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Esempio 1.

Rotazione di $\frac{\pi}{2}$ in \mathbb{R}^2 è operatore massimale monotono ma non sottodifferenziale (si controlla immediatamente che non è autoaggiunto). Lo scrivo

$$(x', y') = A(x, y) \text{ se}$$

$$\begin{cases} x' = -y \\ y' = x \end{cases}$$

È monotono $A(x, y) \cdot (x, y) = 0 \quad \forall (x, y)$, lineare e continuo, dunque massimale monotono.

Ci si può divertire a provare che non è ciclicamente monotono.

Problema di Cauchy

$$\begin{cases} x'(t) - y(t) = f(t) \\ y'(t) + x(t) = g(t) \\ x(0) = x_0, \quad y(0) = y_0 \end{cases}$$

Se $f, g \in L^1(0, T)$ allora la soluzione debole è assolutamente continua: $x, y \in C^0([0, T])$ e inoltre

$$x(t) = x_0 + \int_0^t (y(s) + f(s)) ds$$

$$y(t) = y_0 + \int_0^t (g(s) - x(s)) ds$$

Tentativi per provare che A non è ciclicamente monotono.

$$(x_0, y_0), \dots, (x_n, y_n) \equiv (x_0, y_0)$$

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$$(-y_0, x_0), \dots, (-y_n, x_n)$$

$$\sum_{i=1}^n \left(((-y_i, x_i), (x_i, y_i) - (x_{i-1}, y_{i-1})) \right) \stackrel{?}{\geq} 0$$

$$= \sum_{i=1}^n (y_i x_{i-1} - x_i y_{i-1})$$

$$= y_1 x_0 + y_2 x_1 + \dots + y_n x_{n-1}$$

$$- x_1 y_0 - x_2 y_1 - \dots - x_n y_{n-1}$$

~~$$x_0^2 - y_0^2 - x_1^2 - y_1^2 - \dots - x_n^2 - y_n^2$$~~

provo con $n=2$
ma chiaramente (!)
non funziona

$$\begin{cases} y_1 x_0 + y_2 x_1 \\ - x_1 y_0 - x_2 y_1 \end{cases}$$

basta controllare
per $n=3$

$$\begin{aligned} & y_1 x_3 + y_2 x_1 + y_3 x_2 \\ & - x_1 y_3 - x_2 y_1 - x_3 y_2 \end{aligned}$$

$$y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1)$$

Ese. $y_1 = 0, y_2 = 0, y_3 = 1$

basta che $x_2 < x_1$

Esempio 2

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Ω dominio limitato e regolare di \mathbb{R}^N .

β grafo massimale monotono in $\mathbb{R} \times \mathbb{R}$, $0 \in \beta(0)$

Sia $j : \mathbb{R} \rightarrow (-\infty, +\infty]$ la funzione convessa propria s.c.i. tale che $\beta = \partial j$.

Mi interesso all'operatore

$$Au = -\Delta u + \beta(u)$$

con dominio

$$D(A) = \left\{ u \in H^2(\Omega) : \exists \xi \in L^2(\Omega) \text{ con } \xi \in \beta(u) \text{ a.e. in } \Omega, \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma \right\}$$

nello spazio $H = L^2(\Omega)$. E' ben definito.

E' monotono. Sospetto che sia ie sottodifferenziale della funzione

$$\varphi(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} j(u) dx & \text{se } u \in H^1(\Omega) \text{ e } j(u) \in L^1(\Omega) \\ +\infty & \text{altrove.} \end{cases}$$

Si noti che $D(A) \subsetneq D(\varphi)$. In effetti, se esiste $\xi \in L^2(\Omega)$ tale che $\xi \in \beta(u)$ q.o. in Ω , allora

$$j(u) + \xi(v-u) \leq j(v) \quad \forall v \in \mathbb{R}$$

e dunque $j(u) \in L^1(\Omega)$ in quanto

$$0 \leq j(u) \leq j(0) + \xi u$$

e posso assumere $0 = j(0) = \min j$.

