



Weierstrass Institute for  
Applied Analysis and Stochastics



# Recent results on optimal control for Cahn-Hilliard/Navier-Stokes systems with nonlocal interactions

Sergio Frigeri

ERC Group “Entropy Formulation of Evolutionary Phase Transitions”

Supported by the FP7-IDEAS-ERC-StG Grant “EntroPhase”

- An isothermal model for the flow of a **mixture of two**
  - viscous
  - incompressible
  - Newtonian fluids
  - of equal density
- Avoid problems related to interface singularities
  - ⇒ use a **diffuse interface model**
  - ⇒ the classical sharp interface replaced by a **thin interfacial region**
- A partial mixing of the macroscopically immiscible fluids is allowed
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- The original idea of diffuse interface model for fluids: HOHENBERG and HALPERIN, '77
  - ⇒ **H-model**
  - Later, GURTIN ET AL., '96: continuum mechanical derivation based on microforces
  
- Models of two-phase or two-component fluids are receiving growing attention (e.g., ABELS, BOYER, GARCKE, GRÜN, GRASSELLI, LOWENGRUB, TRUSKINOVSKI, ...)

In  $\Omega \times (0, \infty)$ ,  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{v}$$

$$\operatorname{div}(\mathbf{u}) = 0$$

$$\varphi_t + \mathbf{u} \cdot \nabla \varphi = \operatorname{div}(m(\varphi) \nabla \mu)$$

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- $\mu$ : **chemical potential** (Cahn-Hilliard), first variation of the (total Helmholtz) free energy

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- $F$  double-well potential: Helmholtz free energy density

- Singular

$$F(s) = -\frac{\theta_c}{2} s^2 + \frac{\theta}{2} \left( (1+s) \log(1+s) + (1-s) \log(1-s) \right)$$

for all  $s \in (-1, 1)$ , with  $0 < \theta < \theta_c$

- Regular

$$F(s) = (1 - s^2)^2 \quad \forall s \in \mathbb{R}$$

- **Nonlocal free energy** rigorously justified by Giacomini and Lebowitz ('97 & '98) as macroscopic limit of microscopic phase segregation models

$$E(\varphi) = \frac{1}{4} \int_{\Omega} \int_{\Omega} J(x-y)(\varphi(x) - \varphi(y))^2 dx dy + \int_{\Omega} F(\varphi(x)) dx$$

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- First analytical results on nonlocal CH: Giacomini & Lebowitz '97 and '98; Gajewski '02; Gajewski & Zacharias '03
- Several other contributions on nonlocal Allen-Cahn equations and phase-field systems (notably by Bates et al. and Sprekels et al.)

$$\varphi_t + \mathbf{u} \cdot \nabla \varphi = \operatorname{div} (m(\varphi) \nabla \mu)$$

$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

$$\mathbf{u}_t - 2\operatorname{div}(\nu(\varphi) D\mathbf{u}) + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{v}$$

$$\operatorname{div}(\mathbf{u}) = 0$$

subject to

$$\frac{\partial \mu}{\partial n} = 0 \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0, \infty)$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega$$

- Mass is conserved

$$\overline{\varphi(t)} := |\Omega|^{-1} \int_{\Omega} \varphi(x, t) dx = \overline{\varphi}_0$$

### ■ Constant mobility+ regular potential

- $\exists$  **global weak sols in 2D-3D** (Colli, F. & Grasselli, J. Math. Anal. Appl. '12)
- global attractor in 2D and trajectory attractor in 3D (F. & Grasselli, J. Dynam Differential Equations '12)

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### ■ Degenerate mobility+ singular potential

- $\exists$  and regularity of global weak sols in 2D-3D, global attractor in 2D (F., Grasselli & Rocca, Nonlinearity '15)



### More recent results

- **Constant mobility+ regular or singular potential & degenerate mobility + singular potential**
  - **Uniqueness of global weak sols in 2D**
  
- **Constant mobility, nonconstant viscosity +regular potential**
  - $\exists$  global unique strong sols in 2D, regularity of global attractor in 2D, convergence to equilibria of weak sols in 2D
  - weak-strong uniqueness in 2D
  - Connectedness and regularity of global attractor,  $\exists$  exponential attractor in 2D.

