Adv. Math. Meth. for Engineers, June 27, 2023

Ex 1)

a) The function $f(x,y) = (y-1)\cos x \in C^1(\mathbb{R}\times\mathbb{R})$
$\Rightarrow \forall (x_0, y_0) \in \mathbb{R}^2 \ \exists!$ local solution
$y : \mathcal{U}(x_0) \to \mathbb{R}$ of the Cauchy problem.
Moreover, f is linear in y, with continuous coefficient $\cos x =: g(x) \Rightarrow \exists!$ global solution $y : \mathbb{R} \to \mathbb{R} \ \forall (x_0, y_0) \in \mathbb{R}^2$.

b) The equation can be solved either with the method of separation of variables or with the resolution formula for linear ODES. We get for the ODE : $y - 1 = c\, e^{\operatorname{sen} x}$, $c \in \mathbb{R}$.
If we impose the i.c. $y(0) = -1 \Rightarrow$ we get $c = -2 \Rightarrow$ the solution is
$y(x) = 1 - 2\, e^{\operatorname{sen} x}$, $x \in \mathbb{R}$.

c) The solution is bounded because :
$|y(x)| \leq 1 + 2\,|e^{\operatorname{sen} x}| \leq 1 + 2e \quad \forall x \in \mathbb{R}$

Ex2) The caracteristic equation is

$$\lambda^2 + 1 = 0$$

for the homogeneous equation $\Rightarrow$

$$\lambda_{1,2} = \pm i \Rightarrow y(x) = c_1 \cos x + c_2 \operatorname{sen} x,$$

$c_1, c_2 \in \mathbb{R}$.

We search now for a particular solution of the form $y_p(x) = A \cos 2x + B \operatorname{sen} 2x$, $A, B \in \mathbb{R}$. Computing y_p' and y_p'' and substituting into the ODE, we get

$$A = 0, B = -\frac{1}{3} \Rightarrow y_p(x) = -\frac{1}{3} \operatorname{sen}(2x)$$

$$\Rightarrow y(x) = c_1 \cos x + c_2 \operatorname{sen} x - \frac{1}{3} \operatorname{sen} 2x,$$

$c_1, c_2 \in \mathbb{R}$

Imposing the condition $y(0) = 0$, we get $c_1 = 0 \Rightarrow \bar{y}(x) = c_2 \operatorname{sen} x - \frac{1}{3} \operatorname{sen} 2x$, $c_2 \in \mathbb{R}$.

All solutions are bounded because we have:

$$|y(x)| \leq |c_1 \cos x| + |c_2 \operatorname{sen} x| + \left|-\frac{1}{3} \operatorname{sen}(2x)\right|$$

$$\leq |c_1| + |c_2| + \frac{1}{3}.$$

Ex3) $f_n(x) \to \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x>0. \end{cases}$

Moreover, $\forall x>0$ and $\forall n \in \mathbb{N}$ we have

$$|f_n(x)| \le \frac{e^{-x}}{1+nx} \le e^{-x} \in L^1(0,+\infty) \Rightarrow$$

by the Lebesgue dominated conv. thm.
we have that

$$\lim_{n\to\infty} \int_0^{+\infty} f_n(x)\,dx = 0.$$

Ex 4)

b) $$\delta_h * f = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(t)\,dt =$$

$$= \frac{1}{h}\int_{x-h/2}^{x+h/2} (at+b)\,dt =$$

$$= \frac{axh+bh}{h} = (ax+b) = f(x)$$

c) $$\delta_h * f = \frac{1}{h}\int_{x-h/2}^{x+h/2} (a\,q(x)+p_1(x))\,dx$$

where $p_1(x) = bx+c$ and $q(x) = x^2$

$$\Rightarrow \int_{x-h/2}^{x+h/2} t^2\,dt = x^2 + \frac{h^2}{12} \Rightarrow$$

we get the result applying point b).