

Asymptotic analysis of some isothermal models for nematic liquid crystal flows

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PDEs for multiphase advanced materials-ADMAT 2012
Cortona, September 17-21 2012

Supported by the FP7-IDEAS-ERC-StG Grant



“EntroPhase”#256872

Nematic liquid crystals

- Materials consisting of molecules with elongated shape
- Fluid has anisotropic properties over a limited temperature range: molecules lined up in a specific direction (uniaxial), but no positional order
- the director \mathbf{d} is the average, over a small volume element, of unit vectors representing the long axis of each molecule



Solido



Cristallo Liquido



Liquido

A simplified model for nematic liquid crystals (F.-H. Lin & C. Liu '95, '96)

Simplification of the original Ericksen-Leslie system with thermal and e.m. effects neglected

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = -\lambda \operatorname{div}(\nabla \mathbf{d} \otimes \nabla \mathbf{d}) + \mathbf{h}$$

$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \eta(\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d}))$$

$$\operatorname{div} \mathbf{u} = 0$$

- $W(\mathbf{d})$ double-well regular potential, e.g.
 $W(\mathbf{d}) = (|\mathbf{d}|^2 - 1)^2$
- W relaxation of the constraint $|\mathbf{d}| = 1$

Reasonable b.c. are

- no-slip for \mathbf{u} , Dirichlet for \mathbf{d} (strong anchoring)
- free-slip for \mathbf{u} , hom. Neumann for \mathbf{d}

A corrected model (H. Sun & C. Liu, '09)

In $\Omega \times (0, \infty)$, $\Omega \subset \mathbb{R}^3$

$$\begin{aligned} \mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p &= \operatorname{div}(\nu(\nabla \mathbf{u} + \nabla^T \mathbf{u})) - \operatorname{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d}) \\ &\quad - \operatorname{div}(\alpha(\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \otimes \mathbf{d} - (1 - \alpha)\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) + \mathbf{h} \\ \mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} - \alpha \mathbf{d} \cdot \nabla \mathbf{u} + (1 - \alpha)\mathbf{d} \cdot \nabla^T \mathbf{u} &= (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \\ \operatorname{div}(\mathbf{u}) &= 0 \end{aligned}$$

- $\alpha \in [0, 1]$ related to the shape of liquid crystal molecules
 $\alpha = 1$ rod-like, $\alpha = 1/2$ spherical, $\alpha = 0$ disc-like
- b.c. considered: periodic, or no-slip for \mathbf{u} +hom. Neumann or non hom. Dirichlet for \mathbf{d}

$$\partial_{\mathbf{n}} \mathbf{d} = 0 \quad \text{or} \quad \mathbf{d} = \mathbf{g} \quad \text{on } \partial \Omega$$

Remarks

- Lin-Liu model neglects kinematic transport: liquid crystal molecules assumed small
- Stretching term $\mathbf{d} \cdot \nabla \mathbf{u}$ included and a new component $\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d}))$ added in the stress tensor to ensure energy balance
- Mathematical difficulty: lack of maximum principle and of L^∞ -estimate for \mathbf{d}

- Well-posedness

- H. Sun & C. Liu 09': \exists strong sols, periodic b.c., in 2D (3D, ν large)
- H. Wu, X. Xu & C. Liu '10: uniqueness and continuous dependence on in. data of strong sols, periodic b.c., in 2D
- C. Cavaterra & E. Rocca '12: \exists weak sols in 3D, Neumann or non-hom. Dirichlet b.c. for \mathbf{d} and no-slip b.c. for \mathbf{u}
- E. Feireisl, M Frémond, E. Rocca & G. Schimperna '11: \exists weak sols for non-isothermal system, Neumann b.c. for \mathbf{d} , free-slip b.c. for \mathbf{u}

- Asymptotic behavior

- H. Wu, X. Xu & C. Liu '10: conv. to eq. strong sols, conv. rate, periodic bc, in 2D (3D, ν large)
- H. Petzeltová, E. Rocca & G. Schimperna '12: conv. to eq. weak sols, conv. rate, Neumann bc for \mathbf{d} in 2D and 3D
- M. Grasselli & H. Wu '11: smooth global attractor in 2D for strong sols, with finite fractal dimension, periodic b.c.