Last results in: F., Gal & Grasselli, WIAS Preprint '14

**Theorem (Colli, F. & Grasselli '12)**

Assume  $J \in W^{1,1}(\mathbb{R}^d)$  and that  $\mathbf{v} \in L^2(0, T; H_{div}^1(\Omega)'), \mathbf{u}_0 \in L_{div}^2(\Omega)^d, \varphi_0 \in L^2(\Omega)$  with  $F(\varphi_0) \in L^1(\Omega)$ . Then,  $\forall T > 0 \exists$  a weak sol  $[\mathbf{u}, \varphi]$  on  $[0, T]$  s.t.

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, T; L_{div}^2(\Omega)^d) \cap L^2(0, T; H_{div}^1(\Omega)^d), & \mathbf{u}_t &\in L^{4/d}(0, T; H_{div}^1(\Omega)') \\ \varphi &\in L^\infty(0, T; L^4(\Omega)) \cap L^2(0, T; H^1(\Omega)), & \varphi_t &\in L^2(0, T; H^1(\Omega)') \\ \mu &\in L^2(0, T; H^1(\Omega)) \end{aligned}$$



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which satisfies the energy inequality (identity if  $d = 2$ )

$$\mathcal{E}(\mathbf{u}(t), \varphi(t)) + \int_0^t (\nu \|\nabla \mathbf{u}(\tau)\|^2 + \|\nabla \mu(\tau)\|^2) d\tau \leq \mathcal{E}(\mathbf{u}_0, \varphi_0) + \int_0^t \langle \mathbf{v}(\tau), \mathbf{u}(\tau) \rangle d\tau$$

for all  $t > 0$ , where we have set

$$\mathcal{E}(\mathbf{u}(t), \varphi(t)) = \frac{1}{2} \|\mathbf{u}(t)\|^2 + \frac{1}{4} \int_\Omega \int_\Omega J(x-y) (\varphi(x, t) - \varphi(y, t))^2 dx dy + \int_\Omega F(\varphi(t))$$

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- **The nonlocal term implies that  $\varphi$  is not as regular as for the standard (local) CHNS system:**  $\varphi \in L^2(H^1)$  (nonlocal), instead of  $\varphi \in L^\infty(H^1)$  (local)  $\implies$  regularity results and uniqueness of weak sols in 2D difficult issues

- We need stronger assumptions on  $J$ . In particular  $J \in W^{2,1}(\mathbb{R}^2)$  or  $J$  **admissible**

### Definition (J. Bedrossian, N. Rodríguez & A. Bertozzi '11)

A kernel  $J \in W_{loc}^{1,1}(\mathbb{R}^2)$  is admissible if the following conditions are satisfied:

- (A1)  $J \in C^3(\mathbb{R}^d \setminus \{0\})$ ;
- (A2)  $J$  is radially symmetric,  $J(x) = \tilde{J}(|x|)$  and  $\tilde{J}$  is non-increasing;
- (A3)  $\tilde{J}''(r)$  and  $\tilde{J}'(r)/r$  are monotone on  $(0, r_0)$  for some  $r_0 > 0$ ;
- (A4)  $|D^3 J(x)| \leq C_d |x|^{-d-1}$  for some  $C_d > 0$

**Newtonian and Bessel kernels are admissible for all  $d \geq 2$**

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**Newtonian and Bessel kernels are admissible for all  $d \geq 2$**

