Master Program in Electronic Engineering

Advanced Mathematical Methods for Engineers

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1. Let $k \in \mathbf{R}$, consider the following Cauchy Problem

$$\begin{cases} y'(x) + xy(x) + x\sqrt{y(x)} = 0 \\ y(0) = k. \end{cases}$$

a) Discuss local and global existence and uniqueness of solutions, depending on k.

b) Find explicitly the solution in case $k > 0$ and draw its graph.

2. Find the solutions of the following linear ODE system

$$\begin{cases} x' = x + 3y + 2z \\ y' = y \\ z' = 2y + z\,. \end{cases}$$

3. Let, for $x \geq 0$ and $n \in \mathbf{N}$,

$$g_n(x) := \frac{x^3}{n + x^4} \quad \text{and} \quad f_n(x) := g_n(x) \arctan\left(\frac{1}{nx^2 + 2}\right).$$

Then

a) Find the pointwise limit f of f_n as n tends to $+\infty$.

b) Compute $\sup_{x \in [0,+\infty)} |g_n(x)|$.

c) Prove that $f_n \to f$ in $C^0([0, +\infty))$ (with the sup-norm) as $n \to +\infty$.

4. Let $f \in \mathcal{D}(\mathbf{R})$, $\psi \in C^\infty(\mathbf{R})$ and check that

a) $(\psi f)' = \psi' f + \psi f'$ in $\mathcal{D}'(\mathbf{R})$;

b) $x\delta' = -\delta$ in $\mathcal{D}'(\mathbf{R})$.