∃ weak sols in 3D

Consider, e.g., hom. Neumann b.c. for \mathbf{d} (no-slip for \mathbf{u})

$$(A1) \quad W = W_1 + W_2 \quad W \in C^2(\mathbb{R}^3), \quad W_1 \text{ convex}, \quad W_2 \in C^{1,1}$$

$$(A2) \quad \mathbf{h} \in L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div}) \quad \mathbb{R}^+ := [0, \infty)$$

Theorem (C. Cavaterra & E. Rocca '12)

Assume (A1), (A2) and that

$$\mathbf{u}_0 \in \mathbf{V}_{div}, \quad \mathbf{d}_0 \in H^1(\Omega)^3, \quad W(\mathbf{d}_0) \in L^1(\Omega)$$

Then, \exists a weak sol $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ corresponding to $\mathbf{u}_0, \mathbf{d}_0$ s.t.

$$\mathbf{u} \in L^\infty_{loc}(\mathbb{R}^+; \mathbf{H}_{div}) \cap L^2_{loc}(\mathbb{R}^+; \mathbf{V}_{div})$$

$$\mathbf{u}_t \in L^2_{loc}(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3)$$

$$\mathbf{d} \in L^\infty_{loc}(\mathbb{R}^+; H^1(\Omega)^3) \cap L^2_{loc}(\mathbb{R}^+; H^2(\Omega)^3), \quad W(\mathbf{d}) \in L^\infty_{loc}(\mathbb{R}^+; L^1(\Omega))$$

$$\mathbf{d}_t \in L^2_{loc}(\mathbb{R}^+; L^{3/2}(\Omega)^3)$$

Theorem (C. Cavaterra & E. Rocca '12)

and satisfying the energy inequality

$$\begin{aligned} \mathcal{E}(\mathbf{w}(t)) + \int_s^t \left(\|\Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d})\|^2 + \nu \|\nabla \mathbf{u}\|^2 \right) d\tau \\ \leq \mathcal{E}(\mathbf{w}(s)) + \int_s^t \langle \mathbf{h}, \mathbf{u} \rangle d\tau \end{aligned}$$

for all $t \geq s$, for a.e. $s \in (0, \infty)$, including $s = 0$. We have set

$$\mathcal{E}(\mathbf{w}(t)) := \frac{1}{2} \|\mathbf{u}(t)\|^2 + \frac{1}{2} \|\nabla \mathbf{d}(t)\|^2 + \int_{\Omega} W(\mathbf{d}(t)), \quad \mathbf{w} = [\mathbf{u}, \mathbf{d}]$$

Remark

The space of test functions for weak sols is $W_{0,div}^{1,3}(\Omega)$
(uniqueness or strong-weak uniqueness in 2D not known)

Trajectory attractor approach (V.V. Chepyzhov & M.I. Vishik)

Abstract evolution equation in a Banach space E

$$w_t = A_\sigma w, \quad \sigma \in \Sigma, \quad \Sigma \text{ space of symbols}$$

- *Space of sols*: sols $w : \mathbb{R}^+ \rightarrow E$ sought in a space \mathcal{W}_{loc}^+ endowed with local convergence topology Θ_{loc}^+ (weak or strong)
- *Trajectory space*: for each $\sigma \in \Sigma$, \mathcal{K}_σ^+ is the set of *some* sols from \mathcal{W}_{loc}^+ and $\mathcal{K}_\Sigma^+ := \cup_{\sigma \in \Sigma} \mathcal{K}_\sigma^+$
- The translation semigroup $\{T(t)\}$ acts on \mathcal{K}_Σ^+ (if the family $\{\mathcal{K}_\sigma^+\}_{\sigma \in \Sigma}$ is translation coordinated)
- Introduce a subspace \mathcal{W}_b^+ of \mathcal{W}_{loc}^+ : usually Banach, but also metric space with metric $\rho_{\mathcal{W}_b^+}$. We assume that $\mathcal{K}_\sigma^+ \subset \mathcal{W}_b^+$, for all $\sigma \in \Sigma$. The subspace \mathcal{W}_b^+ is used to define bounded subsets of the trajectory space \mathcal{K}_Σ^+

