On a non-isothermal diffuse interface model for two-phase flows of fluids

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Abstract

We study a non-isothermal diffuse interface model for the flow of two viscous incompressible Newtonian fluids of the same density in a bounded domain $\Omega \subset \mathbb{R}^3$. We derive the model problem following the general approach proposed by M. Frémond and find a weak solution to our model problem under suitable assumptions on the data.

Diffuse interface models: motivation

In classical models the interface between the two fluids is assumed to be a lower dimensional sufficiently smooth surface; in this case capillarity phenomena are related to contact angle conditions and a jump condition for the stress tensor across the interface. This classical description fails when some parts of the interface merge or reconnect (developing singularities) due to droplet formation or coalescence of several droplets. Thus, in order to be able to describe flows beyond these kinds of singularities an alternative approach can be used, based on a class of models called *diffuse interface models*. In this case the classical sharp interface is replaced by a thin interfacial region of finite "thickness", measured by a small parameter $\varepsilon > 0$. Therefore a partial mixing of the macroscopically immiscible fluids is taken into account; in this respect a variable φ is introduced representing the order parameter related to the concentration of the fluids (for instance the concentration difference or the concentration of one component). Diffusional effects are also included in the model.

The original idea of the diffuse interface model goes back to Hohenberg and Halperin [4] and it is referred with the name "model H". Later Gurtin et al. [3] gave a continuum mechanical derivation based on the concept of microforces. For a review of the development of diffuse-interface models and their applications we refer to [2], see also [1].

- [1] H. Abels: *Lecture notes, Max Planck Institute for Mathematics in the Sciences, No. 36/2007*, 2007.
- [2] D.M. Anderson, G.B. MacFadden, A.A. Wheeler: *Annual review of fluid mechanics*, 1998.
- [3] M.E. Gurtin, D. Polignone, J. Viñals: Math. Models Methods Appl. Sci., (1996).
- [4] P.C. Hohenberg, B.I. Halperin: Rev. Mod. Phys., (1977).

Quantities involved and assumptions

▷ $\mathbf{u}$ mean velocity and θ absolute temperature

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▷ φ order parameter related to the concentration of the fluids and μ chemical potential

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▷ $\mathbb{S} = \nu(\theta) D\mathbf{u}$ stress tensor (only dissipative part)

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▷ $F(\varphi)$ energy density function - **assumption**: $F(\varphi)$ is the classical double well potential, for instance $F(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2$

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▷ $\kappa(\theta) = 1 + \theta^\beta$ thermal conductivity - **assumption**: $\beta > 2$

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▷ $\nu(\theta)$ viscosity of the mixture - **assumption**: $0 < \underline{\nu} \le \nu(\theta) \le \overline{\nu}$ for all $\theta \ge 0$

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▷ $c_V(\theta) = \theta^\delta$ specific heat - **assumption**: $\frac{1}{2} < \delta < 1$

The equations of the model

INCOMPRESSIBILITY

$$\operatorname{div} \mathbf{u} = 0$$

CONSERVATION OF MOMENTUM

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x \tilde{p} = \operatorname{div} \mathbb{S} + (\mu + \theta)\nabla_x \varphi$$

MODIFIED CAHN-HILLIARD SYSTEM

$$\begin{cases} \varphi_t + \mathbf{u} \cdot \nabla_x \varphi = \triangle \mu \\ \mu = -\triangle \varphi + F'(\varphi) - \theta \end{cases}$$

EQUATION FOR THE TEMPERATURE

$$c_V(\theta)\theta_t + \theta \frac{D\varphi}{Dt} + \mathbf{u} \cdot \nabla_x \theta - \operatorname{div}(\kappa(\theta)\nabla_x \theta) = \nu(\theta)|D\mathbf{u}|^2 + |\nabla_x \mu|^2$$

