}  \tag{6.38}\\
& -\int_{0}^{t}\left(k(0) \Delta \lambda\left(\chi_{n}(s)\right), \Delta w_{n}(s)\right)_{H} d s-\int_{0}^{t}\left(\left(k^{\prime} * \Delta \lambda\left(\chi_{n}\right)\right)(s), \Delta w_{n}(s)\right)_{H} d s
\end{align*}
$$

and the computation of $\Delta \lambda\left(\chi_{n}\right)$ provides

$$
\Delta \lambda\left(\chi_{n}\right)=\lambda^{\prime \prime}\left(\chi_{n}\right)\left|\nabla \chi_{n}\right|^{2}+\lambda^{\prime}\left(\chi_{n}\right) \Delta \chi_{n} .
$$

The sum (6.38) splits into six terms, three of which, e.g.

$$
\begin{equation*}
I_{15,1}(t):=\left(\left(k *\left(\lambda^{\prime \prime}\left(\chi_{n}\right)\left|\nabla \chi_{n}\right|^{2}\right)\right)(t), \Delta w_{n}(t)\right)_{H} \tag{6.39}
\end{equation*}
$$

are not trivial, unless $\lambda^{\prime \prime}$ identically vanishes. By $(3.21) I_{15,1}(t)$ can be treated this way,

$$
\begin{gathered}
\left|I_{15,1}(t)\right| \leq c\left(\int_{0}^{t}\left|\nabla \chi_{n}(s)\right|^{2} d s,\left|\Delta w_{n}(t)\right|\right)_{H} \\
\leq c \int_{0}^{t}\left(\left|\nabla \chi_{n}(s)\right|^{2},\left|\Delta w_{n}(t)\right|\right)_{H} d s \leq c \int_{0}^{t}\left\|\Delta w_{n}(t)\right\|_{H}\left\|\left|\nabla \chi_{n}(s)\right|^{2}\right\|_{H} d s \\
\leq c\left\|\Delta w_{n}(t)\right\|_{H} \int_{0}^{t}\left\|\nabla \chi_{n}(s)\right\|_{L^{4}(\Omega)}^{2} d s .
\end{gathered}
$$

With the help of (4.13), bound (6.31) entails

$$
\left|I_{15,1}(t)\right| \leq c\left\|\Delta w_{n}(t)\right\|_{H}\left\|\chi_{n}\right\|_{L^{2}(0, T ; W)}^{2} \leq \frac{k(0)}{16}\left\|\Delta w_{n}(t)\right\|_{H}^{2}+c
$$

Since the other five terms deriving from (6.38) can be handled similarly, we infer the estimate

$$
\begin{equation*}
\left|I_{15}(t)\right| \leq \frac{k(0)}{8}\left\|\Delta w_{n}(t)\right\|_{H}^{2}+c\left(1+\int_{0}^{t}\left\|\Delta w_{n}(s)\right\|_{H}^{2} d s\right) \tag{6.40}
\end{equation*}
$$

Integrating by parts in (6.35), we have (see also (5.14))

$$
I_{16}(t)=-\left(g_{n}^{1}(t), \Delta w_{n}(t)\right)_{H}+\int_{0}^{t}\left(\partial_{t} g_{n}^{1}(s), \Delta w_{n}(s)\right)_{H} d s
$$

which allows us to derive

$$
\begin{equation*}
\left|I_{16}(t)\right| \leq \frac{k(0)}{8}\left\|\Delta w_{n}(t)\right\|_{H}^{2}+c+\int_{0}^{t}\left\|\partial_{t} g_{n}^{1}(s)\right\|_{H}\left\|\Delta w_{n}(s)\right\|_{H} d s \tag{6.41}
\end{equation*}
$$

To deal with $I_{17}(t)$ (cf. (6.36)) we can use the Green formulas. It results that

$$
\begin{equation*}
\left|I_{17}(t)\right| \leq \int_{0}^{t}\left\|\nabla g_{n}^{2}(s)\right\|_{H}\left\|\partial_{t} \nabla w_{n}(s)\right\|_{H} d s \tag{6.42}
\end{equation*}
$$

Collecting inequalities (6.37) and (6.40-42), from (6.32) we conclude

$$
\begin{gathered}
\frac{1}{2}\left\|\partial_{t} \nabla w_{n}(t)\right\|_{H}^{2}+\frac{k(0)}{8}\left\|\Delta w_{n}(t)\right\|_{H}^{2} \\
\leq \frac{1}{2}\left\|\nabla \eta_{0 n}\right\|_{H}^{2}+c\left(1+\int_{0}^{t}\left\|\Delta w_{n}(s)\right\|_{H}^{2} d s\right) \\
+\int_{0}^{t}\left(\left\|\partial_{t} g_{n}^{1}(s)\right\|_{H}\left\|\Delta w_{n}(s)\right\|_{H}+\left\|g_{n}^{2}(s)\right\|_{V}\left\|\partial_{t} \nabla w_{n}(s)\right\|_{H}\right) d s
\end{gathered}
$$

for any time $t \in[0, T]$. Now, it suffices to recall (6.1-4) and to apply the generalized Gronwall lemma in the form of [3]. Thus we get

$$
\left\|\partial_{t} \nabla w_{n}(t)\right\|_{H}^{2}+\left\|\Delta w_{n}(t)\right\|_{H}^{2} \leq c
$$

for any $t \in[0, T]$ and any $n$, whence, on account of (6.24), it comes out that

$$
\begin{equation*}
\left\|w_{n}\right\|_{W^{1, \infty}(0, T ; V) \cap L^{\infty}(0, T ; W)} \leq c . \tag{6.43}
\end{equation*}
$$

Fourth a priori estimate. From (5.12), thanks to (6.24-25), (6.31), (6.43), (6.3-4) and arguing as in the proof of (6.40), we deduce

$$
\begin{align*}
\left|\left(\partial_{t}^{2} w_{n}(t), v\right)_{H}\right| \leq & c\left\|\Delta w_{n}(t)\right\|_{H}\|v\|_{H}+c\left\|\Delta w_{n}\right\|_{L^{2}(0, T ; H)}\|v\|_{H}  \tag{6.44}\\
& +c\left\|\Delta \lambda\left(\chi_{n}\right)\right\|_{L^{2}(0, T ; H)}\|v\|_{H}+c\left\|g_{n}(t)\right\|_{H}\|v\|_{H} \\
\leq & c\left(1+\left\|g_{n}^{2}(t)\right\|_{H}\right)\|v\|_{H}
\end{align*}
$$

for any $v \in V_{n}$ and any $t \in[0, T]$. Therefore, choosing $v=\partial_{t}^{2} w_{n}(t)$, we achieve that $\left\|\partial_{t}^{2} w_{n}(t)\right\|_{H} \leq c\left(1+\left\|g_{n}^{2}(t)\right\|_{H}\right)$ and consequently

$$
\begin{equation*}
\left\|\partial_{t}^{2} w_{n}\right\|_{L^{1}(0, T ; H)} \leq c \tag{6.45}
\end{equation*}
$$

First limit. Here we let $n \rightarrow \infty$ and show that the solution of the approximating problem tends to a solution of Problem (3.5-14), where $\beta$ is replaced with $\beta_{\varepsilon}$.