Trajectory attractor approach

- $\mathcal{A}_\Sigma \subset \mathcal{W}_{loc}^+$ *uniform (w.r.t. $\sigma \in \Sigma$) trajectory attractor* if
 - 1) \mathcal{A}_Σ is compact in Θ_{loc}^+
 - 2) \mathcal{A}_Σ is a uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for $\{\mathcal{K}_\sigma^+\}_{\sigma \in \Sigma}$ in the topology Θ_{loc}^+
 - 3) \mathcal{A}_Σ is the minimal compact and uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for the family $\{\mathcal{K}_\sigma^+\}_{\sigma \in \Sigma}$ in Θ_{loc}^+
- If \mathcal{A}_Σ exists, it is unique
- If $T(t)$ is continuous in Θ_{loc}^+ , then $T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma, \forall t \in \mathbb{R}^+$
- For existence of \mathcal{A}_Σ : prove $\exists P \subset \mathcal{W}_{loc}^+$ compact and uniformly (w.r.t. $\sigma \in \Sigma$) attracting in Θ_{loc}^+

Trajectory attractor approach

To this aim we need

- A dissipative estimate of the form

$$\rho_{\mathcal{W}_b^+}(T(t)w, w_0) \leq \Lambda_0 \left(\rho_{\mathcal{W}_b^+}(w, w_0) \right) e^{-kt} + \Lambda_1, \quad \forall t \geq t_0$$

for every $w \in \mathcal{K}_\Sigma^+$

$\Lambda_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ locally bdd; k, Λ_0, Λ_1 independent of w

- That the ball

$$B_{\mathcal{W}_b^+}(w_0, 2\Lambda_1) := \{w \in \mathcal{W}_b^+ : \rho_{\mathcal{W}_b^+}(w, w_0) \leq 2\Lambda_1\}$$

is compact in Θ_{loc}^+

If, in addition, $\{\mathcal{K}_\sigma^+\}_{\sigma \in \Sigma}$ is (Θ_{loc}^+, Σ) -closed, Σ compact, then $\mathcal{A}_\Sigma \subset \mathcal{K}_\Sigma^+$ and

$$\mathcal{A}_\Sigma = \mathcal{A}_\omega(\Sigma)$$

+further properties of the trajectory attractor



GENERAL SMOOTH POTENTIALS

Banach-metric setting

Set $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ and introduce the space

$$\mathcal{W}_{loc}^+ := \left\{ \mathbf{w} \in L_{loc}^\infty(\mathbb{R}^+; \mathbf{H}_{div} \times H^1(\Omega)^3) \cap L_{loc}^2(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \right. \\ \left. \mathbf{u}_t \in L_{loc}^2(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L_{loc}^2(\mathbb{R}^+; L^{3/2}(\Omega)^3) \right\}$$

endowed with the topology Θ_{loc}^+ of local weak convergence.

In \mathcal{W}_{loc}^+ we consider the following metric subspace

$$\mathcal{W}_b^+ := \left\{ \mathbf{w} \in L^\infty(\mathbb{R}^+; \mathbf{H}_{div} \times H^1(\Omega)^3) \cap L_{tb}^2(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \right. \\ \left. \mathbf{u}_t \in L_{tb}^2(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L_{tb}^2(\mathbb{R}^+; L^{3/2}(\Omega)^3), \right. \\ \left. W(\mathbf{d}) \in L^\infty(\mathbb{R}^+; L^1(\Omega)) \right\}$$

used to define bounded subsets of the space of trajectories \mathcal{K}_Σ^+ .

Definition

For every $\mathbf{h} \in L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div})$ the trajectory space $\mathcal{K}_{\mathbf{h}}^+$ with external force \mathbf{h} is the set of all weak sols $\mathbf{w} = [\mathbf{u}, \mathbf{d}]$ satisfying the energy inequality for all $t \geq s$ and for a.a. $s \in (0, \infty)$

Set $\mathcal{K}_{\Sigma}^+ := \cup_{\mathbf{h} \in \Sigma} \mathcal{K}_{\mathbf{h}}^+$. We have $\mathcal{K}_{\Sigma}^+ \subset \mathcal{W}_b^+$. We take

$$\Sigma = \mathcal{H}_+(\mathbf{h}_0) := \left[\{T(t)\mathbf{h}_0, t \geq 0\} \right]_{L^2_{loc,w}(\mathbb{R}^+; \mathbf{V}'_{div})}$$

