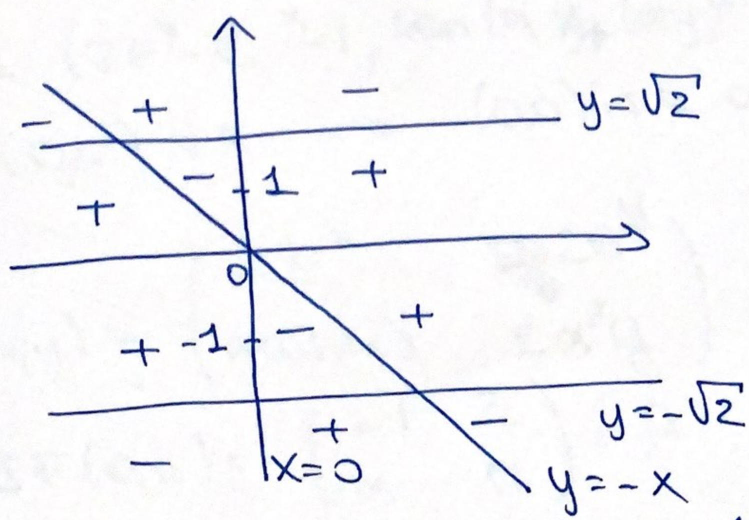
Advanced Math. Meth. for Engineers

February 24, 2022

1) $f(x,y) = \arctan((2-y^2)(x^2+xy)) \in C^\infty(\mathbb{R}^2)$ and it is bounded in $\mathbb{R}^2$ $\Rightarrow$ the Cauchy problem $\forall K \in \mathbb{R}$ has a unique global solution $y: \mathbb{R} \to \mathbb{R}$ and dom $(y) \equiv \mathbb{R}$.

$y = \pm\sqrt{2}$ are the stationary solutions, so the other solutions cannot intersect them due to the uniqueness.

The lines $x = 0$ and $y = -x$ are the lines of stationary points. The sign of y' is the following indeed:



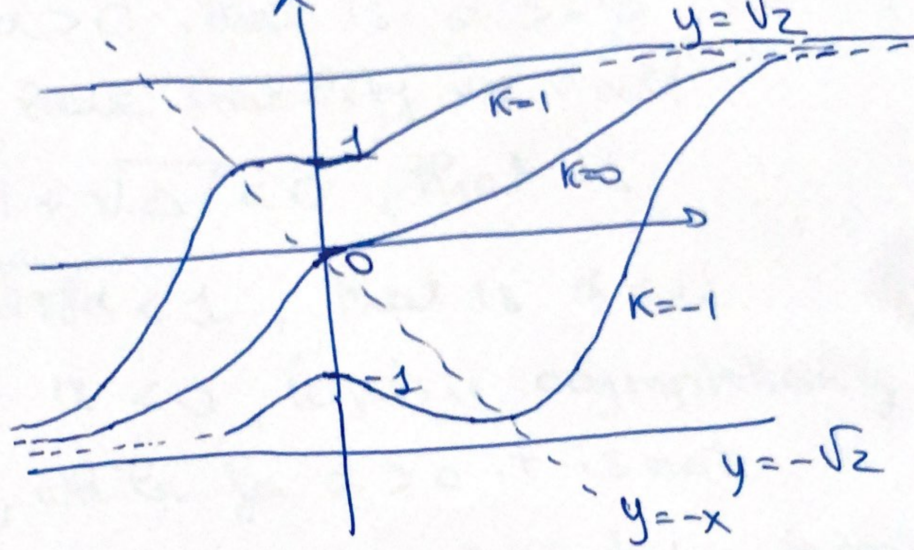
The solutions stemming from $(0,0)$ and $(0,\pm 1)$ must be included in the sector $-\sqrt{2} < y < \sqrt{2}$ and so the $\lim_{x\to\pm\infty} y(x) < +\infty$ and so

$\lim_{x\to\pm\infty} y'(x) = 0$ (due to the asymptot thm)

And so we get $\ell = \lim_{x \to \pm\infty} y(x) = \pm\sqrt{2}$

$\Rightarrow$ $y = \pm\sqrt{2}$ are orizontal asymptots.

Here are the graphs:



2) $F(x,y) = (2e^y - e^x - 1, \operatorname{sen}(\alpha x) + (\alpha y)^2)$

$F(0,0) = (0,0)\ \forall \alpha \Rightarrow (0,0)$ is a critical point.

$$JF(x,y) = \begin{pmatrix} -e^x & 2e^y \\ \alpha\cos(\alpha x) & 2\alpha^2 y \end{pmatrix}$$

$$\Rightarrow JF(0,0) = \begin{pmatrix} -1 & 2 \\ \alpha & 0 \end{pmatrix} = A$$

$\det JF(0,0) \neq 0$

$0 = \det(A - \lambda Id) = (-1-\lambda)(-\lambda) - 2\alpha$

$= \lambda + \lambda^2 - 2\alpha \quad \Rightarrow$

$\Delta = 1 + 8\alpha, \quad \lambda_{1,2} = \dfrac{-1 \pm \sqrt{\Delta}}{2}$

If $1+8\alpha \leq 0$, that is $\alpha \leq -1/8$ $\Rightarrow$

$Re(\lambda_1) = Re(\lambda_2) = -\frac{1}{2} < 0$ $\Rightarrow$

$(0,0)$ is asymptotically stable

If $1+8\alpha > 0$, that is $\alpha > -1/8$, in order to have stability we need

$-1+\sqrt{\Delta} < 0$, that is

$\sqrt{1+8\alpha} < 1$, that is $\alpha < 0$.

So if $\alpha < 0$, $(0,0)$ is asymptotically stable, while for $\alpha > 0$ it is not.