### Lemma (J. Bedrossian, N. Rodríguez & A. Bertozzi '11)

Let  $J$  be admissible and  $\chi = \nabla J * \psi$ . Then, for all  $p \in (1, \infty)$ , there exists  $C_p > 0$  such that

$$\|\nabla \chi\|_{L^p(\Omega)} \leq C_p \|\psi\|_{L^p(\Omega)}$$

## Theorem (F., Grasselli &amp; Krejčí '13)

Assume that  $J \in W^{2,1}(\mathbb{R}^2)$  or  $J$  admissible and that

$$\mathbf{v} \in L^2(0, T; L^2_{div}(\Omega)^2) \quad \mathbf{u}_0 \in H^1_{div}(\Omega)^2 \quad \varphi_0 \in H^2(\Omega)$$

Then,  $\forall T > 0 \exists$  **unique** strong sol  $[\mathbf{u}, \varphi]$  on  $[0, T]$  s.t.

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, T; H^1_{div}(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2), & \mathbf{u}_t &\in L^2(0, T; L^2_{div}(\Omega)^2) \\ \varphi &\in L^\infty(0, T; H^2(\Omega)), & \varphi_t &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \end{aligned}$$

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Only recently (F., Gal & Grasselli, WIAS Preprint '14) we included

■ **Nonconstant viscosity**

$$\nu = \nu(\varphi), \quad \nu \text{ loc. Lipschitz on } \mathbb{R}, \quad 0 < \nu_1 \leq \nu(\varphi) \leq \nu_2$$

**How to handle with nonconstant viscosity to get regularity results?**

- We cannot rely on NS regularity in 2D to get  $\mathbf{u} \in L^2(0, T; H^2(\Omega)^2)$ . Indeed

$$\varphi \text{ weak sol, } \quad \mathbf{u} \in H^2(\Omega)^2 \cap H_{div}^1(\Omega)^2 \implies \operatorname{div}(\nu(\varphi)D\mathbf{u}) \in L^{2-\epsilon}(\Omega)^2$$

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- Approach: (nonloc CH)  $\times \mu_t$  and avoid the use of the  $H^2$  – norm of  $\mathbf{u}$ . We deduce

$$\begin{aligned} \frac{d}{dt} \|\nabla \mu\|^2 + c_0 \|\varphi_t\|^2 &\leq Q(R) (\|\mathbf{u}\|^2 \|\nabla \mathbf{u}\|^2) \|\nabla \varphi\|^2 + c \|\mathbf{u}\|^2 \|\nabla \mathbf{u}\|^2 \|\nabla \varphi\|^2 + Q(R) \\ &+ c (\|\nabla a\|_{L^\infty(\Omega)}^2 + Q(R)) \|\nabla \varphi\|^2 + Q(R) \sum_{i,j=1}^2 \|\partial_{ij}^2 a\|^2 \\ &+ c \sum_{i,j=1}^2 \|\partial_i(\partial_j J * \varphi)\|^2 + c \|J\|_{W^{1,1}(\mathbb{R}^2)}^2 \|\varphi_t\|_{H^1(\Omega)}^2, \quad \|\varphi\|_{L^\infty(Q)} \leq R \end{aligned}$$



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$$+ c (\|\nabla a\|_{L^\infty(\Omega)}^2 + Q(R)) \|\nabla \varphi\|^2 + Q(R) \sum_{i,j=1}^2 \|\partial_{ij}^2 a\|^2$$

$$+ c \sum_{i,j=1}^2 \|\partial_i(\partial_j J * \varphi)\|^2 + c \|J\|_{W^{1,1}(\mathbb{R}^2)}^2 \|\varphi_t\|_{H^1(\Omega)}^2, \quad \|\varphi\|_{L^\infty(Q)} \leq R$$

$$\implies \varphi \in L^\infty(0, T; H^1(\Omega)), \quad \varphi_t \in L^2(0, T; L^2(\Omega)), \quad \mu \in L^\infty(0, T; H^1(\Omega))$$