\mathbf{h}_0 translation bounded in $L^2_{loc}(\mathbb{R}^+; \mathbf{V}'_{div})$, i.e.

$$\|\mathbf{h}_0\|_{L^2_{tb}(\mathbb{R}^+; \mathbf{V}'_{div})}^2 := \sup_{t \geq 0} \int_t^{t+1} \|\mathbf{h}_0(\tau)\|_{\mathbf{V}'_{div}}^2 d\tau < \infty$$

$\Leftrightarrow \mathbf{h}_0$ translation compact (tr.-c.) in $L^2_{loc,w}(\mathbb{R}^+; \mathbf{V}'_{div})$, i.e., Σ compact in $L^2_{loc,w}(\mathbb{R}^+; \mathbf{V}'_{div})$

Trajectory attractor in 3D-Hom. Neumann b.c. for \mathbf{d}

(A3) W satisfies (A1) and $\exists c_0 \geq 0, c_1 > 0, c_2 \in \mathbb{R}$ and $\delta > 0$ s.t. for all $\mathbf{d} \in \mathbb{R}^3$

$$W_1(\mathbf{d}) \leq c_0(1 + |\nabla_{\mathbf{d}} W_1(\mathbf{d})|^2) \quad W_1(\mathbf{d}) \geq c_1 |\mathbf{d}|^{2+\delta} - c_2$$

(A4) $W(\mathbf{d}) \leq b(1 + |\mathbf{d}|^6) \quad \forall \mathbf{d} \in \mathbb{R}^3 \quad b > 0$

Theorem (S.F. & E. Rocca '12)

Assume (A3) and $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \mathbf{V}_{div})$. Then, the semigroup $\{T(t)\}$ acting on $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ possesses the uniform (w.r.t. $\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)}$. This set is strictly invariant, bounded in \mathcal{W}_b^+ and compact in Θ^+_{loc} .

In addition, if (A4) holds and \mathbf{h}_0 is tr.-c. in $L^2_{loc}(\mathbb{R}^+; \mathbf{V}_{div})$ or $\mathbf{h}_0 \in L^2_{tb}(\mathbb{R}^+; \mathbf{H}_{div})$, then $\mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ is closed in Θ^+_{loc} ,

$\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}^+_{\mathcal{H}_+(\mathbf{h}_0)}$ and

$$\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\mathbf{h}_0))}$$

POLYNOMIAL POTENTIALS

Banach-Banach setting

Set $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ and, for $p \geq 2$, introduce the space

$$\mathcal{W}_{p,loc}^+ := \left\{ \mathbf{w} \in L_{loc}^\infty(\mathbb{R}^+; \mathbf{H}_{div} \times (H^1(\Omega)^3 \cap L^p(\Omega)^3)) \right. \\ \left. \mathbf{w} \in L_{loc}^2(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \right. \\ \left. \mathbf{u}_t \in L_{loc}^2(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L_{loc}^2(\mathbb{R}^+; L^{3/2}(\Omega)^3) \right\}$$

endowed with its inductive limit topology $\Theta_{p,loc}^+$.

Bdd subsets of \mathcal{K}_Σ^+ defined w.r.t the Banach subspace of $\mathcal{W}_{p,loc}^+$

$$\mathcal{W}_{p,b}^+ := \left\{ \mathbf{w} \in L^\infty(\mathbb{R}^+; \mathbf{H}_{div} \times (H^1(\Omega)^3 \cap L^p(\Omega)^3)) \right. \\ \left. \mathbf{w} \in L_{tb}^2(\mathbb{R}^+; \mathbf{V}_{div} \times H^2(\Omega)^3) : \right. \\ \left. \mathbf{u}_t \in L_{tb}^2(\mathbb{R}^+; W^{-1,3/2}(\Omega)^3), \mathbf{d}_t \in L_{tb}^2(\mathbb{R}^+; L^{3/2}(\Omega)^3) \right\}$$

Trajectory attractor in 3D-Hom. Neumann b.c. for \mathbf{d}

(A5) $\exists C_1, C_2 > 0$ and $p \in (2, +\infty)$ s.t.

