Adv. Math. Methods

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1) $f(x,y) = \frac{x+2}{x^2+1}\, y - \frac{y^2}{(x^2+1)^{3/2}} \in C^1(\mathbb{R}^2)$

$\Rightarrow \forall a\ \exists!$ local solution $y: U(0) \to \mathbb{R}$.

If $a=0 \Rightarrow y \equiv 0$ is a solution.

If $a \neq 0 \Rightarrow a^2 > 0 \Rightarrow y_a > 0 \quad \forall x \in \operatorname{dom} y_a$

It's a Bernoulli eq., so, if we take $z(x) = 1/y(x)$ we get the linear Cauchy problem:

$$\begin{cases} z' + \frac{x+2}{x^2+1} z = \frac{1}{(x^2+1)^{3/2}} \\ z(0) = 1/a^2 \end{cases}$$

$$\Rightarrow z(x) = \frac{e^{-2\arctan x}}{\sqrt{x^2+1}} \cdot \left\{ \frac{1}{a^2} + \int_0^x \frac{e^{2\arctan \tau}}{\tau^2+1}\, d\tau \right\}$$

$$= \frac{e^{-2\arctan x}}{\sqrt{x^2+1}} \cdot \left\{ \frac{1}{a^2} + \frac{1}{2} e^{2\arctan x} - \frac{1}{2} \right\}$$

$$\Rightarrow y_a(x) = \frac{2 e^{2\arctan x} \sqrt{x^2+1}}{\frac{2}{a^2} - 1 + e^{2\arctan x}}$$

dom $y_a \equiv \mathbb{R}$ iff the denominator is different from 0 $\forall x \in \mathbb{R}$

Since $\operatorname{im}(g) = (e^{-\pi}, e^{\pi})$ we need 2

$g = e^{2\arctan y}$

$$1 - \frac{2}{a^2} \leq e^{-\pi} \quad \text{or} \quad 1 - \frac{2}{a^2} \geq e^{\pi}$$

The second one is never satisfied, while the first one gives:

$$\frac{2}{a^2} \geq 1 - e^{-\pi} \implies a^2 \leq \frac{2e^{\pi}}{e^{\pi}-1}$$

and so we get $|a| \leq \sqrt{\dfrac{2e^{\pi}}{e^{\pi}-1}}$.

2) The system is

$$\begin{cases} x' = (1-y-x)\,x \\ y' = (x-2)\,y \end{cases}$$

$\implies$ the stationary points are

$(0,0)$, $\underbrace{(2,-1)}_{\text{not admissible}}$, $(1,0)$ and

$$DF = \begin{pmatrix} 1-y-2x & -x \\ y & x-2 \end{pmatrix}$$

$$DF\big|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \implies (0,0) \text{ is } \underline{\text{not stable}}$$

$$DF\big|_{(1,0)} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow (1,0)$ is stable.

3) The sequence $\left(1+\frac{x}{n}\right)^n \underset{n\to\infty}{\nearrow} e^x$ for $x \geq 0$

$$\Rightarrow \left(1+\frac{x}{n}\right)^n e^{-\pi x} \mathbb{1}_{[0,n]} \uparrow e^{(1-\pi)x}$$

and so, using Beppo Levi Theorem, we have

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-\pi x}\,dx =$$

$$= \lim_{n\to\infty} \int_0^{+\infty} e^{-(\pi-1)x}\,dx = \frac{1}{\pi-1}$$

4) Particular solutions of $(x^2-1)\tilde{u} = 0$ are $\tilde{u} = c_1\delta_{-1} + c_2\delta_1$

$\Rightarrow (x^2-1)u = \delta_0 \Rightarrow u = \dfrac{\delta_0}{x^2-1}$

because $\operatorname{supp}(\delta_0) = \{0\} \notin \operatorname{supp}\dfrac{1}{x^2-1}$

$$\Rightarrow \left\langle \frac{\delta}{x^2-1}, \varphi \right\rangle = \left\langle \delta, \frac{\varphi}{x^2-1} \right\rangle = -\varphi(0) =$$

(4)

$= - \langle \delta, \varphi \rangle \Rightarrow$

$$\frac{\delta}{x^2-1} = -\delta \text{ in } \mathcal{D}' \Rightarrow$$

$$u = c_1 \delta_{-1} + c_2 \delta_1 - \delta_0$$