- Second step:  $(NS) \times \mathbf{u}_t$ , integrate by parts in time to get

$$\frac{1}{2} \|\mathbf{u}_t\|^2 + \frac{d}{dt} \int_{\Omega} \nu(\varphi) |D\mathbf{u}|^2 + b(\mathbf{u}, \mathbf{u}, \mathbf{u}_t) \leq \frac{1}{2} \|l\|^2 + \int_{\Omega} |D\mathbf{u}|^2 \nu'(\varphi) \varphi_t$$

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$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \nu(\varphi) |D\mathbf{u}|^2 + \frac{1}{8} \|\mathbf{u}_t\|^2 \\ & \leq Q(R, \|\varphi_0\|_V, \|\mathbf{u}_0\|) \left( \|l\|^2 + (\|\mathbf{u}\|^2 + \|\mathbf{u}\|^{p-2}) \|\nabla \mathbf{u}\|^2 \right) \|D\mathbf{u}\|^2 \\ & + \|\varphi_t\|^2 \|D\mathbf{u}\|^2 + \|\nabla \mathbf{u}\|^2 \quad 2 < p < \infty \end{aligned}$$

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Exploiting the regularity obtained at previous step

$$\implies \mathbf{u} \in L^\infty(0, T; H_{div}^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2) \quad \mathbf{u}_t \in L^2(0, T; L_{div}^2(\Omega)^2)$$

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and then also

$$\varphi_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \quad \varphi \in L^\infty(0, T; H^2(\Omega))$$

**Relevant case:** mobility  $m$  degenerates at  $\pm 1$  and singular double-well potential  $F$  on  $(-1, 1)$  (e.g. logarithmic like).

- $\varphi$ -dependent mobility in the original derivation of CH eq. (J.W. Cahn & J.E. Hilliard, 1971). Thermodynamically reasonable choice

$$m(\varphi) = k(1 - \varphi^2)$$

- Key assumption (cf. [Elliot & Garcke '96], [Gajewski & Zacharias '03], [Giacomin & Lebowitz '97,'98])

$$mF'' \in C([-1, 1])$$

**Relevant case:** mobility  $m$  degenerates at  $\pm 1$  and singular double-well potential  $F$  on  $(-1, 1)$  (e.g. logarithmic like).

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- **local CH eq.** : Elliot & Garcke '96 ( $\exists$ ), Schimperna & Zelik '13 ( $\exists$ , asymptotic behavior, separation)
- **nonlocal CH eq.**: Giacomin & Lebowitz '97,'98, Gajewski & Zacharias '03 ( $\exists$  and uniqueness), Londen & Petzeltová '11, '11 (conv. to eq., separation)
- **local CHNS**: Boyer '99 ( $\exists$ ), Abels, Depner & Garcke '13 (unmatched densities,  $\exists$ )

We are not able to control  $\nabla\mu$  in some  $L^p$  space; hence we reformulate the definition of weak sol in such a way that  $\mu$  does not appear any more.

### Notion of weak sol

Let  $\mathbf{u}_0 \in L^2_{div}(\Omega)^d$ ,  $\varphi_0 \in L^2(\Omega)$  with  $F(\varphi_0) \in L^1(\Omega)$  and  $0 < T < +\infty$  be given.  $[\mathbf{u}, \varphi]$  is a weak sol on  $[0, T]$  if

■  $\mathbf{u}, \varphi$  satisfy

$$\begin{aligned} \mathbf{u} &\in L^\infty(0, T; L^2_{div}(\Omega)^d) \cap L^2(0, T; H^1_{div}(\Omega)^d), & \mathbf{u}_t &\in L^{4/d}(0, T; H^1_{div}(\Omega)') \\ \varphi &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), & \varphi_t &\in L^2(0, T; H^1(\Omega)') \end{aligned}$$

and

$$\varphi \in L^\infty(Q_T) \quad |\varphi(x, t)| \leq 1 \quad \text{a.e. } (x, t) \in Q_T := \Omega \times (0, T)$$