Estimates (6.43) and (6.24) for $\left\{w_{n}\right\}$, estimates (6.25) and (6.31) for $\left\{\chi_{n}\right\}$, and well-known weak or weak* compactness results ensure the existence of a pair ( $w^{\varepsilon}, \chi^{\varepsilon}$ ) such that, at least for a subsequence of $n \rightarrow \infty$,

$$
\begin{align*}
w_{n} \stackrel{*}{\rightharpoonup} w^{\varepsilon} & \text { in } L^{\infty}(0, T ; W)  \tag{6.46}\\
\partial_{t} w_{n} \stackrel{*}{\rightharpoonup} \partial_{t} w^{\varepsilon} & \text { in } L^{\infty}(0, T ; V)  \tag{6.47}\\
\chi_{n} \stackrel{*}{\rightharpoonup} \chi^{\varepsilon} & \text { in } L^{\infty}(0, T ; V)  \tag{6.48}\\
\chi_{n} \rightharpoonup \chi^{\varepsilon} & \text { in } H^{1}(0, T ; H) \cap L^{2}(0, T ; W) . \tag{6.49}
\end{align*}
$$

In view of the next subsection, note that, as the constants in (6.43), (6.25), and (6.31) have the dependences specified in Warning 6.1 (in particular, they do not depend on $\varepsilon$ ), for any $\varepsilon>0$ it happens that

$$
\begin{align*}
& \left\|w^{\varepsilon}\right\|_{W^{1, \infty}(0, T ; V) \cap L^{\infty}(0, T ; W)} \leq c  \tag{6.50}\\
& \left\|\chi^{\varepsilon}\right\|_{H^{1}(0, T ; H) \cap L^{\infty}(0, T ; V) \cap L^{2}(0, T ; W)} \leq c . \tag{6.51}
\end{align*}
$$

Now, it is a standard matter to see that (6.46-49) and the generalized Ascoli theorem (cf., e.g., [40, Cor. 4, Sec. 8]) entail

$$
\begin{align*}
w_{n} \rightarrow w^{\varepsilon} & \text { in } C^{0}([0, T] ; V)  \tag{6.52}\\
\chi_{n} \rightarrow \chi^{\varepsilon} & \text { in } C^{0}([0, T] ; H) \cap L^{2}(0, T ; V) \tag{6.53}
\end{align*}
$$

Moreover, (6.46-47) and the boundedness property (6.45) enable us to obtain the further strong convergence (see [40] again)

$$
\begin{equation*}
\partial_{t} w_{n} \rightarrow \partial_{t} w^{\varepsilon} \quad \text { in } \quad L^{p}(0, T ; H) \quad \forall p \in[1, \infty[. \tag{6.54}
\end{equation*}
$$

Therefore, as the functions $\lambda, \lambda^{\prime}, \beta_{\varepsilon}$ and $\gamma$ are Lipschitz continuous, one can easily conclude that

$$
\begin{align*}
\lambda\left(\chi_{n}\right) \rightarrow \lambda\left(\chi^{\varepsilon}\right) & \text { in } C^{0}([0, T] ; H)  \tag{6.55}\\
\lambda^{\prime}\left(\chi_{n}\right) \rightarrow \lambda^{\prime}\left(\chi^{\varepsilon}\right) & \text { in } C^{0}([0, T] ; H)  \tag{6.56}\\
\beta_{\varepsilon}\left(\chi_{n}\right) \rightarrow \beta_{\varepsilon}\left(\chi^{\varepsilon}\right) & \text { in } C^{0}([0, T] ; H)  \tag{6.57}\\
\gamma\left(\partial_{t} w_{n}, \chi_{n}\right) \rightarrow \gamma\left(\partial_{t} w^{\varepsilon}, \chi^{\varepsilon}\right) & \text { in } L^{2}(0, T ; H) . \tag{6.58}
\end{align*}
$$

So, if we remind (6.3-4) and set $\xi^{\varepsilon}=\beta_{\varepsilon}\left(\chi^{\varepsilon}\right)$, reasoning as in [32, pp. 13-14], one checks that the triplet ( $w^{\varepsilon}, \chi^{\varepsilon}, \xi^{\varepsilon}$ ) fulfills

$$
\begin{align*}
& w_{t t}^{\varepsilon}-k(0) \Delta w^{\varepsilon}=g^{1}+g^{2}+k^{\prime} * \Delta w^{\varepsilon}-k * \Delta \lambda\left(\chi^{\varepsilon}\right) \quad \text { a.e. in } Q  \tag{6.59}\\
& \mu \chi_{t}^{\varepsilon}-\nu \Delta \chi^{\varepsilon}+\xi^{\varepsilon}=\gamma\left(w_{t}^{\varepsilon}, \chi^{\varepsilon}\right)+\lambda^{\prime}\left(\chi^{\varepsilon}\right)\left(w_{t}^{\varepsilon}-\lambda\left(\chi^{\varepsilon}\right)\right) \quad \text { a.e. in } Q \tag{6.60}
\end{align*}
$$

as well as the boundary and initial conditions (3.12-14). All this makes ( $w^{\varepsilon}, \chi^{\varepsilon}, \xi^{\varepsilon}$ ) a solution to Problem (3.5-14) with $\beta$ replaced by $\beta_{\varepsilon}$.

Remark 6.2. Observe that $\left(w^{\varepsilon}, \chi^{\varepsilon}, \xi^{\varepsilon}\right)$ is the limit of the whole sequence of approximating solutions, because of uniqueness, which of course holds for $\left(w^{\varepsilon}, \chi^{\varepsilon}, \xi^{\varepsilon}\right)$ too.

Remark 6.3. If $\beta$ is a Lipschitz continuous function, then the existence proof is complete since we have just taken $\beta_{\varepsilon}=\beta$ in this case.

