11.3 Some Classical Spaces of Sequences

Given a sequence \( x = (x_1, x_2, \ldots, x_k, \ldots) \), set
\[
\|x\|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, \quad 1 \leq p < \infty,
\]
\[
\|x\|_{\infty} = \sup_k |x_k|
\]
and consider the corresponding spaces
\[
\ell^p = \{ x; \|x\|_p < \infty \}, \quad 1 \leq p < \infty,
\]
\[
\ell^\infty = \{ x; \|x\|_{\infty} < \infty \},
\]
which are Banach spaces for the \( \ell^p \) (resp. \( \ell^\infty \)) norms. This can be established directly (and is quite easy); or one can rely on Theorem 4.8 applied to \( \Omega = \mathbb{N} \) equipped with the counting measure, \( \mu(E) = \) the number of points in a set \( E \subset \mathbb{N} \). Many properties mentioned below are consequences of general results from Chapter 4. For the convenience of the reader, we also present some direct proofs.

There are two interesting subspaces of \( \ell^\infty \):
\[
c = \left\{ x; \lim_{k \to \infty} x_k \text{ exists} \right\}
\]
and
\[
c_0 = \left\{ x; \lim_{k \to \infty} x_k = 0 \right\}.
\]
They are both equipped with the \( \ell^\infty \) norm. Clearly \( c_0 \subset c \subset \ell^\infty \) with \( c_0 \) closed in \( c \), and \( c \) closed in \( \ell^\infty \).

Hölder’s inequality takes the form
\[
(5) \quad \left| \sum_{k=1}^{\infty} x_k y_k \right| \leq \|x\|_p \|y\|_{p'} \quad \forall x \in \ell^p, \quad \forall y \in \ell^{p'} \text{ with } \frac{1}{p} + \frac{1}{p'} = 1.
\]

The space \( \ell^2 \) is a Hilbert space equipped with the scalar product
\[
(x, y) = \sum_{k=1}^{\infty} x_k y_k.
\]

It is clear that \( \ell^p \subset c_0 \) with
\[
\|x\|_{\infty} \leq \|x\|_p \quad \forall p, \quad 1 \leq p < \infty, \quad \forall x \in \ell^p,
\]
and this yields \( \ell^p \subset \ell^q \) when \( 1 \leq p \leq q \leq \infty \), with
\[ \|x\|_q \leq \|x\|_p \quad \forall x \in \ell^p. \]

**Proposition 11.15.** The space \( \ell^p \) is reflexive, and even uniformly convex, for \( 1 < p < \infty \).

*Proof.* Apply Theorem 4.10 and Exercise 4.12 with \( \Omega = \mathbb{N} \).

**Proposition 11.16.** The spaces \( c, c_0 \), and \( \ell^p \), with \( 1 \leq p < \infty \), are separable.

*Proof.* Let
\[
D = \{ x = (x_k); x_k \in \mathbb{Q} \quad \forall k, \text{ and } x_k = 0 \text{ for } k \text{ sufficiently large}\}.
\]

It is clear that \( D \) is countable; moreover, \( D \) is dense in \( \ell^p \) when \( 1 \leq p < \infty \) and in \( c_0 \). The set \( D + \lambda(1, 1, 1, \ldots) \), with \( \lambda \in \mathbb{Q} \), is countable and dense in \( c \).

**Proposition 11.17.** The space \( \ell^\infty \) is not separable.

*Proof.* Assume that \( A \subset \ell^\infty \) is countable. We will check that \( A \) cannot be dense in \( \ell^\infty \). Write \( A = (a_k^*) \), where each \( a_k^* \in \ell^\infty \), so that \( a_k^* = (a_k^*1, a_k^*2, \ldots) \). For each integer \( k \) set
\[
b_k = \begin{cases} 
a_k^* + 1 & \text{if } |a_k^*| \leq 1, \\
0 & \text{if } |a_k^*| > 1. 
\end{cases}
\]

Note that \( b = (b_k) \in \ell^\infty \) and \( |b_k - a_k^*| \geq 1 \forall k \). Therefore,
\[
\|b - a_k^*\|_{\infty} \geq |b_k - a_k^*| \geq 1 \quad \forall k,
\]
and thus \( b \notin \overline{A} \).

**Proposition 11.18.** Let \( 1 \leq p < \infty \). Given any \( \phi \in (\ell^p)^* \), there exists a unique \( u \in \ell^{p'} \) such that
\[
\langle \phi, x \rangle = \sum_{k=1}^{\infty} u_k x_k \quad \forall x \in \ell^p.
\]

Moreover,
\[
\|u\|_{p'} = \|\phi\|_{(\ell^p)^*}.
\]

*Proof.* Let \( e_k = (0, 0, \ldots, 1, 0, 0, \ldots) \). Set \( u_k = \phi(e_k) \). We claim that \( u = (u_k) \in \ell^{p'} \) and
\[
\langle \phi, x \rangle = \sum_{k=1}^{\infty} u_k x_k \quad \forall x \in \ell^p.
\]

Inequality (6) is clear when \( p = 1 \), since
\[
|u_k| \leq \|\phi\|_{(\ell^1)^*} ||e_k||_1 \leq \|\phi\|_{(\ell^1)^*} \quad \forall k.
\]