■  $\forall \psi \in H^1(\Omega), \forall \mathbf{v} \in H^1_{div}(\Omega)^d$  and for a.e.  $t \in (0, T)$

$$\langle \varphi_t, \psi \rangle + \int_\Omega (m(\varphi)F''(\varphi) + m(\varphi)a) \nabla\varphi \cdot \nabla\psi + \int_\Omega m(\varphi)(\varphi\nabla a - \nabla J * \varphi) \cdot \nabla\psi = \langle \mathbf{u}\varphi, \nabla\psi \rangle$$

$$\langle \mathbf{u}_t, \mathbf{v} \rangle + \nu \langle \nabla\mathbf{u}, \nabla\mathbf{v} \rangle + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle (a\varphi - J * \varphi)\nabla\varphi, \mathbf{v} \rangle + \langle \mathbf{h}, \mathbf{v} \rangle$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \varphi(0) = \varphi_0$$



**Theorem (F., Grasselli & Rocca '15)**

Let  $M \in C^2(-1, 1)$  s.t.  $m(s)M''(s) = 1$ ,  $M(0) = M'(0) = 0$ . Let

$$\mathbf{u}_0 \in L^2_{div}(\Omega)^d, \quad \varphi_0 \in L^\infty(\Omega), \quad F(\varphi_0) \in L^1(\Omega), \quad M(\varphi_0) \in L^1(\Omega)$$

Then,  $\forall T > 0 \exists$  a weak sol  $z := [\mathbf{u}, \varphi]$  on  $[0, T]$  s.t.  $\overline{\varphi(t)} = \overline{\varphi_0} \forall t \in [0, T]$ . In addition,  $z$  satisfies the *energetic inequality* (identity if  $d = 2$ )

$$\begin{aligned} & \frac{1}{2} (\|\mathbf{u}(t)\|^2 + \|\varphi(t)\|^2) + \int_0^t \int_\Omega (m(\varphi)F''(\varphi) + am(\varphi)) |\nabla\varphi|^2 + \nu \int_0^t \|\nabla\mathbf{u}\|^2 \leq \frac{1}{2} \|\mathbf{u}_0\|^2 \\ & + \frac{1}{2} \|\varphi_0\|^2 + \int_0^t \int_\Omega (a\varphi - J * \varphi) \mathbf{u} \cdot \nabla\varphi + \int_0^t \int_\Omega m(\varphi) (\nabla J * \varphi - \varphi \nabla a) \cdot \nabla\varphi + \int_0^t \langle \mathbf{h}, \mathbf{u} \rangle \end{aligned}$$

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**Remark: a comparison with the constant mobility case**

- Condition  $|\overline{\varphi_0}| < 1$  not required (different weak sol formulation w.r.t. the case of const. mob.)
- Therefore, if  $F$  is bdd (e.g.  $F$  is the log pot) and at  $t = 0$  the fluid is in a pure phase, i.e.  $\varphi_0 = 1$  a.e. in  $\Omega$ , and furthermore  $\mathbf{u}_0 = \mathbf{u}(0)$  is given in  $L^2(\Omega)^d_{div}$ , then the couple  $\mathbf{u} = \mathbf{u}(x, t)$ ,  $\varphi = \varphi(x, t) = 1$  a.e. in  $\Omega$ , a.a.  $t$ , where  $\mathbf{u}$  is a sol of NS with non-slip b.c. explicitly satisfies the weak formulation.

## Theorem (F., Grasselli &amp; Rocca '15)

Let  $\varphi_0$  be s.t.

$$F'(\varphi_0) \in L^2(\Omega)$$

Then,  $\exists$  weak sol  $z = [\mathbf{u}, \varphi]$  that also satisfies

$$\mu \in L^\infty(0, T; L^2(\Omega)) \quad \nabla \mu \in L^2(0, T; L^2(\Omega)^d)$$

As a consequence,  $z = [\mathbf{u}, \varphi]$  also satisfies the weak formulation and the energy inequality (identity for  $d = 2$ ) of the non degenerate mobility case

$$\mathcal{E}(\mathbf{u}(t), \varphi(t)) + \int_0^t (\nu \|\nabla \mathbf{u}(\tau)\|^2 + \|\sqrt{m(\varphi)} \nabla \mu(\tau)\|^2) d\tau \leq \mathcal{E}(\mathbf{u}_0, \varphi_0) + \int_0^t \langle \mathbf{v}(\tau), \mathbf{u}(\tau) \rangle d\tau$$

for all  $t > 0$ , where we have set

$$\mathcal{E}(\mathbf{u}(t), \varphi(t)) = \frac{1}{2} \|\mathbf{u}(t)\|^2 + \frac{1}{4} \int_\Omega \int_\Omega J(x-y) (\varphi(x, t) - \varphi(y, t))^2 dx dy + \int_\Omega F(\varphi(t))$$

### Constant mobility + regular potentials

#### Theorem (F., Gal & Grasselli '14)

Let  $\mathbf{u}_0 \in L^2_{div}(\Omega)^2$ ,  $\varphi_0 \in L^2(\Omega)$  with  $F(\varphi_0) \in L^1(\Omega)$ . Then,  $\exists$  a **unique** weak sol  $[\mathbf{u}, \varphi]$  corresponding to  $[\mathbf{u}_0, \varphi_0]$

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- **Uniqueness of sol and  $\exists$  of the global attractor for the local CH with degenerate mobility are open issues**



## Consequences

- the nonlocal CHNS system generates a **semigroup**  $S(t)$  of *closed* operators:

$[\mathbf{u}(t), \varphi(t)] = S(t)[\mathbf{u}_0, \varphi_0]$  on the (metric) phase-space

$$\mathcal{X}_\eta = L^2_{div}(\Omega)^2 \times \mathcal{Y}_\eta \quad \mathcal{Y}_\eta = \{\varphi \in L^2(\Omega) : F(\varphi) \in L^1(\Omega), |\bar{\varphi}| \leq \eta\}$$

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- Smoothing property** for the difference of two sols in  $L_{div}^2(\Omega)^2 \times L^2(\Omega)$

## Theorem (F., Gal &amp; Grasselli '14)

For every  $\eta \geq 0$  the dynamical system  $(\mathcal{X}_\eta, S(t))$  possesses an **exponential attractor**  $\mathcal{M}_\eta$ , i.e., a compact set in  $\mathcal{X}_\eta$  s.t.

- (i) *Positively invariance*:  $S(t)\mathcal{M} \subset \mathcal{M} \forall t \geq 0$
- (ii) *Finite dimensionality*:  $\dim_F \mathcal{M} < \infty$
- (iii) *Exponential attraction*:  $\exists J : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  increasing and  $\kappa > 0$  s.t.,  $\forall R > 0$  and  $\forall \mathcal{B} \subset \mathcal{X}_\eta$  with  $\sup_{z \in \mathcal{B}} \mathbf{d}_{\mathcal{X}_\eta}(z, 0) \leq R$  there holds

$$\text{dist}(S(t)\mathcal{B}, \mathcal{M}) \leq J(R)e^{-\kappa t}$$

## Constant mobility+regular potential

Problem **(CP)**: minimize the **cost functional**

$$J(y, \mathbf{v}) := \frac{\beta_1}{2} \|\mathbf{u} - \mathbf{u}_Q\|_{L^2(Q)^2}^2 + \frac{\beta_2}{2} \|\varphi - \varphi_Q\|_{L^2(Q)}^2 + \frac{\beta_3}{2} \|\mathbf{u}(T) - \mathbf{u}_\Omega\|^2 \\ + \frac{\beta_4}{2} \|\varphi(T) - \varphi_\Omega\|^2 + \frac{\gamma}{2} \|\mathbf{v}\|_{L^2(Q)^2}^2$$