Proviamo facilmente che $A \subseteq \partial\varphi$ in quanto, per ogni $u \in D(A)$ e $\xi \in L^2(\Omega)$ tale che $\xi \in \beta(u)$ q.o. abbiamo

$$\begin{aligned} \int_{\Omega} (-\Delta u + \xi)(v - u) &\leq \int_{\Omega} \nabla u \cdot \nabla (v - u) + \int_{\Omega} \xi(v - u) \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \int_{\Omega} (j(v) - j(u)) \\ &\quad \forall v \in D(\varphi). \end{aligned}$$

Di contro, φ è convessa propria ($0 \in D(\varphi)$) e s.c.i. in $L^2(\Omega)$. Infatti se $u_n \rightarrow u$ in $L^2(\Omega)$ e

$$\varphi(u_n) \leq \lambda \quad \text{per un valore } \lambda \in \mathbb{R},$$

si prova facilmente che anche $\varphi(u) \leq \lambda$ (sottolivelli chiusi).

Dunque, per provare che $A \equiv \partial\varphi$ basta controllare la massimalità di A , cioè basta provare che $R(I+A) = L^2(\Omega)$. Sia pertanto $f \in L^2(\Omega)$ e cerchiamo soluzioni del problema

$$\begin{cases} u - \Delta u + \beta(u) \ni f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{su } \Gamma. \end{cases}$$

Approssimiamo β con la regolarizzata Yosida β_ϵ che poi troncheremo a livello n : applichiamo il teorema di punto fisso di Schauder (V.Banach, $K \subset V$ convesso e compatto di V , $f: K \rightarrow K$ continua \Rightarrow ammette punto fisso). Dato punto fisso, stima uniforme,

passiamo al limite prima per $n \rightarrow \infty$ e troviamo soluzione di

$$\begin{cases} u - \Delta u + \beta_\varepsilon(u) = f \text{ in } \Omega \\ \frac{\partial u}{\partial n} = 0 \text{ su } \Gamma \end{cases}$$

Testiamo per u e per $\beta_\varepsilon(u)$, troviamo stime

$$\|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)} + \|\beta_\varepsilon(u)\|_{L^2(\Omega)} \leq C.$$

Passiamo al limite per $\varepsilon \rightarrow 0$ per compattezza e chiusura forte-debole dell'operatore β esteso a $L^2(\Omega)$.

Equazione di Allen-Cahn

$$\begin{cases} u_t - \Delta u + \beta(u) - \gamma(u) = f & \text{in } \Omega \times (0, T) \\ u(0) = u_0 \text{ e cond. } \frac{\partial u}{\partial n} = 0 \text{ in } \Gamma \times (0, T) \end{cases}$$

Esempi di $\beta = \gamma$.

Se $u_0 \in D(\varphi)$ e $f \in L^2(0, T; L^2(\Omega))$ abbiamo una soluzione forte con

$$u \in W^{1,2}(0, T; L^2(\Omega))$$

e inoltre $u \in L^2(0, T; H^2(\Omega))$, $\beta(u) \in L^2(0, T; L^2(\Omega))$.

La perturbazione lipschitziana si tratta col teorema delle contrazioni.

Esempio 3.

2. Subdifferential mappings

Using Eq. (2.27), a similar estimate is obtained for $\partial^2 u / \partial x_n^2$. Then going back to $u(x)$ we see that

$$\|u_\lambda\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u_\lambda\|_{H^1(\Omega)})$$

Taking a partition of unity subordinate to a finite system of $\{V\}$ and using the interior estimates (2.26), one finally obtains estimate (2.23). Thus the proof of Proposition 2.9 is complete.

Example 5: Let $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ be usual Sobolev spaces on Ω . As mentioned in Chapter I, $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$. Moreover the Laplace operator $-\Delta$ is the canonical isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. $H^{-1}(\Omega)$ is a Hilbert space with an inner product

$$(u, v) = (J^{-1}u, v), \quad \forall u, v \in H^{-1}(\Omega)$$

where $J = -\Delta$ and $(,)$ is the usual pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Let j be a lower-semicontinuous proper convex function from R^1 into $]-\infty, +\infty]$ and let $\partial j = \beta$. We shall assume that

$$\lim_{|r| \rightarrow \infty} j(r)/|r| = \infty. \quad (2.28)$$

Let $\varphi : H^{-1}(\Omega) \rightarrow]-\infty, +\infty]$ be defined by

$$\varphi(u) = \begin{cases} \int_{\Omega} j(u(x)) dx & \text{if } u \in L^1(\Omega) \text{ and } j(u) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