We now turn to the case \( 1 < p < \infty \). Fix an integer \( N \). Then for every \( x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots) \) we have
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\[ \sum_{k=1}^{N} u_k x_k = \phi \left( \sum_{k=1}^{N} x_k e_k \right) \leq \| \phi \|_{(\ell^p)^*} \| x \|_p. \]

Choosing \( x_k = |u_k|^{p'-2} u_k \) yields

\[ \left( \sum_{k=1}^{N} |u_k|^p \right)^{1/p'} \leq \| \phi \|_{(\ell^p)^*}. \]

As \( N \to \infty \) we see that \( u \in \ell^{p'} \) and (6) holds. Moreover,

\[ \phi(x) = \sum_{k=1}^{\infty} u_k x_k \quad \forall x \in D, \]

where \( D \) is defined in the proof of Proposition 11.16. Since \( D \) is dense in \( \ell^p \) we obtain

\[ \phi(x) = \sum_{k=1}^{\infty} u_k x_k \quad \forall x \in \ell^p. \]

Hölder’s inequality yields

\[ |\phi(x)| \leq \| u \|_{p'} \| x \|_p \quad \forall x \in \ell^p, \]

and therefore \( \| \phi \|_{(\ell^p)^*} \leq \| u \|_{p'} \). Combining with (6), we obtain

\[ \| \phi \|_{(\ell^p)^*} = \| u \|_{p'}. \]

The uniqueness of \( u \) is obvious.

**Proposition 11.19.** Given any \( \phi \in (c_0)^* \), there exists a unique \( u \in \ell^1 \) such that

\[ \langle \phi, x \rangle = \sum_{k=1}^{\infty} u_k x_k \quad \forall x \in c_0. \]

Moreover,

\[ \| u \|_1 = \| \phi \|_{(c_0)^*}. \]

**Proof.** This is an easy adaptation of the proof of Proposition 11.18 (with \( p = \infty \) and \( p' = 1 \)); the last part of the proof holds since \( D \) is dense in \( c_0 \) (but not in \( \ell^\infty \)).

**Proposition 11.20.** Given \( \phi \in (c)^* \), there exists a unique pair \((u, \lambda) \in \ell^1 \times \mathbb{R}\) such that

\[ \langle \phi, x \rangle = \sum_{k=1}^{\infty} u_k x_k + \lambda \lim_{k \to \infty} x_k \quad \forall x \in c. \]

Moreover,

\[ \| u \|_1 + |\lambda| = \| \phi \|_{(c)^*}. \]

**Proof.** Applying Proposition 11.19 to \( \phi|_{c_0} \), we find some \( u \in \ell^1 \) such that
\[ \phi(y) = \sum_{k=1}^{\infty} u_k y_k \quad \forall y \in c_0. \]

If \( x \in c \) write \( x = y + ae \), where \( e = (1, 1, 1, \ldots) \), \( a = \lim_{k \to \infty} x_k \), and \( y \in c_0 \). Then

\[
\phi(x) = \sum_{k=1}^{\infty} u_k y_k + a\phi(e) = \sum_{k=1}^{\infty} u_k (x_k - a) + a\phi(e) = \sum_{k=1}^{\infty} u_k x_k + \lambda a,
\]

where \( \lambda = \phi(e) - \sum_{k=1}^{\infty} u_k \).

Conversely, given any \( u \in \ell^1 \) and \( \lambda \in \mathbb{R} \), the functional

\[
\phi(x) = \sum_{k=1}^{\infty} u_k x_k + \lambda \lim_{k \to \infty} x_k, \quad x \in c,
\]

defines an element of \((c)^*\). We claim that

\[
\|\phi\|_{(c)^*} = \|u\|_1 + |\lambda|.
\]

It is clear that

\[
\|\phi\|_{(c)^*} \leq \|u\|_1 + |\lambda|.
\]

Choosing \( x = (x_k) \) in (8), where \( N \) is a fixed integer and

\[
x_k = \begin{cases} 
\text{sign} (u_k), & 1 \leq k \leq N, \\
\text{sign} (\lambda), & k > N,
\end{cases}
\]

yields

\[
\phi(x) = \sum_{k=1}^{N} |u_k| + \text{sign} (\lambda) \sum_{k=N+1}^{\infty} u_k + |\lambda| \leq \|\phi\|_{(c)^*}.
\]

As \( N \to \infty \) we obtain

\[
\|u\|_1 + |\lambda| \leq \|\phi\|_{(c)^*},
\]

which, together with (10), gives (9).

**Proposition 11.21.** The spaces \( \ell^1 \), \( \ell^\infty \), \( c \), and \( c_0 \) are not reflexive.

**Proof.** From Propositions 11.19 and 11.18 we know that \((c_0)^*\) is \( \ell^1 \) and \((\ell^1)^*\) is \( \ell^\infty \). Therefore the identity map from \( c_0 \) into \( \ell^\infty \) corresponds to the canonical injection \( J : c_0 \to (c_0)^{**} \) defined in Section 1.3. Since it is not surjective, we conclude that \( c_0 \) is not reflexive. Applying Corollary 3.21, we deduce that \( \ell^1 \) and \( \ell^\infty \) are not reflexive. Moreover, \( c \) cannot be reflexive; otherwise, \( c_0 \), which is a closed subspace of \( c \), would be reflexive by Proposition 3.20.

The following table summarizes the main properties discussed above: