Master Program in Electronic Engineering

Advanced Mathematical Methods for Engineers

February 20, 2018

1. Let $b \in \mathbf{R}^+$ and consider the following Cauchy Problem

$$\begin{cases} y'(x) = \dfrac{2y(x)}{x} + 3x^b \\ y(1) = 2. \end{cases}$$

1.1) Discuss existence and uniqueness of solutions.

1.2) Find explicitly the solution y_b, depending on b.

1.3) Find the values of b such that $\lim_{x \to +\infty} y_b(x) = +\infty$.

2. Find all solutions X the ODE system $X' = AX$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ -2 & 1 & -1 \end{bmatrix}$$

3. Consider in $(0, +\infty)$ the sequence of functions

$$f_n(x) = \frac{\sin(nx)}{nx^{3/2}}$$

and prove that

a) $f_n \in L^1(0, +\infty)$ for every $n \in \mathbf{N}$,

b) $f_n \to 0$ as $n \to \infty$ pointwise in $(0, +\infty)$,

c) $\int_0^\infty f_n(x)dx \to 0$ as $n \to \infty$.

4. Suppose to have the following orthonormal system of polynomials (the so-called *Laguerre* polynomials):

$$L_n(x) := \frac{e^x}{n!} \frac{d^n}{dx^n}(e^{-x}x^n),$$

and suppose to know that this is a basis in $L^2(0, +\infty)$ endowed with the scalar product $(u, v) := \int_0^\infty u(x)v(x)e^{-x}dx$.

Compute the polynomial of degree ≤ 2 which approximate better the function $f(x) = e^{x/4}$ in $L^2(0, +\infty)$ endowed with the scalar product $(u, v) := \int_0^\infty u(x)v(x)e^{-x}dx$.
(**Suggestion**: compute first the three *Laguerre* polynomials: L_0, L_1, L_2).