3) The $n \mapsto \operatorname{sen}(n\sqrt{x})$ is bounded on $(0,+\infty)$ and $n \mapsto \frac{1}{x(n+\sqrt{x})}$ goes to 0 as $n \to +\infty$

and so $f_n(x) := \frac{\operatorname{sen}(n\sqrt{x})}{x(n+\sqrt{x})} \xrightarrow[n\to\infty]{} 0$.

Moreover $\forall z$ $|\operatorname{sen}(z)| \leq z$, hence if $x \in [0,1]$ we have

$$|f_n(x)| \leq \frac{n\sqrt{x}}{x(n+\sqrt{x})} = \frac{1}{\sqrt{x}}\left(\frac{n}{n+\sqrt{x}}\right) \leq \underset{L^1(0,1)}{\underset{\cap}{\frac{1}{\sqrt{x}}}}$$

For $x > 1$ we have

$$|f_n(x)| \leq \frac{1}{x(n+\sqrt{x})} \leq \frac{1}{x^{3/2}} \in L^1(1,+\infty)$$

$\Rightarrow$ on $(0,+\infty)$ we have

$$|f_n(x)| \leq g(x) := \begin{cases} \frac{1}{\sqrt{x}}, & x \in (0,1] \\ \frac{1}{x^{3/2}}, & x > 1 \end{cases}$$

and $g \in L^1(0,+\infty)$ => by the dominated convergence theorem we have

$$\lim_{n\to\infty} \int_0^{+\infty} f_n(x)\,dx = \int_0^{+\infty} f(x)\,dx = 0.$$

4) we search for solutions:

$u(x,t) = v(x)\,w(t)$ which solves:

$$w'(t) - \lambda D w(t) = 0$$

i.e. $w(t) = c e^{\lambda D t}$, $c \in \mathbb{R}$ and

$$\begin{cases} v''(x) + \lambda^2 v(x) = 0 \\ v'(0) = 0 \\ v'(L) = -\gamma v(L). \end{cases}$$

We have 3 cases:

a) $\lambda = \mu^2 > 0 \Rightarrow v(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

and $\begin{cases} \mu c_1 - \mu c_2 = 0 \\ (\mu+\gamma) e^{\mu L} c_1 - (\mu - \gamma) e^{-\mu L} c_2 = 0 \end{cases}$

and being $\mu[(\mu+\gamma)e^{\mu L} - (\mu-\gamma)e^{-\mu L}] \neq 0$

we get $c_1 = c_2 = 0$ (only trivial solutions)

b) $\lambda = -\mu^2 < 0 \Rightarrow v'(x) = -\mu c_1 \operatorname{sen}(\mu x) + \mu c_2 \cos(\mu x)$

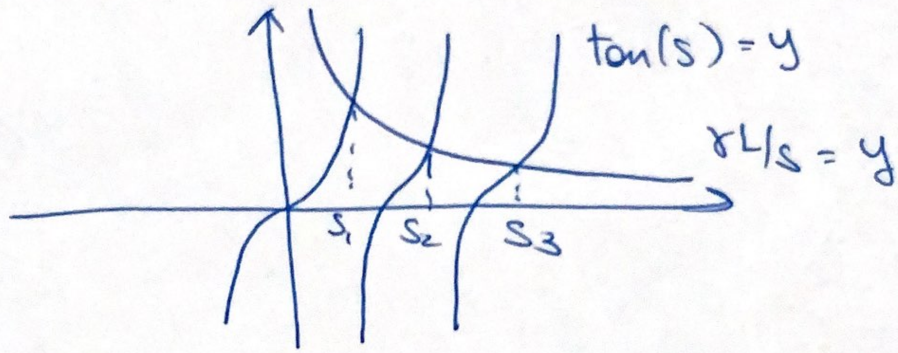
$\Rightarrow v(x) = c_1 \cos(\mu x) + c_2 \operatorname{sen}(\mu x)$

$v'(0) = 0 \Rightarrow c_2 = 0$

$v'(L) = -\gamma v(L) \Rightarrow \mu \operatorname{sen}(\mu L) = \gamma \cos(\mu L)$

i.e. $\tan(\mu L) = \gamma/\mu$.

Letting $s = \mu L \Rightarrow \tan(s) = \gamma L/s \quad , s > 0$



$0 < s_1 = \mu_1 L < s_2 = \mu_2 L < \dots$ and

$(m-1)\pi < \mu_m L < m\pi \Rightarrow \mu_m \sim \frac{m\pi}{L}$ as $m \to \infty$

$\Rightarrow \tan(\mu_m L), \text{sen}(\mu_m L) \to 0$ as $m \to \infty$

$\Rightarrow \lambda_m = -\mu_n^2 = -\frac{s_m^2}{L^2}$ and

$v_m(x) = C\cos(\mu_n x) \Rightarrow$

$$u(x,t) = \sum_{m=1}^{\infty} a_n e^{-D\mu_m^2 t}\cos(\mu_n x).$$

Imposing the initial condition:

$$u(x,0) = \sum_{m=1}^{\infty} a_n \cos(\mu_n x) = g(x) \text{ on } [0,L]$$

we get (writing $g(x) = \sum_{m=1}^{\infty} g_n \cos(\mu_n x)$)

$$u(x,t) = \sum_{m=1}^{\infty} g_n e^{-D\mu_n^2 t}\cos(\mu_n x)$$

where $g_n = \frac{1}{\beta_m}\int_0^L g(x)\cos(\mu_n x)\,dx$

with $\beta_m = \int_0^L v_m^2(x)\,dx = \frac{L}{2} + \frac{\text{sen}(2\mu_n L)}{4\mu_n}$