Let us point out one consequence of our procedure. Since $\varepsilon$ is fixed, the function $\phi_{\varepsilon}$ has a quadratic growth and (6.53), (6.26) yield

$$
\begin{align*}
& \phi_{\varepsilon}\left(\chi_{n}\right) \rightarrow \phi_{\varepsilon}\left(\chi^{\varepsilon}\right) \quad \text { in } C^{0}\left([0, T] ; L^{1}(\Omega)\right)  \tag{6.61}\\
& \left\|\phi_{\varepsilon}\left(\chi^{\varepsilon}\right)\right\|_{L^{\infty}\left(0, T ; L^{1}(\Omega)\right)} \leq c . \tag{6.62}
\end{align*}
$$

Conclusion of the proof. Finally, we let $\varepsilon \rightarrow 0$ and show that $\left(w^{\varepsilon}, \chi^{\varepsilon}, \xi^{\varepsilon}\right)$ tends to a solution of Problem (3.5-14).

Recalling estimates (6.50-51) and assumptions (2.1-2), (6.2), by comparison in equation (6.59) one realizes that

$$
\left\|w_{t t}^{\varepsilon}-g^{2}\right\|_{L^{\infty}(0, T ; H)} \leq c
$$

Then we can find a pair $(w, \chi)$ satisfying (3.5-7) such that, possibly taking a subsequence, $\left(w^{\varepsilon}, \chi^{\varepsilon}\right)$ converges to ( $w, \chi$ ) weakly or weakly* or strongly in the appropriate spaces, as in the previous subsection, with the additional weak* convergence

$$
w_{t t}^{\varepsilon}-g^{2} \stackrel{*}{\rightharpoonup} w_{t t}-g^{2} \quad \text { in } \quad L^{\infty}(0, T ; H)
$$

and, consequently, the strong convergence

$$
w_{t}^{\varepsilon} \rightarrow w_{t} \quad \text { in } \quad C^{0}([0, T] ; H)
$$

which improves (6.54). Moreover, the passage to the limit is even simpler than before provided we change the procedure regarding the deduction of (3.8-9).

Here is the new argument. By (2.2) and (3.3), a comparison in (6.60) gives

$$
\begin{equation*}
\left\|\xi^{\varepsilon}\right\|_{L^{2}(Q)}=\left\|\beta_{\varepsilon}\left(\chi^{\varepsilon}\right)\right\|_{L^{2}(Q)} \leq c \tag{6.63}
\end{equation*}
$$

Thus we may suppose

$$
\begin{equation*}
\xi^{\varepsilon} \rightharpoonup \xi \quad \text { in } L^{2}(Q) \tag{6.64}
\end{equation*}
$$

for some $\xi \in L^{2}(Q)$, so that, on account of (6.64) and (6.53), we get

$$
\lim _{\varepsilon \rightarrow 0} \iint_{Q} \xi^{\varepsilon} \chi^{\varepsilon}=\iint_{Q} \xi \chi
$$

In view of [4, Prop. 1.1, p. 42], this ensures (3.9) and completes the proof that ( $w, \chi, \xi$ ) solves Problem (3.5-14).

In order to achieve Theorem 3.2, it remains to check (3.16) and (2.22). We now show (2.22), postponing (3.16) in the last section.

Note that (6.62) reads

$$
\int_{\Omega} \phi_{\varepsilon}\left(\chi^{\varepsilon}(t)\right) \leq c \quad \forall t \in[0, T] \quad \forall \varepsilon>0 .
$$

Since $\chi^{\varepsilon}$ converges strongly to $\chi$ in $C^{0}([0, T] ; V)$, for any fixed $t \in[0, T]$ we can choose a sequence $\left\{\varepsilon_{k}\right\}$ tending to 0 such that the corresponding sequence $\left\{\chi^{\varepsilon_{k}}(t)\right\}$ converges to $\chi(t)$ a.e. in $\Omega$. Hence, by virtue of (5.9) and (6.62) we have that

$$
\phi_{\varepsilon_{k}}\left(\chi^{\varepsilon_{k}}(t)\right) \rightarrow \phi(\chi(t)) \quad \text { a.e. in } \Omega \quad \text { and } \quad \int_{\Omega} \phi_{\varepsilon_{k}}\left(\chi^{\varepsilon_{k}}(t)\right) \leq c .
$$

Then, since $\phi_{\varepsilon} \geq 0$, we can use the Fatou lemma and claim that the last estimate holds also for $\phi(\chi(t))$, whence (2.22) follows.

Remark 6.4. Note that the whole family $\left\{\left(w^{\varepsilon}, \chi^{\varepsilon}, \xi^{\varepsilon}\right)\right\}$ converges to the solution of Problem (3.5-14), thanks to the uniqueness result proved in Section 4.

Remark 6.5. The restriction to the dimension is due to the use of (4.13) in estimating the integral (6.34). Therefore, it is clear that our proof holds unchanged up to dimension 4 since (4.13) remains true if $\Omega \subset \mathbb{R}^{4}$. Moreover, if $\lambda^{\prime \prime}$ vanishes, then the inequality (4.13) is no longer needed in the derivation of (6.40). This implies that the proof works in any dimension if $\lambda$ is linear.
7. Proof of Theorem 3.3. At this point, we know that the solution to the approximating problem (5.10-15) tends, first as $n \rightarrow \infty$ and then as $\varepsilon \rightarrow 0$, to the solution to Problem (3.5-14). Thus, in order to prove Theorem 3.3, it is enough to establish the corresponding a priori estimates on the solution to the approximating problem, i.e.

$$
\begin{equation*}
\left\|\chi_{n}\right\|_{W^{1, \infty}(0, T ; H) \cap H^{1}(0, T ; V) \cap L^{\infty}(0, T ; W)} \leq c \tag{7.1}
\end{equation*}
$$

Owing to the further assumptions (2.23-24), we can require that

$$
\begin{equation*}
\left\|\chi_{0 n}\right\|_{W} \leq C \tag{7.2}
\end{equation*}
$$

in addition to (6.1), (6.3-4). Note that (7.2) makes sense because $\chi_{0 n} \in V_{n}$ and

$$
V_{n} \subset\left\{v \in W: \partial_{\mathrm{n}} v=0 \text { on } \Gamma\right\} \quad \forall n \in \mathbb{N} .
$$

Moreover, the constants $c$ are allowed to depend on $\left\|\chi_{0}\right\|_{W}$ and $\left\|\beta^{0}\left(\chi_{0}\right)\right\|_{H}$ besides all the other quantities specified in Warnings 3.5 and 6.1.