PROPOSITION 2.10. *The function φ is convex and lower-semicontinuous on $H^{-1}(\Omega)$. Moreover, $f \in \partial\varphi(u)$, if and only if, $f \in J\beta(u)$.*

Proof. First we shall prove that φ is lower-semicontinuous on $H^{-1}(\Omega)$. Let $\{u_n\} \subset H^{-1}(\Omega) \cap L^1(\Omega)$ be such that $u_n \rightarrow u$ in $H^{-1}(\Omega)$ and $\int_{\Omega} j(u_n(x)) dx \leq \lambda$.

We must prove that $\int_{\Omega} j(u(x)) dx \leq \lambda$. We have already seen in the proof of

Proposition 2.8 that the function $u \rightarrow \int_{\Omega} j(u(x)) dx$ is lower-semicontinuous in $L^1(\Omega)$. Since every convex and lower-semicontinuous function on a Banach space is also weakly lower-semicontinuous, it suffices to show that the sequence $\{u_n\}$ is weakly compact in $L^1(\Omega)$. By Dunford-Pettis criterion for weak compactness in L^1 -spaces (see e.g. N. Dunford and J. Schwartz [1],

P. 294) it suffices to prove that the integrals $\int |u_n| dx$ are uniformly abso-

lutely continuous, i.e. there exists for every $\epsilon > 0$ some δ such that $\int_T |u_n| dx < \epsilon$

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whenever $\text{meas}(T) < \delta$. Let $M > 2\lambda/\varepsilon$ and let R be such that $j(r)/|r| \geq M$ for $|r| > R$. Here we have used condition (2.28). If $\delta < \varepsilon/2R$ then we have

$$\begin{aligned} \int_T |u_n(x)| dx &\leq \int_{\{x \in T; |u_n(x)| \geq R\}} |u_n(x)| dx + \int_{\{x \in T; |u_n(x)| < R\}} |u_n(x)| dx \leq \\ &\leq M^{-1} \int_{\Omega} j(u_n(x)) dx + R\delta < \varepsilon \end{aligned}$$

as desired.

Now let $A : H^{-1}(\Omega) \rightarrow H^{-1}(\Omega)$ be the operator defined by

$$Au = \{Jv; v \in H_0^1(\Omega) \text{ and } v(x) \in \beta(u(x)), \text{ a.e. } x \in \Omega\}$$

where $D(A)$ is the set of all $u \in H^{-1}(\Omega) \cap L^1(\Omega)$ with the property that there is some $v \in H_0^1(\Omega)$ such that $v(x) \in \beta(u(x))$ a.e. on Ω . We must prove that $A = \partial\varphi$.

We need the following lemma.

LEMMA 2.1. *Let $f \in H^{-1}(\Omega) \cap L^1(\Omega)$ and let $v \in H_0^1(\Omega)$. Let $g \in L^1(\Omega)$ and let h be measurable on Ω such that*

$$f(x)v(x) \geq h(x) \geq g(x), \quad \text{a.e. } x \in \Omega. \quad (2.29)$$

Then $h \in L^1(\Omega)$ and $(f, v) \geq \int_{\Omega} h(x) dx$.

Proof of Lemma 2.1. Define,

$$v_n = \begin{cases} n & \text{if } v \geq n \\ v & \text{if } |v| \leq n \\ -n & \text{if } v \leq -n. \end{cases}$$

Let $h_n = h \frac{v_n}{v}$ and let $g_n = g \frac{v_n}{v}$. Multiplying condition (2.29) by $\frac{v_n}{v}$ we obtain

$$fv_n \geq h_n \geq g_n.$$

Hence $0 \leq h_n - g_n \leq fv_n - g_n$. Clearly, $h_n - g_n \rightarrow h - g$, a.e. on Ω as $n \rightarrow \infty$ and

$$\int_{\Omega} (h_n - g_n) dx \leq \int_{\Omega} fv_n dx - \int_{\Omega} g_n dx = (f, v_n) - \int_{\Omega} g_n dx. \quad (2.30)$$

