Some recent results on nonlocal Cahn-Hilliard-Navier-Stokes systems

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Local Cahn-Hilliard-Navier-Stokes systems

In $\Omega \times (0, \infty)$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$

$$
\begin{aligned}
& u_t + (u \cdot \nabla) u - \mathrm{div}(\nu(\varphi) D u) + \nabla \pi = \mu \nabla \varphi + h \\
& \mathrm{div}(u) = 0 \\
& \varphi_t + u \cdot \nabla \varphi = \mathrm{div}\, (m(\varphi) \nabla \mu) \\
& \mu = -\epsilon \Delta \varphi + \epsilon^{-1} F'(\varphi)
\end{aligned}
$$

μ chemical potential, first variation of the free energy

$$
E(\varphi) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} F(\varphi) \right) dx
$$

Rigorous derivation by Gurtin, Polignone and Viñals '96

Local Cahn-Hilliard-Navier-Stokes systems

- F double-well potential
  - Regular, e.g.
$$F(s) = (1 - s^2)^2, \qquad \forall s \in \mathbb{R}$$
  - Singular, e.g.
$$F(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2$$
    for all $s \in (-1, 1)$, with $\theta < \theta_c$
- Mathematical results by V.N. Starovoitov ('97), F. Boyer ('99), Abels '09, Abels and Feireisl '08 (existence of weak and strong solutions, uniqueness and regularity) and by Abels '09, Gal & Grasselli '09 , Zhao, Wu & Huang '09 (convergence to single equilibria), Abels '09, Gal & Grasselli '09, '10 and '11 (attractors).

Nonlocal Cahn-Hilliard-Navier-Stokes systems

- **Nonlocal free energy** (van der Waals)

$$\mathcal{E}(\varphi) = \frac{1}{4}\int_\Omega \int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \eta \int_\Omega F(\varphi(x))dx$$

  where $J : \mathbb{R}^d \to \mathbb{R}$ s.t. $J(x) = J(-x)$
  Local free energy is an approximation of the nonlocal one

- **Nonlocal chemical potential**

$$\mu = a\varphi - J * \varphi + \eta F'(\varphi)$$

  where

$$(J * \varphi)(x) := \int_\Omega J(x-y)\varphi(y)dy \quad a(x) := \int_\Omega J(x-y)dy$$

Nonlocal Cahn-Hilliard-Navier-Stokes systems

Consider in $\Omega \times (0, \infty)$ ($\Omega \subset \mathbb{R}^d$ bounded, $d = 2, 3$)

$$
\begin{aligned}
&\varphi_t + u \cdot \nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu) \\
&\mu = a\varphi - J * \varphi + F'(\varphi) \\
&u_t - \mathrm{div}(\nu(\varphi)Du) + (u \cdot \nabla)u + \nabla\pi = \mu\nabla\varphi + h \\
&\mathrm{div}(u) = 0
\end{aligned}
$$

subject to

$$
\begin{aligned}
&\frac{\partial\mu}{\partial n} = 0 \quad u = 0 \qquad \text{on} \quad \partial\Omega \times (0, \infty) \\
&u(0) = u_0 \quad \varphi(0) = \varphi_0 \qquad \text{in} \quad \Omega
\end{aligned}
$$

First mathematical results

- Existence of dissipative global weak sols in 2D-3D with regular (polynomial growth of arbitrary order) potentials and constant mobility (Colli, F., Grasselli, J. Math. Anal. Appl. '12)
- Asymptotic behavior of weak sols in 2D (global attractor for the associated generalized semiflow) and in 3D (trajectory attractor) with regular potential and constant mobility (F., Grasselli, J. Dynam Differential Equations '12)
- Singular potentials: existence of weak sols in 2D-3D with constant mobility and asymptotic behavior, i.e., global attractor in 2D and trajectory attractor in 3D (F., Grasselli, Dyn. Partial Differ. Equ. '12)

$\exists$ weak sols (regular potentials, constant mobility)

(H1) $J \in W^{1,1}(\mathbb{R}^d)$ s.t. $a(x) = \int_\Omega J(x-y)dy \geq 0$

(H2) $F \in C^2(\mathbb{R})$ and $\exists c_0 > 0$ s.t.

$$F''(s) + a(x) \geq c_0, \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega$$

(H3) $\exists c_1 > 0$, $c_2 > 0$ and $p > 2$ s.t.

$$F''(s) + a(x) \geq c_1|s|^{p-2} - c_2, \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega$$

(H4) $\exists c_3 > 0$, $c_4 \geq 0$ and $r \in (1,2]$ s.t.

$$|F'(s)|^r \leq c_3|F(s)| + c_4, \quad \forall s \in \mathbb{R}$$

(H5) $h \in L^2_{loc}(\mathbb{R}^+; H^1_{div}(\Omega)')$ $\qquad \mathbb{R}^+ := [0,\infty)$

$\exists$ weak sols (regular potentials, constant mobility)

Theorem (Colli, F. & Grasselli '11)