$$C_1(|\mathbf{d}|^p - 1) \leq W(\mathbf{d}) \leq C_2(1 + |\mathbf{d}|^p), \quad \forall \mathbf{d} \in \mathbb{R}^3$$

$\mathcal{K}_{p,\mathbf{h}}^+$ and $\mathcal{K}_{p,\mathcal{H}_+(\mathbf{h}_0)}^+ := \bigcup_{\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)} \mathcal{K}_{p,\mathbf{h}}^+$ are the trajectory spaces

Theorem (S.F. & E. Rocca '12)

Assume (A3), (A5) and $\mathbf{h}_0 \in L_{tb}^2(\mathbb{R}^+; \mathbf{V}_{div})$. Then, $\{T(t)\}$ acting on $\mathcal{K}_{p,\mathcal{H}_+(\mathbf{h}_0)}^+$ possesses the uniform (w.r.t. $\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)$) trajectory attractor $\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)}$. This set is strictly invariant, bounded in $\mathcal{W}_{p,b}^+$, compact in $\Theta_{p,loc}^+$.

In addition, if \mathbf{h}_0 is tr.-c. in $L_{loc}^2(\mathbb{R}^+; \mathbf{V}_{div})$ or $\mathbf{h}_0 \in L_{tb}^2(\mathbb{R}^+; \mathbf{H}_{div})$, then $\mathcal{K}_{p,\mathcal{H}_+(\mathbf{h}_0)}^+$ is closed in $\Theta_{p,loc}^+$, $\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}_{p,\mathcal{H}_+(\mathbf{h}_0)}^+$ and

$$\mathcal{A}_{p,\mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{p,\omega(\mathcal{H}_+(\mathbf{h}_0))}$$

Trajectory attractor in 3D-Non hom. Dirichlet b.c. for \mathbf{d}

If $\mathbf{g} \in H_{loc}^1(\mathbb{R}^+; H^{-1/2}(\Gamma)^3) \cap L_{loc}^2(\mathbb{R}^+; H^{3/2}(\Gamma)^3)$ (same assumptions for \mathbf{u}_0 , \mathbf{d}_0 and \mathbf{h}), then \exists a weak sol $\mathbf{w} := [\mathbf{u}, \mathbf{d}]$ satisfying

$$\begin{aligned} \mathcal{E}(\mathbf{w}(t)) + \int_s^t \left(\| -\Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d}) \|^2 + \nu \|\nabla \mathbf{u}\|^2 \right) d\tau \\ \leq \mathcal{E}(\mathbf{w}(s)) + \int_s^t \langle \mathbf{g}_t, \partial_n \mathbf{d} \rangle_{H^{-1/2}(\Gamma)^3 \times H^{1/2}(\Gamma)^3} d\tau + \int_s^t \langle \mathbf{h}, \mathbf{u} \rangle d\tau \end{aligned}$$

for all $t \geq s$, for a.e. $s \in (0, \infty)$, including $s = 0$
Symbol space for the Dirichlet datum \mathbf{g}

$$\mathcal{H}_+(\mathbf{g}_0) := [\{T(t)\mathbf{g}_0, t \geq 0\}]_{\Xi_{loc,w}^+}$$

$$\Xi_{loc,w}^+ := \{\mathbf{g} \in C(\mathbb{R}^+; H^{3/2}(\Gamma)^3) : \mathbf{g}_t \in L_{loc,w}^2(\mathbb{R}^+; H^{-1/2}(\Gamma)^3)\}$$

Trajectory attractor in 3D-Non hom. Dirichlet b.c. for \mathbf{d}

Consider, e.g., general smooth potentials. Trajectory spaces

$$\mathcal{K}_{\mathbf{g},\mathbf{h}}^+ \quad \text{and} \quad \mathcal{K}_{\mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0)}^+ := \bigcup_{\mathbf{g} \in \mathcal{H}_+(\mathbf{g}_0), \mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)} \mathcal{K}_{\mathbf{g},\mathbf{h}}^+$$

Theorem (S.F. & E. Rocca '12)

Assume (A3) and that \mathbf{g}_0 is tr.-c. in $C(\mathbb{R}^+; H^{3/2}(\Gamma)^3)$ with $\partial_t \mathbf{g}_0 \in L_{tb}^2(\mathbb{R}^+; H^{-1/2}(\Gamma)^3)$, and $\mathbf{h}_0 \in L_{tb}^2(\mathbb{R}^+; \mathbf{V}_{div})$. Then, $\{T(t)\}$ acting on $\mathcal{K}_{\mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0)}^+$ possesses the uniform (w.r.t.