(We recall that $(,)$ denotes the pairing between H_0^1 and H^{-1} .) On the other hand, $v_n \rightarrow v$ in $H_0^1(\Omega)$ and $g_n \rightarrow g$ in $L^1(\Omega)$. Again using Fatou's lemma,

from condition (2.30) we conclude that $h - g \in L^1(\Omega)$ and

$$\int_{\Omega} (h - g) dx \leq (f, v) - \int_{\Omega} g dx$$

which concludes the proof of Lemma 2.1.

Now we continue the proof of Proposition 2.10 by showing that $A \subset \partial\varphi$. Let $f \in Au$. This implies that $f = Jv$ where $v \in H_0^1(\Omega)$ and $v(x) \in \beta(u(x))$ a.e. $x \in \Omega$. Let $w \in H^{-1}(\Omega) \cap L^1(\Omega)$ be such that $j(w) \in L^1(\Omega)$ (i.e. $w \in D(\varphi)$). Since $\beta = \partial j$ we have

$$j(w) - j(u) \geq v(w - u), \text{ a.e. on } \Omega.$$

We apply Lemma 2.1. with $f = u - w$, $h = j(u) - j(w)$ and $g = -c_1|u| - c_2 - j(w)$ where c_1 and c_2 are some constants such that $j(r) \geq -c_1|r| - c_2$ for all $r \in R^1$ (see Proposition 2.1). Hence $j(u) \in L^1(\Omega)$ and

$$\int_{\Omega} j(w) dx - \int_{\Omega} j(u) dx \geq (v, w - u) = (J^{-1}f, w - u)$$

which shows that $f \in \partial\varphi(u)$. The arbitrariness of f implies that $A \subset \partial\varphi$. Next we shall prove that A is maximal monotone in $H^{-1}(\Omega)$. Let f_0 be given in $H^{-1}(\Omega)$. We must show that there are $u \in H^{-1}(\Omega) \cap L^1(\Omega)$ and $v \in H_0^1(\Omega)$ such that

$$u + Jv = f_0, \quad v(x) \in \beta(u(x)), \text{ a.e. } x \in \Omega$$

or equivalently

$$\gamma(v) + Jv = f_0 \tag{2.31}$$

where $\gamma = \beta^{-1}$. Consider the approximate equation

$$\gamma_\lambda(v_\lambda) + Jv_\lambda = f_0 \tag{2.32}$$

where $\gamma_\lambda = \lambda^{-1}(1 - (1 + \lambda\gamma)^{-1})$. We have already seen that γ_λ is Lipschitz on $X = R^1$. Recalling that $J = -\Delta$ it follows using standard arguments that Eq. (2.32) has a unique solution $v_\lambda \in H_0^1(\Omega)$ for every $\lambda > 0$. We multiply Eq. (2.32) by v_λ and integrate over Ω . We obtain

$$\int_{\Omega} |\operatorname{grad} v_\lambda|^2 dx + \int_{\Omega} \gamma_\lambda(v_\lambda)v_\lambda dx = (f_0, v_\lambda). \tag{2.33}$$

Without any loss of generality we may assume that $0 \in \gamma(0)$. (We note that condition (2.28) implies that $D(\gamma) = R^1$ in virtue of Proposition 2.6.) Then the last inequality implies that v_λ is bounded in $H_0^1(\Omega)$ as $\lambda \rightarrow 0$. Thus a subsequence (denoted again v_λ) can be extracted from $\{v_\lambda\}$ such that $v_\lambda \rightarrow v$ in $H_0^1(\Omega)$, and $v_\lambda \rightarrow v$ in $L^2(\Omega)$. Extracting further subsequences if necessary we may

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also assume that

$$\begin{aligned} v_\lambda(x) &\rightarrow v(x), \quad \text{a.e. } x \in \Omega \\ (1 + \lambda\gamma)^{-1}v_\lambda(x) &\rightarrow v(x), \quad \text{a.e. } x \in \Omega \end{aligned} \tag{2.34}$$

because $\lim_{\lambda \rightarrow 0} (1 + \lambda\gamma)^{-1}v = v$, for every $v \in \overline{D(\gamma)} = R^1$.