where  $y := [\mathbf{u}, \varphi]$  solves

$$\begin{aligned} \mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi &= \mu \nabla \varphi + \mathbf{v} \\ \varphi_t + \mathbf{u} \cdot \nabla \varphi &= \Delta \mu \\ \mu &= a\varphi - J * \varphi + F'(\varphi) \\ \operatorname{div}(\mathbf{u}) &= 0 \\ \partial_n \mu = 0 \quad \mathbf{u} = 0 &\quad \text{on } \partial\Omega \\ \mathbf{u}(0) = \mathbf{u}_0 \quad \varphi(0) = \varphi_0 & \end{aligned}$$

**(nlocCHNS)**

and the external body force density  $\mathbf{v}$ , which plays the role of the **control**, belongs to a suitable closed, bounded and convex subset of the **space of controls**

$$\mathcal{V} := L^2(0, T; L_{div}^2(\Omega)^2)$$

- Introducing the space

$$\mathcal{H} := [L^\infty(0, T; H_{div}^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2)] \times L^\infty(0, T; H^2(\Omega))$$

then, the **control-to-state map**

$$S : \mathcal{V} \rightarrow \mathcal{H}, \quad \mathbf{v} \in \mathcal{V} \mapsto S(\mathbf{v}) := y := [\mathbf{u}, \varphi] \in \mathcal{H}$$

where  $y := [\mathbf{u}, \varphi]$  is the unique strong sol to Problem **(nloc CHNS)** corresponding to  $\mathbf{v} \in \mathcal{V}$  and to fixed initial data  $\mathbf{u}_0 \in H_{div}^1(\Omega)^2$ ,  $\varphi_0 \in H^2(\Omega)$ , is well defined

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$$\mathcal{V}_{ad} := \{ \mathbf{v} \in \mathcal{V} : v_{a,i}(x, t) \leq v_i(x, t) \leq v_{b,i}(x, t), \text{ a.e. } (x, t) \in Q, \quad i = 1, 2 \}$$

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- Introducing the **reduced cost functional**  $f(\mathbf{v}) := J(S(\mathbf{v}), \mathbf{v})$ , for all  $\mathbf{v} \in \mathcal{V}$ , then

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## Theorem

Problem **(CP)** admits a sol  $\bar{\mathbf{v}} \in \mathcal{V}_{ad}$ , with associated state  $\bar{y} := [\bar{\mathbf{u}}, \bar{\varphi}] := S(\bar{\mathbf{v}})$



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$$\xi_t - \nu \Delta \xi + (\bar{\mathbf{u}} \cdot \nabla) \xi + (\xi \cdot \nabla) \bar{\mathbf{u}} + \nabla \tilde{\pi} = (a\eta - J * \eta + F''(\bar{\varphi})\eta) \nabla \bar{\varphi} + \bar{\mu} \nabla \eta + \mathbf{h}$$

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$$\operatorname{div}(\xi) = 0$$

$$\xi = 0, \quad \frac{\partial}{\partial \mathbf{n}} (a\eta - J * \eta + F''(\bar{\varphi})\eta) = 0 \quad \text{on } \Sigma := \partial\Omega \times (0, T)$$

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For every  $\mathbf{h} \in \mathcal{V}$  the linearized problem above has a unique sol satisfying

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**Remark.** States  $\bar{y} = [\bar{\mathbf{u}}, \bar{\varphi}]$  need to be **strong sols** to (nloc CHNS)

**Differentiability of the control-to-state operator.** Set

$$\mathcal{Z} := [C([0, T]; L^2_{div}(\Omega)^2) \cap L^2(0, T; H^1_{div}(\Omega)^2)] \times [C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))]$$

