Adv. Math. Meth. for Engineers June 28, 22

1) a) Since it is a linear ODE with continuous coefficients, the cauchy problem has a unique global solution $y : \mathbb{R} \to \mathbb{R}$

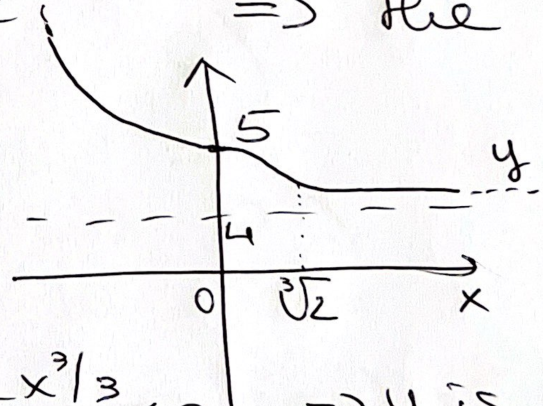
b) we can compute explicitely the solution as

$$y(x) = e^{-x^3/3}\left(4e^{x^3/3} + c\right)$$

and imposing the initial condition we get :

$$y(x) = 4 + e^{-x^3/3} \Rightarrow \text{the}$$

graph is



$y'(x) = -x^2 e^{-x^3/3} < 0 \Rightarrow y$ is decreasing in $\mathbb{R}$

$y''(x) = x \cdot (x^3 - 2)e^{-x^3/2} \geq 0$

for $x \leq 0$ and $x \geq \sqrt[3]{2}$

c) $y(1) = 4e^{-1/3}$ $\quad y'(1) = -e^{-1/3}$

$\Rightarrow 4(y(1) + y'(1)) = 16$

2) $\begin{cases} x' = x(y-1) \\ y' = y(2x+y-5) \end{cases}$

The stationary points are :

$(0,0)$, $(0,5)$, $(2,1)$

$$DF = A = \begin{pmatrix} y-1 & x \\ 2y & 2y+2\cancel{x}-5 \end{pmatrix}$$

$$DF(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

$\Rightarrow$ $(0,0)$ is asymptotically stable

$$DF(0,5) = \begin{pmatrix} 4 & 0 \\ 10 & 5 \end{pmatrix}$$

$$(DF-\lambda I)(0,5) = \begin{pmatrix} 4-\lambda & 0 \\ 10 & 5-\lambda \end{pmatrix}$$

$\lambda_1 = 4$, $\lambda_2 = 5$ $\Rightarrow$ $(0,5)$ is unstable

$$DF(2,1) = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$$

$$(DF-\lambda I)(2,1) = \begin{pmatrix} -\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix}$$

$0 = |\det \downarrow| = -\lambda \cdot (1-\lambda) - 4$

$= \lambda^2 - \lambda - 4$

$0 = -\lambda + \lambda^2 - 4 \iff$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

$$Re(\lambda_2) = \frac{1-\sqrt{17}}{2} < 0 \implies$$

(2, 1) is unstable.

3) a) $f_n'(x) = n^2 e^{-nx}\left(\frac{1}{2\sqrt{x}} - n\sqrt{x}\right) \geq 0$

$\iff \quad x \leq \frac{1}{2n} < 1 \implies$

$x_{max} = \frac{1}{2n} \implies \|f_n\|_\infty = f_n\left(\frac{1}{2n}\right) = \frac{1}{\sqrt{2}} n^{3/2} e^{-1/2}$

b) $\lim\limits_{n\to\infty} f_n(x) = 0 = f(x)$
at x fixed in $[0, +\infty)$

c) On $[0, +\infty)$ the convergence is not uniform because $\|f_n\|_\infty \xrightarrow[n\to\infty]{} +\infty$
but f_n are decreasing on $[1, +\infty) \implies$

d) $\|f_n\|_{\infty \text{ on } [1,+\infty)} = n^2 e^{-n} \xrightarrow[n\to\infty]{} 0$
So we have uniform convergence on $[1, +\infty)$.

4) $\langle \mathcal{F}(e^{ix}), \varphi \rangle = \langle e^{ix}, \mathcal{F}(\varphi)(x) \rangle$

$$= \int_{-\infty}^{+\infty} \left(e^{ix} \int_{-\infty}^{+\infty} e^{-ixt} \varphi(t)\, dt \right) dx$$

$$= 2\pi \, \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(e^{i\, x \cdot 1} \cdot \int_{-\infty}^{+\infty} e^{-ixt} \varphi(t)\, dt \right) dx$$

$$\Rightarrow \mathcal{F}(e^{ix})(\omega) = 2\pi\, \varphi(1) = 2\pi\, \delta_1 = 2\pi\, \delta(\omega - 1)$$

Being $\operatorname{sen} x = \dfrac{e^{ix} - e^{-ix}}{2i}$,

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2} \Rightarrow$

$$\mathcal{F}(\operatorname{sen} x) = \frac{\pi}{i} (\delta_1 - \delta_{-1})$$

$$\mathcal{F}(\cos x) = \pi (\delta_1 + \delta_{-1})$$

Moreover $\langle \mathcal{F}(\operatorname{sign} x), \varphi(x) \rangle = \langle \frac{2}{i} PV\left(\frac{1}{x}\right), \varphi \rangle$

$\cancel{= \int_{-\infty}^{+\infty}}$ $\quad \mathcal{F}(\operatorname{sign})(\omega) = \frac{2}{i} PV\left(\frac{1}{\omega}\right)$

because $\quad$ if $u = \operatorname{sign} x \Rightarrow \underbrace{\hat{u'}(\omega)}_{2\hat{\delta}(\omega)} = 2$

$\Rightarrow i\omega \hat{u}(\omega) = 2 \Rightarrow \hat{u}(\omega) = \frac{2}{i} PV\left(\frac{1}{\omega}\right) + c\delta, \; c \in \mathbb{R}$

but u is odd $\underset{c=0}{\Rightarrow} \hat{u}(\omega) = \frac{2}{i} PV\left(\frac{1}{\omega}\right)$