We set $f_\lambda = \gamma_\lambda(v_\lambda)$ and $w_\lambda = (1 + \lambda\gamma)^{-1}v_\lambda$. We note that $f_\lambda(x) \in \gamma(w_\lambda(x))$ a.e. $x \in \Omega$ and $f_\lambda \in L^2(\Omega)$. In particular this implies that $f_\lambda v_\lambda \in L^1(\Omega)$ and by (2.33)

$$\int_{\Omega} f_\lambda v_\lambda \, dx \leq C, \quad \forall \lambda > 0 \tag{2.35}$$

where C is some positive constant. On the other hand, for some $v_0 \in D(j)$ we have

$$j(f_\lambda(x)) \leq j(v_0) + (f_\lambda(x) - v_0)v, \quad \forall v \in \beta(f_\lambda(x)).$$

Next by (2.34) and (2.35),

$$\int_{\Omega} j(f_\lambda(x)) \, dx \leq C.$$

Using once again the Dunford-Pettis theorem, we find that $\{f_\lambda\}$ is weakly compact in $L^1(\Omega)$. Consequently we may assume that

$$f_\lambda \rightarrow f \quad \text{in } L^1(\Omega) \text{ as } \lambda \rightarrow 0.$$

Passing to limit through $\{\lambda\}$ in Eq. (2.32) we obtain $f + Jv = f_0$. To conclude the proof it remains to be shown that $f(x) \in \gamma(v(x))$ a.e. $x \in \Omega$. It suffices to prove that for every N , $f(x) \in \gamma(v(x))$ a.e. $x \in \Omega_N = \{x \in \Omega; |v(x)| \leq N\}$. According to Egorov's lemma for every $\epsilon > 0$ there is $E \subset \Omega_N$ such that meas. $E < \epsilon$, $v_\lambda(x) \rightarrow v(x)$ uniformly on Ω_N/E . (We note that $v_\lambda \rightarrow v$ a.e. on Ω in virtue of (2.34).) Thus no loss of generality results in assuming that $v_\lambda \rightarrow v$ uniformly on Ω and $v \in L^\infty(\Omega)$. Let $g : R^1 \rightarrow]-\infty, +\infty]$ be such that $\partial g = \gamma$. Then

$$\int_{\Omega} f_\lambda(x)(w_\lambda(x) - \tilde{v}(x)) \, dx \geq \int_{\Omega} g(w_\lambda(x)) \, dx - \int_{\Omega} g(\tilde{v}(x)) \, dx$$

for every $\tilde{v} \in L^\infty(\Omega)$. Again using Fatou's lemma we get

$$\int_{\Omega} f(x)(v(x) - \tilde{v}(x)) \, dx \geq \int_{\Omega} g(v(x)) \, dx - \int_{\Omega} g(\tilde{v}(x)) \, dx$$

⁷⁰ as $f_\lambda \rightarrow f$ weakly in $L^1(\Omega)$.