Assume (H1)–(H5). Then, if $u_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ *with* $F(\varphi_0) \in L^1(\Omega)$, *for every* $T > 0$ $\exists$ *a weak sol* $[u, \varphi]$ *on* $[0, T]$ *corresponding to* u_0 *and* φ_0 *s.t.*

$$u \in L^\infty(0, T; L^2_{div}(\Omega)^d) \cap L^2(0, T; H^1_{div}(\Omega)^d)$$
$$\varphi \in L^\infty(0, T; L^p(\Omega)) \cap L^2(0, T; H^1(\Omega))$$
$$u_t \in L^{4/d}(0, T; H^1_{div}(\Omega)')$$
$$\varphi_t \in L^2(0, T; H^1(\Omega)') \quad \text{if} \quad d = 2 \quad \text{or} \quad d = 3 \text{ and } p \geq 3$$
$$\mu \in L^2(0, T; H^1(\Omega))$$

Theorem (Colli, F. & Grasselli '11)

s.t. the energy inequality

$$\mathcal{E}(u(t),\varphi(t)) + \int_s^t (\nu\|\nabla u(\tau)\|^2 + \|\nabla\mu(\tau)\|^2)d\tau$$
$$\leq \mathcal{E}(u(s),\varphi(s)) + \int_s^t \langle h, u(\tau)\rangle d\tau$$

holds for all $t \geq s$ *and for a.a.* $s \in (0,\infty)$, *including* $s = 0$
We have set

$$\mathcal{E}(u(t),\varphi(t)) = \frac{1}{2}\|u(t)\|^2$$
$$+\frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t)-\varphi(y,t))^2 dxdy + \int_\Omega F(\varphi(t))$$

STRONG SOLS IN 2D (reg. pots, const. mob. visc.)

Theorem (F., Grasselli, Krejčí, '13)

Let $h \in L^2_{loc}(\mathbb{R}^+; L^2_{div}(\Omega)^2)$, (H1)-(H4) be satisfied and in addition that $F \in C^3(\mathbb{R})$, $J \in W^{2,1}(\mathbb{R}^2)$. If

$$u_0 \in H^1_{div}(\Omega), \qquad \varphi_0 \in H^2(\Omega),$$

then, for every given $T > 0$, there exists a unique strong solution $[u, \varphi]$ such that

$$u \in L^\infty(0, T; H^1_{div}(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2),$$
$$u_t \in L^2(0, T; L^2_{div}(\Omega)^2),$$
$$\varphi \in L^\infty(0, T; H^2(\Omega)),$$
$$\varphi_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

Moreover, a continuous dependence estimate w.r.t. data u_0, φ_0, h in $L^2_{div}(\Omega)^2 \times H^1(\Omega)' \times L^2_{loc}(\mathbb{R}^+; L^2_{div}(\Omega)^2)$ holds.

Consequences

- Regularity of the global attractor

$$\mathcal{A}_{m_0} \subset \mathcal{B}(\Lambda(m_0)),$$

where $\mathcal{B}(\Lambda(m_0))$ is a ball of radius $\Lambda(m_0)$ in

$$\mathbb{Y}_{m_0} := H^1_{div}(\Omega)^2 \times Y_{m_0}, \quad Y_{m_0} := \{\varphi \in H^2(\Omega) : \; |(\varphi, 1)| \leq m_0\}$$

- Convergence to equilibria of weak sols $z := [u, \varphi]$ with F analytic

$$z(t) \to z_\infty \quad \text{in} \;\; L^2_{div}(\Omega)^2 \times L^2(\Omega), \qquad \text{as} \;\; t \to \infty,$$

where $z_\infty = [0, \varphi_\infty]$ solves

$$a\varphi_\infty - J * \varphi_\infty + F'(\varphi_\infty) = \mu_\infty, \qquad \mu_\infty = \overline{F'(\varphi_\infty)}$$

NONCONSTANT MOBILITY, $\exists$ weak sols

Relevant case: mobility m degenerates at ± 1 and singular double-well potential F on $(-1,1)$ (e.g. logarithmic like). More precisely, we assume (cf. [Elliot, Garcke, '96], [Gajewski, Zacharias, '03], [Giacomin, Lebowitz, '97,'98]

(A1) $m \in C^1([-1,1])$, $m \geq 0$ and $m(s) = 0$ iff $s = \pm 1$, $F \in C^2(-1,1)$ and

$$mF'' \in C([-1,1])$$

(A2) $F = F_1 + F_2$, $F_2 \in C^2([-1,1])$ and there exist $a_2 > 4(a^* - a_* - b_2)$, $b_2 := \min_{[-1,1]} F_2''$, and $\epsilon_0 > 0$ such that

$$F_1''(s) \geq a_2, \qquad \forall s \in (-1, -1+\epsilon_0] \cup [1-\epsilon_0, 1)$$

(A3) There exists $\epsilon_0 > 0$ such that F_1'' is non-decreasing in $[1-\epsilon_0, 1)$ and non-increasing in $(-1, -1+\epsilon_0]$

(A4) There exists $c_0 > 0$ such that

$$F''(s) + a(x) \geq c_0, \qquad \forall s \in (-1,1), \quad \text{a.a. } x \in \Omega$$

Examples of m and F

(A1)–(A4) are satisfied in the following physically relevant case

$$m(s) = k_1(1 - s^2)$$
$$F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\big((1+s)\log(1+s) + (1-s)\log(1-s)\big),$$

$0 < \theta < \theta_c$. Indeed, we have $mF_1'' = k\theta > 0$ and (A4) holds iff $\inf_\Omega a > \theta_c - \theta$.