Set $t=0$ in (5.13). Taking (5.14-15) into account, we obtain

$$
\begin{align*}
& \mu\left(\partial_{t} \chi_{n}(0), v\right)_{H}+\int_{\Omega} \nabla \chi_{0 n} \cdot \nabla v+\left(\beta_{\varepsilon}\left(\chi_{0 n}\right), v\right)_{H}  \tag{7.3}\\
& =\left(\gamma\left(\eta_{0 n}, \chi_{0 n}\right)+\lambda^{\prime}\left(\chi_{0 n}\right)\left(\eta_{0 n}-\lambda\left(\chi_{0 n}\right)\right), v\right)_{H}
\end{align*}
$$

for any $v \in V_{n}$. Then, recalling (3.3), (2.2), and (7.2), one easily infers

$$
\left\|\partial_{t} \chi_{n}(0)\right\|_{H} \leq c\left(\left\|\Delta \chi_{0 n}\right\|_{H}+\left\|\beta_{\varepsilon}\left(\chi_{0 n}\right)\right\|_{H}+\left\|\eta_{0 n}\right\|_{H}+\left\|\chi_{0 n}\right\|_{H}+1\right)
$$

Remarking that (2.24) and (5.8) entail

$$
\left\|\beta_{\varepsilon}\left(\chi_{0 n}\right)\right\|_{H} \leq\left\|\beta_{\varepsilon}^{\prime}\right\|_{L^{\infty}(\mathbb{R})}\left\|\chi_{0 n}-\chi_{0}\right\|_{H}+\left\|\beta_{\varepsilon}\left(\chi_{0}\right)\right\|_{H} \leq 1+\left\|\beta^{0}\left(\chi_{0}\right)\right\|_{H}
$$

(cf. Warning 6.1), we have that

$$
\begin{equation*}
\left\|\partial_{t} \chi_{n}(0)\right\|_{H} \leq c \tag{7.4}
\end{equation*}
$$

Next, differentiating (5.13) we get

$$
\begin{gathered}
\mu\left(\partial_{t}^{2} \chi_{n}(t), v\right)_{H}+\nu \int_{\Omega} \partial_{t} \nabla \chi_{n}(t) \cdot \nabla v+\left(\beta_{\varepsilon}^{\prime}\left(\chi_{n}(t)\right) \partial_{t} \chi_{n}(t), v\right)_{H} \\
=\left(\gamma_{1}\left(\partial_{t} w_{n}(t), \chi_{n}(t)\right) \partial_{t}^{2} w_{n}(t), v\right)_{H}+\left(\gamma_{2}\left(\partial_{t} w_{n}(t), \chi_{n}(t)\right) \partial_{t} \chi_{n}(t), v\right)_{H} \\
+\left(\lambda^{\prime \prime}\left(\chi_{n}(t)\right)\left(\partial_{t} w_{n}(t)-\lambda\left(\chi_{n}(t)\right)\right) \partial_{t} \chi_{n}(t), v\right)_{H} \\
+\left(\lambda^{\prime}\left(\chi_{n}(t)\right)\left(\partial_{t}^{2} w_{n}(t)-\lambda^{\prime}\left(\chi_{n}(t)\right) \partial_{t} \chi_{n}(t)\right), v\right)_{H}
\end{gathered}
$$

for any $v \in V_{n}$. Choosing $v=\partial_{t} \chi_{n}(t)$ and integrating lead to

$$
\begin{align*}
& \frac{\mu}{2}\left\|\partial_{t} \chi_{n}(t)\right\|_{H}^{2}+\nu \int_{0}^{t}\left\|\partial_{t} \nabla \chi_{n}(s)\right\|_{H}^{2} d s+\iint_{Q_{t}} \beta_{\varepsilon}^{\prime}\left(\chi_{n}\right)\left|\partial_{t} \chi_{n}\right|^{2}  \tag{7.5}\\
& =\frac{\mu}{2}\left\|\partial_{t} \chi_{n}(0)\right\|_{H}^{2}+I_{18}(t)+I_{19}(t)+I_{20}(t)
\end{align*}
$$

where

$$
\begin{align*}
I_{18}(t) & :=\int_{0}^{t}\left(\left(\gamma_{1}\left(\partial_{t} w_{n}(s), \chi_{n}(s)\right)+\lambda^{\prime}\left(\chi_{n}(s)\right)\right) \partial_{t}^{2} w_{n}(s), \partial_{t} \chi_{n}(s)\right)_{H} d s  \tag{7.6}\\
I_{19}(t) & :=\iint_{Q_{t}}\left(\gamma_{2}\left(\partial_{t} w_{n}, \chi_{n}\right)-\left|\lambda^{\prime}\left(\chi_{n}\right)\right|^{2}\right)\left|\partial_{t} \chi_{n}\right|^{2}  \tag{7.7}\\
I_{20}(t) & :=\iint_{Q_{t}} \lambda^{\prime \prime}\left(\chi_{n}\right)\left(\partial_{t} w_{n}-\lambda\left(\chi_{n}\right)\right)\left|\partial_{t} \chi_{n}\right|^{2} \tag{7.8}
\end{align*}
$$

for any $t \in[0, T]$. Owing to (3.3), (2.2), and (6.44), from (7.6) we infer

$$
\begin{equation*}
\left|I_{18}(t)\right| \leq c \int_{0}^{t}\left(1+\left\|g_{n}^{2}(s)\right\|_{H}\right)\left\|\partial_{t} \chi_{n}(s)\right\|_{H} d s \tag{7.9}
\end{equation*}
$$

Similarly, for (7.7) we see that

$$
\begin{equation*}
\left|I_{19}(t)\right| \leq c \int_{0}^{t}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}^{2} d s \tag{7.10}
\end{equation*}
$$

Regarding (7.8), we easily obtain

$$
\begin{equation*}
\left|I_{20}(t)\right| \leq c \iint_{Q_{t}}\left(1+\left|\chi_{n}\right|+\left|\partial_{t} w_{n}\right|\right)\left|\partial_{t} \chi_{n}\right|^{2} \tag{7.11}
\end{equation*}
$$

Letting $s \in[0, t]$, observe that, for instance, the Hölder inequality gives

$$
\left\|\partial_{t} w_{n}(s)\left|\partial_{t} \chi_{n}(s)\right|^{2}\right\|_{L^{1}(\Omega)} \leq\left\|\partial_{t} w_{n}(s)\right\|_{L^{4}(\Omega)}\left\|\partial_{t} \chi_{n}(s)\right\|_{L^{4}(\Omega)}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}
$$

Thus, by (4.13), it is straightforward to deduce

$$
\left\|\partial_{t} w_{n}(s)\left|\partial_{t} \chi_{n}(s)\right|^{2}\right\|_{L^{1}(\Omega)} \leq c\left\|\partial_{t} w_{n}(s)\right\|_{V}\left\|\partial_{t} \chi_{n}(s)\right\|_{V}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}
$$