By a standard argument, we obtain the pointwise inequality

$$f(x)(v(x) - \tilde{v}) \geq g(v(x)) - g(\tilde{v}) \quad \text{a.e. } \Omega$$

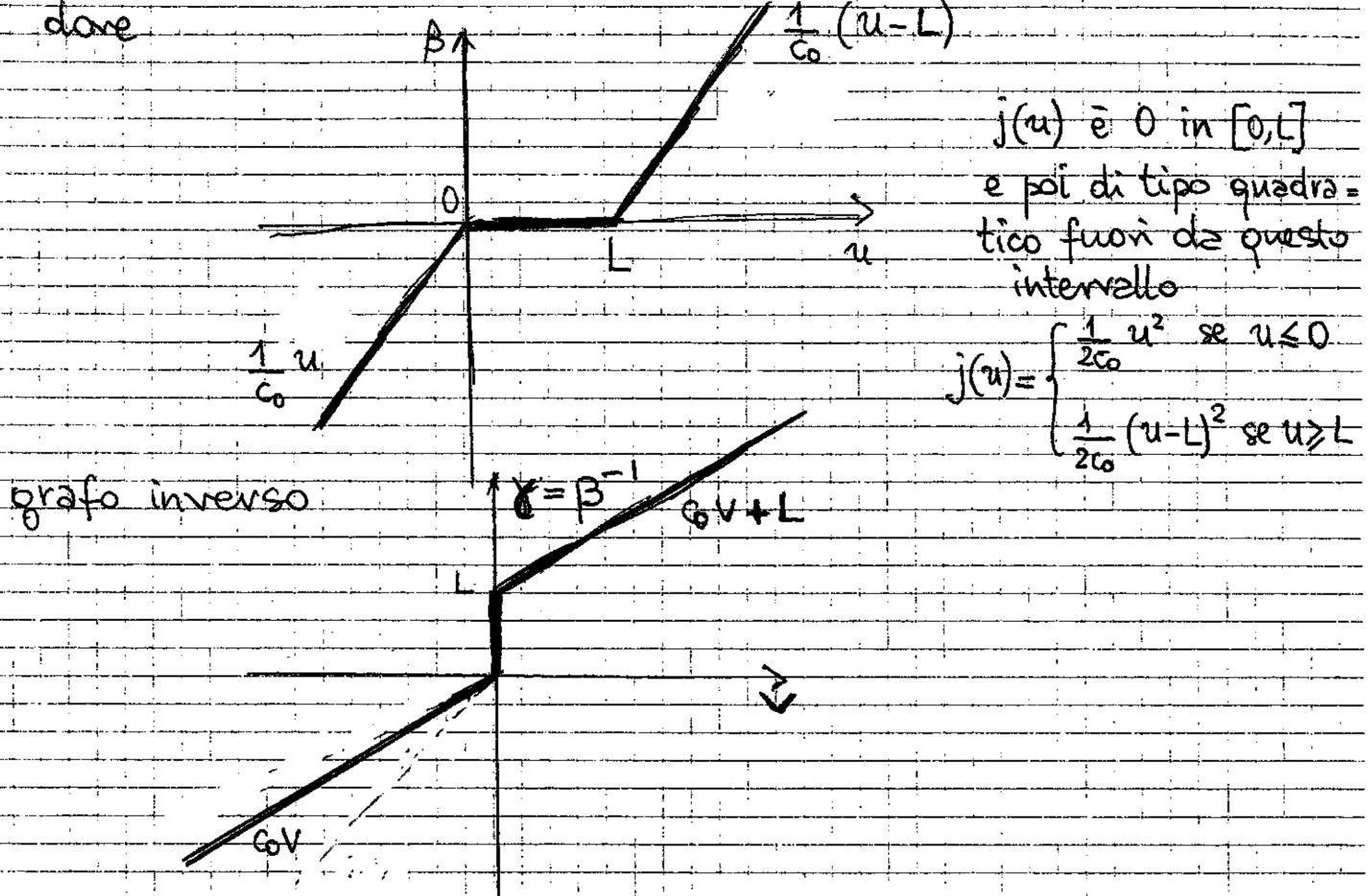
$\forall v \in \mathbb{R} \Rightarrow f(x) \in \partial g(x) = \gamma(v(x))$ a.e. and proof is complete.

Problema di Stefan

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$$\begin{cases} u_t - \Delta \beta(u) = f & \text{in } \Omega \times (0, T) \\ u=0 & \text{su } \partial\Omega \times (0, T) \\ u(0)=u_0 & \text{in } \Omega \end{cases}$$

dove



$$(\gamma(v))_t - \Delta v = f$$

$$\gamma(v) = c_0 v + L H(v)$$

dove H denota il grafo di Heaviside.

v rappresenta usualmente la temperatura

e posso introdurre variabile di fase $X \in H(v)$.

Dunque $u = c_0 v + L X$ è l'entalpia.

Soluzione $u \in W^{1,2}(0, T; H^1(\Omega))$ se $u_0 \in D(\varphi)$, $f \in L^2(0, T; H^1(\Omega))$ e da qui si ha che $\beta(u) \in L^2(0, T; H_0^1(\Omega))$