$[\mathbf{g}, \mathbf{h}] \in \mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0)}$. This set is strictly invariant, bounded in \mathcal{W}_b^+ and compact in Θ_{loc}^+ .

In addition, if (A4) holds, if \mathbf{g}_0 is tr.-c. in Ξ_{loc}^+ and if \mathbf{h}_0 is tr.-c. in $L_{loc}^2(\mathbb{R}^+; \mathbf{V}_{div})$ or $\mathbf{h}_0 \in L_{tb}^2(\mathbb{R}^+; \mathbf{H}_{div})$, then $\mathcal{K}_{\mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0)}^+$ is closed in Θ_{loc}^+ , $\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0)} \subset \mathcal{K}_{\mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0)}^+$ and

$$\mathcal{A}_{\mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\mathbf{g}_0) \times \mathcal{H}_+(\mathbf{h}_0))}$$

Robustness of the trajectory attractor

Open issue: eventual regularization and energy identity for weak sols in 2D \Rightarrow existence of trajectory attractor in 2D for the *strong* topology of \mathcal{W}_{loc}^+ not known
Consider then the problem P_ϵ

$$\begin{aligned} & \mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p \\ &= 2\nu \operatorname{div}(D\mathbf{u}) + \epsilon \operatorname{div}(|D\mathbf{u}|^{q-2} D\mathbf{u}) - \operatorname{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d}) \\ & - \operatorname{div}(\alpha(\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \otimes \mathbf{d} - (1 - \alpha)\mathbf{d} \otimes (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) + \mathbf{h} \\ & \mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} - \alpha \mathbf{d} \cdot \nabla \mathbf{u} + (1 - \alpha)\mathbf{d} \cdot \nabla^T \mathbf{u} = (\Delta \mathbf{d} - \nabla_{\mathbf{d}} W(\mathbf{d})) \\ & \operatorname{div}(\mathbf{u}) = 0 \end{aligned}$$

where $q > 3$

Preliminary results

- *Strong trajectory attractor for P_ϵ*

Every weak sol to P_ϵ satisfies the energy identity

$$\frac{d}{dt} \mathcal{E}(\mathbf{w}(t)) + \| -\Delta \mathbf{d} + \nabla_{\mathbf{d}} W(\mathbf{d}) \|^2 + \nu \|\nabla \mathbf{u}\|^2 + \epsilon \|D\mathbf{u}\|_{L^q}^q = \langle \mathbf{h}, \mathbf{u} \rangle$$

Then, under assumption (A1) and

$$|\partial_{d_i d_j}^2 W(\mathbf{d})| \leq c(1 + |\mathbf{d}|^{4-\sigma}) \quad \forall \mathbf{d} \in \mathbb{R}^3 \quad \forall i, j \quad \sigma > 0$$

Problem P_ϵ admits a (unique) strong trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\mathbf{h}_0)}^\epsilon$ (uniform w.r.t. $\mathbf{h} \in \mathcal{H}_+(\mathbf{h}_0)$) in the space of weak sols $\mathcal{Z}_{q,loc}^+$ with its strong topology $\Theta_{q,loc,s}^+$

- *Convergence of the family of strong trajectory attractors*

Introducing a suitable space $\tilde{\mathcal{Z}}_{q,loc}^+$ of sols of both P_ϵ and P_0 , endowed with its weak topology $\tilde{\Theta}_{q,loc}^+$, so that $\mathcal{A}_{\mathcal{H}_+}^\epsilon, \mathcal{A}_{\mathcal{H}_+} \subset \tilde{\mathcal{Z}}_{q,loc}^+$, we have

$$\mathcal{A}_{\mathcal{H}_+}^\epsilon \rightarrow \mathcal{A}_{\mathcal{H}_+} \quad \text{in } \tilde{\Theta}_{q,loc}^+$$