TOTAL ENERGY BALANCE

$$\partial_t \left(\frac{|\mathbf{u}|^2}{2} + F(\varphi) + \frac{|\nabla \varphi|^2}{2} + Q(\theta) \right) + \operatorname{div} \left(\frac{|\mathbf{u}|^2}{2}\mathbf{u} + \theta \mathbf{u} + \tilde{p}\mathbf{u} - \varphi_t \nabla_x \varphi \right)$$
$$-\operatorname{div}(\kappa(\theta)\nabla_x \theta) = \operatorname{div}(\mathbb{S}\mathbf{u} + \mu \nabla_x \mu)$$

where Q is an antiderivative of c_V

ENTROPY INEQUALITY

$$(\Lambda(\theta) + \varphi)_t + \mathbf{u} \cdot \nabla_x(\log(\theta)) + \mathbf{u} \cdot \nabla_x \varphi - \operatorname{div}\left(\frac{\kappa(\theta)\nabla\theta}{\theta} \right)$$
$$\ge \frac{\nu(\theta)}{\theta}|\nabla_x \mathbf{u}|^2 + \frac{1}{\theta}|\nabla_x \mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x \theta|^2$$

where $\Lambda(\theta) = \int_1^\theta \frac{c_V(s)}{s} ds$

Main existence theorem

Existence of a weak solution with the following regularity:

$$\mathbf{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;H^1(\Omega;\mathbb{R}^3))$$
$$\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega))$$
$$\mu \in L^2(0,T;H^1(\Omega)) \cap L^{\frac{14}{5}}((0,T)\times\Omega)$$
$$\theta \in L^\infty(0,T;L^{\delta+1}(\Omega)) \cap L^\beta(0,T;L^{3\beta}(\Omega)),$$

A priori bounds

▷ ENERGY ESTIMATES

$||Q(\theta)||_{L^\infty(0,T;L^1(\Omega))} \le c$; $||\mathbf{u}||_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \le c$; $||F(\varphi)||_{L^\infty(0,T;L^1(\Omega))} \le c$;
$||\varphi||_{L^\infty(0,T;H^1(\Omega))} \le c$; $||\theta||_{L^\infty(0,T;L^{\delta+1}(\Omega))} \le c$.

▷ ENTROPY ESTIMATES

$||\theta^{-1/2}\nabla_x \mathbf{u}||_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times3})} \le c$; $||\theta^{-1/2}\nabla_x \mu||_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \le c$;
$\int_0^T \int_\Omega \frac{\kappa(\theta)}{\theta^2}|\nabla_x \theta|^2 \le c$; $||\nabla_x \theta||_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \le c$; $||\theta||_{L^\beta(0,T;L^{3\beta}(\Omega))} \le c$.

▷ TEMPERATURE ESTIMATES

$||D\mathbf{u}||_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times3})} \le c$; $||\nabla_x \mu||_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \le c$.

▷ CONSEQUENCES

$||\varphi||_{L^2(0,T;H^3(\Omega))} \le c$; $||\nabla_x(F'(\varphi))||_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \le c$;
$||\mu||_{L^2(0,T;W^{1,2}(\Omega))\cap L^{\frac{14}{5}}((0,T)\times\Omega)} \le c$.

Non-isothermal models for multiphase processes

▷ DIFFUSE INTERFACE MODELS

M.Eleuteri, E. Rocca, G. Schimperna: in preparation (2013)

▷ THERMO-VISCO-ELASTIC MATERIALS

M. Eleuteri, J. Kopfová, P. Krejčí: Physica B (2012);
M. Eleuteri, J. Kopfová, P. Krejčí: Comm. Pure Appl. Anal., (2013)

▷ SHAPE MEMORY ALLOYS

M. Eleuteri, L. Lussardi, U. Stefanelli: DCDS-S (2013).

▷ LIQUID CRYSTALS

E. Feireisl, M. Frémond, E. Rocca, G. Schimperna: Arch. Ration. Mech. Anal., (2012)
E. Feireisl, E. Rocca, G. Schimperna: Nonlinearity, (2011)
E. Feireisl, E. Rocca, G. Schimperna, A. Zarnescu: preprint arXiv:1207.1643v1 (2012), Comm. Math. Sci. (2013) to appear.

▷ DAMAGE IN VISCO-ELASTIC SOLIDS

E. Rocca, R. Rossi: preprint arXiv:1205.3578v1 (2012)

Perspectives and future work

- the 2D case: better embedding estimates ⇒ more regularity for the solution ⇒ regularization estimates uniform in time ⇒ global attractor?
- logarithmic potential (singular) and non constant mobility (degenerate) in Cahn-Hilliard equation.

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