Then, to control the right hand side of (7.11), we exploit (6.25) and (6.43) in deriving

$$
\int_{\Omega}\left(\left|\chi_{n}(s)\right|+\left|\partial_{t} w_{n}(s)\right|\right)\left|\partial_{t} \chi_{n}(s)\right|^{2} \leq c\left\|\partial_{t} \chi_{n}(s)\right\|_{V}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}
$$

for any $s \in[0, t]$, whence we can conclude that

$$
\begin{equation*}
\left|I_{20}(t)\right| \leq \frac{\nu}{2} \int_{0}^{t}\left\|\partial_{t} \nabla \chi_{n}(s)\right\|_{H}^{2} d s+c \int_{0}^{t}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}^{2} d s \tag{7.12}
\end{equation*}
$$

As $\beta_{\varepsilon}^{\prime} \geq 0$, a combination of (7.4), (7.9-10), and (7.12) with (7.5) yields

$$
\begin{gathered}
\frac{\mu}{2}\left\|\partial_{t} \chi_{n}(t)\right\|_{H}^{2}+\frac{\nu}{2} \int_{0}^{t}\left\|\partial_{t} \nabla \chi_{n}(s)\right\|_{H}^{2} d s \\
\leq c\left(1+\int_{0}^{t}\left(1+\left\|g_{n}^{2}(s)\right\|_{H}\right)\left\|\partial_{t} \chi_{n}(s)\right\|_{H} d s+\int_{0}^{t}\left\|\partial_{t} \chi_{n}(s)\right\|_{H}^{2} d s\right)
\end{gathered}
$$

and the generalized Gronwall lemma of [3] entails

$$
\begin{equation*}
\left\|\partial_{t} \chi_{n}(t)\right\|_{H}^{2}+\left\|\partial_{t} \chi_{n}\right\|_{L^{2}(0, t ; V)}^{2} \leq c \quad \forall t \in[0, T] \tag{7.13}
\end{equation*}
$$

Now, write down equation (5.13) in the form

$$
\begin{equation*}
\nu \int_{\Omega} \nabla \chi_{n}(t) \cdot \nabla v+\left(\beta_{\varepsilon}\left(\chi_{n}(t)\right), v\right)_{H}=\left(F_{n}(t), v\right)_{H} \tag{7.14}
\end{equation*}
$$

with

$$
\begin{equation*}
F_{n}(t)=-\mu \partial_{t} \chi_{n}(t)+\gamma\left(\partial_{t} w_{n}(t), \chi_{n}(t)\right)+\lambda^{\prime}\left(\chi_{n}(t)\right)\left(\partial_{t} w_{n}(t)-\lambda\left(\chi_{n}(t)\right)\right) \tag{7.15}
\end{equation*}
$$

and note that

$$
\begin{equation*}
\left\|F_{n}\right\|_{L^{\infty}(0, T ; H)} \leq c \tag{7.16}
\end{equation*}
$$

thanks to (7.13), (3.3), (2.2), (6.43), and (6.25). Therefore, choosing $v=-\Delta \chi_{n}(t)$ in (7.14) and integrating by parts only in space, it is not difficult (cf. (6.27)) to get

$$
\begin{equation*}
\left\|\Delta \chi_{n}\right\|_{L^{\infty}(0, T ; H)} \leq c \tag{7.17}
\end{equation*}
$$

Then (6.25), (7.13), and (7.17) imply (7.1) and the proof is thus complete.
Remark 7.1. Since (4.13) has been invoked just to estimate the integral $I_{20}$ involving $\lambda^{\prime \prime}$, the claims of Remark 6.5 still hold.
8. Further regularity. In view of the statement of Theorem 3.4, here we start proving that

$$
\begin{align*}
& w \in W^{2, \infty}(0, T ; V) \cap W^{1, \infty}(0, T ; W)  \tag{8.1}\\
& \chi \in W^{1, \infty}(0, T ; V) \cap H^{1}(0, T ; W) \tag{8.2}
\end{align*}
$$

and refer to the last section for the verification of (3.18) and (2.33).
Arguing as before, in order to prove (8.1-2) it suffices to establish the corresponding a priori estimates on the solution to the approximating problem, i.e.

$$
\begin{align*}
& \left\|w_{n}\right\|_{W^{2, \infty}(0, T ; V) \cap W^{1, \infty}(0, T ; W)} \leq c  \tag{8.3}\\
& \left\|\chi_{n}\right\|_{W^{1, \infty}(0, T ; V) \cap H^{1}(0, T ; W)} \leq c . \tag{8.4}
\end{align*}
$$

In this case we assume (2.27-31) on the data of Problem (3.5-14). Moreover, thanks to (2.26) and (2.28) we can require that $\beta$ is globally Lipschitz continuous and take $\beta_{\varepsilon}=\beta$. Finally, in addition to (6.1) and (6.3-4) we ask that

$$
\begin{align*}
& \chi_{0 n} \in H^{3}(\Omega) \cap V_{n} \quad \forall n \in \mathbb{N}  \tag{8.5}\\
& \left\|\eta_{0 n}\right\|_{W}+\left\|\chi_{0 n}\right\|_{H^{3}(\Omega)} \leq C  \tag{8.6}\\
& \left\|g_{n}^{1}\right\|_{W^{2,1}(0, T ; H)}+\left\|g_{n}^{2}\right\|_{W^{1,1}(0, T ; V)}+\left\|g_{n}(0)\right\|_{V} \leq C . \tag{8.7}
\end{align*}
$$

Concerning the dependences of the generic constant $c$, besides what stated in Warning 6.1, in this section we let $c$ depend also on the norms of the data of (P2) related to (2.27-29), (3.17), and (2.31).

Setting for convenience

$$
\begin{equation*}
u_{n}:=\partial_{t} w_{n}, \quad z_{n}:=\partial_{t} \chi_{n} \tag{8.8}
\end{equation*}
$$

and taking the regularity of $\lambda, g_{n}, \beta$, and $\gamma$ into account, it is clear that

$$
\begin{equation*}
u_{n} \in C^{2}\left([0, T] ; V_{n}\right), \quad z_{n} \in C^{1}\left([0, T] ; V_{n}\right) \tag{8.9}
\end{equation*}
$$

and that $\left(u_{n}, z_{n}\right)$ fulfills (cf. (3.20))

$$
\begin{align*}
& \left(\partial_{t}^{2} u_{n}(t), v\right)_{H}+\int_{\Omega} k(0) \nabla u_{n}(t) \cdot \nabla v  \tag{8.10}\\
& =-\int_{\Omega} \nabla\left(k^{\prime} * u_{n}-k(0) \lambda\left(\chi_{n}\right)-k^{\prime} * \lambda\left(\chi_{n}\right)\right)(t) \cdot \nabla v+\left(\partial_{t} g_{n}(t), v\right)_{H} \\
& \mu\left(\partial_{t} z_{n}(t), v\right)_{H}+\nu \int_{\Omega} \nabla z_{n}(t) \cdot \nabla v+\left(\beta^{\prime}\left(\chi_{n}(t)\right) z_{n}(t), v\right)_{H}  \tag{8.11}\\
& =\left(\gamma_{1}\left(u_{n}(t), \chi_{n}(t)\right) \partial_{t} u_{n}(t), v\right)_{H}+\left(\gamma_{2}\left(u_{n}(t), \chi_{n}(t)\right) z_{n}(t), v\right)_{H} \\
& +\left(\lambda^{\prime \prime}\left(\chi_{n}(t)\right)\left(u_{n}(t)-\lambda\left(\chi_{n}(t)\right)\right) z_{n}(t), v\right)_{H} \\
& +\left(\lambda^{\prime}\left(\chi_{n}(t)\right)\left(\partial u_{n}(t)-\lambda^{\prime}\left(\chi_{n}(t)\right) z_{n}(t)\right), v\right)_{H}
\end{align*}
$$

The initial conditions for $u_{n}$ and $z_{n}$ can be easily computed. They are

$$
\begin{equation*}
u_{n}(0)=\eta_{0 n}, \quad \partial_{t} u_{n}(0)=g_{n}(0), \quad \text { and } \quad z_{n}(0)=z_{0 n} \tag{8.12}
\end{equation*}
$$

where

$$
\begin{equation*}
z_{0 n}:=\mu^{-1}\left(\nu \Delta \chi_{0 n}-\beta\left(\chi_{0 n}\right)+\gamma\left(\eta_{0 n}, \chi_{0 n}\right)+\lambda^{\prime}\left(\chi_{0 n}\right)\left(\eta_{0 n}-\lambda\left(\chi_{0 n}\right)\right)\right) \tag{8.13}
\end{equation*}
$$

Let us choose $v=-\partial_{t} \Delta u_{n}(t)$ in (8.10) and $v=-\Delta z_{n}(t)$ in (8.11) and integrate as usual. Adding term by term, we obtain

$$
\begin{align*}
& \frac{1}{2}\left\|\partial_{t} \nabla u_{n}(t)\right\|_{H}^{2}+\frac{k(0)}{2}\left\|\Delta u_{n}(t)\right\|_{H}^{2}+\frac{\mu}{2}\left\|\nabla z_{n}(t)\right\|_{H}^{2}+\nu \int_{0}^{t}\left\|\Delta z_{n}(s)\right\|_{H}^{2} d s  \tag{8.14}\\
& =\frac{1}{2}\left\|\nabla g_{n}(0)\right\|_{H}^{2}+\frac{k(0)}{2}\left\|\Delta \eta_{0 n}\right\|_{H}^{2}+\frac{\mu}{2}\left\|\nabla z_{0 n}\right\|_{H}^{2}+\sum_{j=21}^{30} I_{j}(t)
\end{align*}
$$

where

$$
\begin{align*}
& I_{21}(t):=-\int_{0}^{t}\left(\left(k^{\prime} * \Delta u_{n}\right)(s), \partial_{t} \Delta u_{n}(s)\right)_{H} d s  \tag{8.15}\\
& I_{22}(t):=\int_{0}^{t}\left(k(0) \Delta \lambda\left(\chi_{n}(s)\right), \partial_{t} \Delta u_{n}(s)\right)_{H} d s  \tag{8.16}\\
& I_{23}(t):=\int_{0}^{t}\left(\left(k^{\prime} * \Delta \lambda\left(\chi_{n}\right)\right)(s), \partial_{t} \Delta u_{n}(s)\right)_{H} d s  \tag{8.17}\\
& I_{24}(t):=-\int_{0}^{t}\left(\partial_{t} g_{n}^{1}(s), \partial_{t} \Delta u_{n}(s)\right)_{H} d s  \tag{8.18}\\
& I_{25}(t):=-\int_{0}^{t}\left(\partial_{t} g_{n}^{2}(s), \partial_{t} \Delta u_{n}(s)\right)_{H} d s  \tag{8.19}\\
& I_{26}(t):=\int_{0}^{t}\left(\beta^{\prime}\left(\chi_{n}(s)\right) z_{n}(s), \Delta z_{n}(s)\right)_{H} d s  \tag{8.20}\\
& I_{27}(t):=-\int_{0}^{t}\left(\gamma_{1}\left(u_{n}(s), \chi_{n}(s)\right) \partial_{t} u_{n}(s), \Delta z_{n}(s)\right)_{H} d s  \tag{8.21}\\
& I_{28}(t):=-\int_{0}^{t}\left(\gamma_{2}\left(u_{n}(s), \chi_{n}(s)\right) z_{n}(s), \Delta z_{n}(s)\right)_{H} d s  \tag{8.22}\\
& I_{29}(t):=-\int_{0}^{t}\left(\lambda^{\prime \prime}\left(\chi_{n}(s)\right)\left(u_{n}(s)-\lambda\left(\chi_{n}(s)\right)\right) z_{n}(s), \Delta z_{n}(s)\right)_{H} d s  \tag{8.23}\\
& I_{30}(t):=-\int_{0}^{t}\left(\lambda^{\prime}\left(\chi_{n}(s)\right)\left(\partial_{t} u_{n}(s)-\lambda^{\prime}\left(\chi_{n}(s)\right) z_{n}(s)\right), \Delta z_{n}(s)\right)_{H} d s \tag{8.24}
\end{align*}
$$

for any $t \in[0, T]$. Before proceeding to estimate these integrals, it is worth recalling that, owing to (6.43-44), (8.7), and (7.1), we have

$$
\begin{align*}
& \left\|u_{n}\right\|_{L^{\infty}(0, T ; V)} \leq c, \quad\left\|\partial_{t} u_{n}\right\|_{L^{\infty}(0, T ; H)} \leq c  \tag{8.25}\\
& \left\|\chi_{n}\right\|_{L^{\infty}(Q)} \leq c, \quad\left\|z_{n}\right\|_{L^{\infty}(0, T ; H) \cap L^{2}(0, T ; V)} \leq c \tag{8.26}
\end{align*}
$$

the third bound deriving from (7.1) and the continuous embedding $W \subset C^{0}(\bar{\Omega})$.
The integral $I_{21}$ can be treated with the same arguments we have used to deal with the integral $I_{14}$ given by (6.33). Thus we get

$$
\begin{equation*}
\left|I_{21}(t)\right| \leq \frac{k(0)}{12}\left\|\Delta u_{n}(t)\right\|_{H}^{2}+c \int_{0}^{t}\left\|\Delta u_{n}(s)\right\|_{H}^{2} d s \tag{8.27}
\end{equation*}
$$

After integration by parts in time, $I_{22}$ can be rewritten as $I_{22,1}+I_{22,2}$ where

$$
\begin{aligned}
I_{22,1}(t) & =k(0)\left(\lambda^{\prime \prime}\left(\chi_{n}(t)\right)\left|\nabla \chi_{n}(t)\right|^{2}+\lambda^{\prime}\left(\chi_{n}(t)\right) \Delta \chi_{n}(t), \Delta u_{n}(t)\right)_{H} \\
& -k(0)\left(\lambda^{\prime \prime}\left(\chi_{0 n}\right)\left|\nabla \chi_{0 n}\right|^{2}+\lambda^{\prime}\left(\chi_{0 n}\right) \Delta \chi_{0 n}, \Delta \eta_{0 n}\right)_{H} \\
& -k(0) \int_{0}^{t}\left(\lambda^{\prime \prime \prime}\left(\chi_{n}(s)\right)\left|\nabla \chi_{n}(s)\right|^{2} z_{n}(s), \Delta u_{n}(s)\right)_{H} d s
\end{aligned}
$$

and

$$
\begin{aligned}
I_{22,2}(t) & =-2 k(0) \int_{0}^{t}\left(\lambda^{\prime \prime}\left(\chi_{n}(s)\right) \nabla \chi_{n}(s) \cdot \nabla z_{n}(s), \Delta u_{n}(s)\right)_{H} d s \\
& -k(0) \int_{0}^{t}\left(\lambda^{\prime \prime}\left(\chi_{n}(s)\right) \Delta \chi_{n}(s) z_{n}(s), \Delta u_{n}(s)\right)_{H} d s \\
& -k(0) \int_{0}^{t}\left(\lambda^{\prime}\left(\chi_{n}(s)\right) \Delta z_{n}(s), \Delta u_{n}(s)\right)_{H} d s .
\end{aligned}
$$

We now deal with the nasty terms of the above expansion, using the assumptions on $\lambda$, the continuous embedding $W \subset C^{0}(\bar{\Omega})$, and (4.13). With the help of (8.25-26) and (7.1) we infer that

$$
\begin{aligned}
& \left|\left(\lambda^{\prime \prime \prime}\left(\chi_{n}(s)\right)\left|\nabla \chi_{n}(s)\right|^{2} z_{n}(s), \Delta u_{n}(s)\right)_{H}\right| \\
& \leq c\left\|\nabla \chi_{n}(s)\right\|_{L^{4}(\Omega)}^{2}\left\|z_{n}(s)\right\|_{W}\left\|\Delta u_{n}(s)\right\|_{H} \leq c\left\|z_{n}(s)\right\|_{W}\left\|\Delta u_{n}(s)\right\|_{H}
\end{aligned}
$$

and that

$$
\left|\left(\lambda^{\prime \prime}\left(\chi_{n}(s)\right) \nabla \chi_{n}(s) \cdot \nabla z_{n}(s), \Delta u_{n}(s)\right)_{H}\right| \leq c\left\|z_{n}(s)\right\|_{W}\left\|\Delta u_{n}(s)\right\|_{H}
$$

as well as

$$
\left|\left(\lambda^{\prime \prime}\left(\chi_{n}(s)\right) \Delta \chi_{n}(s) z_{n}(s), \Delta u_{n}(s)\right)_{H}\right| \leq c\left\|z_{n}(s)\right\|_{W}\left\|\Delta u_{n}(s)\right\|_{H}
$$

for any $s \in[0, t]$. Therefore, taking advantage of (8.6), it follows

$$
\begin{align*}
\left|I_{22,1}(t)\right|+\left|I_{22,2}(t)\right| & \leq \frac{k(0)}{12}\left\|\Delta u_{n}(t)\right\|_{H}^{2}  \tag{8.28}\\
& +\frac{\nu}{8} \int_{0}^{t}\left\|\Delta z_{n}(s)\right\|_{H}^{2} d s+c\left(1+\int_{0}^{t}\left\|\Delta u_{n}(s)\right\|_{H}^{2} d s\right)
\end{align*}
$$

Arguing as for $I_{15}, I_{16}$, and $I_{17}$ (see (6.34-36)) and recalling (6.40-42), we obtain

$$
\begin{align*}
& \sum_{j=23}^{25}\left|I_{j}(t)\right| \leq \frac{k(0)}{12}\left\|\Delta u_{n}(t)\right\|_{H}+c\left(1+\int_{0}^{t}\left\|\Delta u_{n}(s)\right\|_{H}^{2} d s\right)  \tag{8.29}\\
& \quad+c \int_{0}^{t}\left(\left\|\partial_{t}^{2} g_{n}^{1}(s)\right\|_{H}\left\|\Delta u_{n}(s)\right\|_{H}+\left\|\partial_{t} \nabla g_{n}^{2}(s)\right\|_{H}\left\|\partial_{t} \nabla u_{n}(s)\right\|_{H}\right) d s
\end{align*}
$$

while (2.28), (3.3), and (8.25-26) imply that

$$
\begin{equation*}
\sum_{j=26}^{28}\left|I_{j}(t)\right| \leq \frac{\nu}{8} \int_{0}^{t}\left\|\Delta z_{n}(s)\right\|_{H}^{2} d s+c \tag{8.30}
\end{equation*}
$$

In order to estimate $I_{29}$, by (4.13) and (8.25) we deduce

$$
\int_{\Omega}\left|u_{n}(s) z_{n}(t) \Delta z_{n}(s)\right| \leq c\left\|z_{n}(s)\right\|_{V}\left\|\Delta z_{n}(s)\right\|_{H} \quad \forall s \in[0, t]
$$

and consequently, thanks to (8.26),

$$
\begin{equation*}
\left|I_{29}(t)\right| \leq \frac{\nu}{8} \int_{0}^{t}\left\|\Delta z_{n}(s)\right\|_{H}^{2} d s+c \tag{8.31}
\end{equation*}
$$

The last term is easier to control, namely

$$
\begin{equation*}
\left|I_{30}(t)\right| \leq \frac{\nu}{8} \int_{0}^{t}\left\|\Delta z_{n}(s)\right\|_{H}^{2} d s+c \tag{8.32}
\end{equation*}
$$