Another example

$$m(s) = k(s)(1 - s^2)^n, \qquad F(s) = -k_2 s^2 + F_1(s)$$

where $k \in C^1([-1,1])$ such that $0 < k_3 \leq k(s) \leq k_4$ for all $s \in [-1,1]$, and F_1 is a $C^2(-1,1)$ convex function such that

$$F_1''(s) = l(s)(1 - s^2)^{-n}, \qquad \forall s \in (-1,1),$$

where $n \geq 1$ and $l \in C^1([-1,1])$

Notion of weak sol: we are not able to control $\nabla\mu$ in some L^p space; hence we reformulate the definition of weak sol in such a way that μ does not appear any more.
Let $u_0 \in L^2_{div}(\Omega)^d$, $\varphi_0 \in L^2(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $0 < T < +\infty$ be given. A couple $[u, \varphi]$ is a weak solution on $[0, T]$ corresponding to $[u_0, \varphi_0]$ if

- u, φ satisfy

$$
\begin{aligned}
& u \in L^\infty(0, T; L^2_{div}(\Omega)^d) \cap L^2(0, T; H^1_{div}(\Omega)^d), \\
& u_t \in L^{4/3}(0, T; H^1_{div}(\Omega)'), \qquad \text{if} \quad d = 3, \\
& u_t \in L^2(0, T; H^1_{div}(\Omega)'), \qquad \text{if} \quad d = 2, \\
& \varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
& \varphi_t \in L^2(0, T; H^1(\Omega)')
\end{aligned}
$$

and

$$\varphi \in L^\infty(Q_T), \qquad |\varphi(x,t)| \leq 1 \quad \text{a.e. } (x,t) \in Q_T := \Omega \times (0,T)$$

- for every $\psi \in H^1(\Omega)$, every $v \in H^1_{div}(\Omega)^d$ and for almost any $t \in (0,T)$ we have

$$\langle \varphi_t, \psi \rangle + \int_\Omega m(\varphi) F''(\varphi) \nabla\varphi \cdot \nabla\psi + \int_\Omega m(\varphi) a \nabla\varphi \cdot \nabla\psi$$
$$+ \int_\Omega m(\varphi)(\varphi \nabla a - \nabla J * \varphi) \cdot \nabla\psi = (u\varphi, \nabla\psi)$$
$$\langle u_t, v \rangle + \nu(\nabla u, \nabla v) + b(u,u,v) = \big((a\varphi - J * \varphi)\nabla\varphi, v\big) + \langle h, v \rangle$$
$$u(0) = u_0, \qquad \varphi(0) = \varphi_0$$

NONCONSTANT MOBILITY, $\exists$ weak sols

Theorem (F., Grasselli, Rocca, '13)

Let $M \in C^2(-1,1)$ defined by $m(s)M''(s) = 1$,
$M(0) = M'(0) = 0$
Assume that (A1)-(A4) and (H1), (H5) are satisfied. Let

$$u_0 \in L^2_{div}(\Omega), \quad \varphi_0 \in L^\infty(\Omega) \quad F(\varphi_0) \in L^1(\Omega) \quad M(\varphi_0) \in L^1(\Omega).$$

Then, for every $T > 0$ there exists a weak solution $z := [u, \varphi]$ on $[0, T]$ corresponding to $[u_0, \varphi_0]$ such that $\overline{\varphi}(t) = \overline{\varphi_0}$ for all $t \in [0, T]$ and $\varphi \in L^\infty(0, T; L^p(\Omega))$, where $p \leq 6$ for $d = 3$ and $2 \leq p < \infty$ for $d = 2$.

Theorem (F., Grasselli, Rocca, '13)

In addition, if $d = 2$, the weak solution $z := [u, \varphi]$ satisfies the energetic equation

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|\varphi\|^2) + \int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 + \int_\Omega am(\varphi)|\nabla\varphi|^2$$

$$+ \nu\|\nabla u\|^2 = \int_\Omega m(\varphi)(\nabla J * \varphi - \varphi\nabla a) \cdot \nabla\varphi$$

$$+ \int_\Omega (a\varphi - J * \varphi)u \cdot \nabla\varphi + \langle h, u\rangle,$$

for almost any $t > 0$. Furthermore, if $d = 3$, the weak solution z satisfies an energetic inequality

Some developments and open issues

In progress

- *uniqueness of the weak sol in 2D* , regular potentials, constant mobility and viscosity; strong-weak uniqueness with variable viscosity in 2D and exponential attractors in 2D (with Grasselli and Gal)
- nonlocal Ladyzhenskaya-Cahn-Hilliard models
- robustness of the trajectory attractor (w.r.t. the approximating the potential)
- strong trajectory attractor in 2D

Open issues

- unmatched densities
- non-isothermal nonlocal CH-NS model
- compressible models