As all previous integrals have been examined, combining (8.14) with (8.27-32) and owing to (8.13), (8.6), and (8.7), we achieve

$$
\begin{aligned}
& \frac{1}{2}\left\|\partial_{t} \nabla u_{n}(t)\right\|_{H}^{2}+\frac{k(0)}{4}\left\|\Delta u_{n}(t)\right\|_{H}^{2}+\frac{\mu}{2}\left\|\nabla z_{n}(t)\right\|_{H}^{2}+\frac{\nu}{2} \int_{0}^{t}\left\|\Delta z_{n}(s)\right\|_{H}^{2} d s \\
& \quad \leq c\left(1+\int_{0}^{t}\left\|\Delta u_{n}(s)\right\|_{H}^{2} d s\right) \\
& \quad+c \int_{0}^{t}\left(\left\|\partial_{t}^{2} g_{n}^{1}(s)\right\|_{H}\left\|\Delta u_{n}(s)\right\|_{H}+\left\|\partial_{t} \nabla g_{n}^{2}(s)\right\|_{H}\left\|\partial_{t} \nabla u_{n}(s)\right\|_{H}\right) d s
\end{aligned}
$$

for any $n$ and any $t \in[0, T]$. Hence the generalized Gronwall lemma gives

$$
\begin{equation*}
\left\|\partial_{t} \nabla u_{n}\right\|_{L^{\infty}(0, T ; H)}+\left\|\Delta u_{n}\right\|_{L^{\infty}(0, T ; H)}+\left\|\nabla z_{n}\right\|_{L^{\infty}(0, T ; H)}+\left\|\Delta z_{n}\right\|_{L^{2}(0, T ; H)} \leq c \tag{8.33}
\end{equation*}
$$

Then, in view of (8.8), the collection of (6.43), (7.1), and (8.33) leads to (8.3-4) and the proof of (8.1-2) is complete.
9. Final remarks on regularity. This section is devoted to check the time continuity (3.16) of Theorem 3.2 and to show the validity of (2.33) and (3.18) in Theorem 3.4, that is,

$$
\begin{align*}
& w \in C^{1}([0, T] ; V) \cap C^{0}([0, T] ; W)  \tag{9.1}\\
& w \in W^{3,1}(0, T ; H) \cap C^{2}([0, T] ; V) \cap C^{1}([0, T] ; W)  \tag{9.2}\\
& \chi \in H^{2}(0, T ; H) \cap C^{1}([0, T] ; V) \tag{9.3}
\end{align*}
$$

Doing this, we definitely conclude the proofs of our results.
Setting

$$
\begin{equation*}
f=g+k^{\prime} * \Delta w-k * \Delta \lambda(\chi) \tag{9.4}
\end{equation*}
$$

note that $w$ solves a linear problem of this type (cf. (3.10), (3.12-13))

$$
\begin{array}{lrl}
u_{t t}-k(0) \Delta u=F & \text { a.e. in } Q & \\
\partial_{\mathrm{n}} u=0 \quad \text { on } \Sigma & \\
u(0)=u_{0} \quad \text { and } & u_{t}(0)=u_{1} \quad \text { in } \Omega \tag{9.7}
\end{array}
$$

where $F=f, u_{0}=0$, and $u_{1}=\eta_{0}$. The variable $u$ here has nothing to do with the $u_{n}$ employed in the previous section. With the help of (2.7), (2.1), (2.2), (3.5), (3.7), (3.4), and also of (4.13), if $\lambda$ is nonlinear, it is not difficult to check that

$$
\begin{align*}
& F \in W^{1,1}(0, T ; H)+L^{1}(0, T ; V)  \tag{9.8}\\
& u_{0} \in W \quad \text { and } \quad \partial_{\mathrm{n}} u_{0}=0 \quad \text { on } \Sigma  \tag{9.9}\\
& u_{1} \in V . \tag{9.10}
\end{align*}
$$

Thus, on account of [17, Lemma 5.1] (see [3] for an abstract approach), (9.1) follows.
In order to prove (9.2), let us verify that

$$
\begin{equation*}
f \in W^{2,1}(0, T ; H)+W^{1,1}(0, T ; V) . \tag{9.11}
\end{equation*}
$$

Recalling (2.29), it is sufficient to realize that

$$
\partial_{t}\left(k^{\prime} * \Delta w-k * \Delta \lambda(\chi)\right) \in W^{1,1}(0, T ; H) .
$$

But this is ensured by (2.1), (8.1), (2.27), and (8.2) (cf. (2.26) as well). Then, by applying the results of [24], one derives that

$$
w \in C^{2}([0, T] ; V) \cap C^{1}([0, T] ; W)
$$

Alternatively, one can remark that, by (3.10), (3.13), and (2.29), $w_{t t}(0)=g(0) \in V$ and, by $(3.17), w_{t}(0) \in W$ with $\partial_{\mathrm{n}} w_{t}(0)=0$ on $\Sigma$. Hence, $w_{t}$ solves $(9.5-7)$ where the data $F=f_{t}, u_{0}=\eta_{0}$, and $u_{1}=g(0)$ satisfy (9.8-10). Moreover, by comparison in (9.5), one gets

$$
w_{t t t}\left(=u_{t t}\right) \in C^{0}([0, T] ; H)+L^{1}(0, T ; V)
$$

and consequently $w$ fulfills (9.2).
It remains to show (9.3). In view of (3.11-14), (3.9), (2.27-28), (3.3), (8.1-2), and (2.31), $\chi_{t}$ solves the following linear parabolic problem

$$
\begin{align*}
& \mu u_{t}-\nu \Delta u=F \quad \text { a.e. in } Q  \tag{9.12}\\
& \partial_{\mathrm{n}} u=0 \quad \text { on } \Sigma  \tag{9.13}\\
& u(0)=u_{0} \quad \text { in } \Omega \tag{9.14}
\end{align*}
$$

with

$$
\begin{aligned}
F= & -\beta^{\prime}(\chi) \chi_{t}+\gamma_{1}\left(w_{t}, \chi\right) w_{t t}+\gamma_{2}\left(w_{t}, \chi\right) \chi_{t} \\
& +\lambda^{\prime \prime}(\chi) \chi_{t}\left(w_{t}-\lambda(\chi)\right)+\lambda^{\prime}(\chi)\left(w_{t t}-\lambda^{\prime}(\chi) \chi_{t}\right) \\
u_{0}= & \nu \Delta \chi_{0}-\beta\left(\chi_{0}\right)+\gamma\left(\eta_{0}, \chi_{0}\right)+\lambda^{\prime}\left(\chi_{0}\right)\left(\eta_{0}-\lambda\left(\chi_{0}\right)\right) .
\end{aligned}
$$

Thanks to (9.2), (8.2), (3.17), and (2.31) it is straightforward to find that

$$
\begin{align*}
& F \in L^{\infty}(0, T ; H)  \tag{9.15}\\
& u_{0} \in V \tag{9.16}
\end{align*}
$$

This implies (cf., e.g., [3])

$$
\chi_{t}(=u) \in H^{1}(0, T ; H) \cap C^{0}([0, T] ; V) \cap L^{2}(0, T ; W)
$$

whence $\chi$ satisfies (9.3).

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