### Theorem

The control-to-state operator  $S : \mathcal{V} \rightarrow \mathcal{Z}$  is Frechét differentiable on  $\mathcal{V}$  and the Frechét derivative  $S'(\bar{\mathbf{v}}) \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$  is given by

$$S'(\bar{\mathbf{v}})\mathbf{k} = [\boldsymbol{\xi}^k, \eta^k], \quad \forall \mathbf{k} \in \mathcal{V},$$

where  $[\boldsymbol{\xi}^k, \eta^k]$  is the unique sol to the linearized system at  $[\bar{\mathbf{u}}, \bar{\varphi}] = S(\bar{\mathbf{v}})$  and corresponding to  $\mathbf{k} \in \mathcal{V}$

Key tool for the proof: **stability estimates**

**Lemma (Stability estimate I — F., Gal & Grasselli '14)**

Let  $\mathbf{u}_{0i} := \mathbf{u}_i(0) \in H_{div}^1(\Omega)^2$ ,  $\varphi_{0i} := \varphi_i(0) \in H^2(\Omega)$ ,  $\mathbf{v}_i \in L^2(0, T; L_{div}^2(\Omega)^2)$  and let  $[\mathbf{u}_i, \varphi_i]$  be the corresponding (unique) strong sols,  $i = 1, 2$ . Then, we have

$$\begin{aligned} & \|\mathbf{u}_2 - \mathbf{u}_1\|_{L^\infty(0, T; L_{div}^2(\Omega)^2)}^2 + \|\mathbf{u}_2 - \mathbf{u}_1\|_{L^2(0, T; H_{div}^1(\Omega)^2)}^2 + \|\varphi_2 - \varphi_1\|_{L^\infty(0, T; L^2(\Omega))}^2 \\ & + \|\varphi_2 - \varphi_1\|_{L^2(0, T; H^1(\Omega))}^2 \leq \Lambda_1 (\|\mathbf{u}_{20} - \mathbf{u}_{10}\|^2 + \|\varphi_{20} - \varphi_{10}\|^2 + \|\mathbf{v}_2 - \mathbf{v}_1\|_{\mathcal{V}}^2) \end{aligned}$$

where

$$\Lambda_1 = \Lambda_1 (\|\nabla \mathbf{u}_{01}\|, \|\varphi_{01}\|_{H^2(\Omega)}, \|\mathbf{v}_1\|_{\mathcal{V}}, \|\nabla \mathbf{u}_{02}\|, \|\varphi_{02}\|_{H^2(\Omega)}, \|\mathbf{v}_2\|_{\mathcal{V}})$$

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**Remak.** To prove Frechét differentiability of  $S : \mathcal{V} \rightarrow \mathcal{Z}$  we need an **improved** stability estimate



Key tool for the proof: **stability estimates**

### Lemma (Stability estimate II)

Let  $\mathbf{u}_{0i} := \mathbf{u}_i(0) \in H_{div}^1(\Omega)^2$ ,  $\varphi_{0i} := \varphi_i(0) \in H^2(\Omega)$ ,  $\mathbf{v}_i \in L^2(0, T; L_{div}^2(\Omega)^2)$  and let  $[\mathbf{u}_i, \varphi_i]$  be the corresponding (unique) strong sols,  $i = 1, 2$ . Then, we have

$$\begin{aligned} & \|\mathbf{u}_2 - \mathbf{u}_1\|_{L^\infty(0, T; L_{div}^2(\Omega)^2)}^2 + \|\mathbf{u}_2 - \mathbf{u}_1\|_{L^2(0, T; H_{div}^1(\Omega)^2)}^2 + \|\varphi_2 - \varphi_1\|_{L^\infty(0, T; H^1(\Omega))}^2 \\ & + \|\varphi_2 - \varphi_1\|_{L^2(0, T; H^2(\Omega))}^2 \leq \Lambda_2 (\|\mathbf{u}_{20} - \mathbf{u}_{10}\|^2 + \|\varphi_{20} - \varphi_{10}\|_{H^1(\Omega)}^2 + \|\mathbf{v}_2 - \mathbf{v}_1\|_{\mathcal{V}}^2) \end{aligned}$$

where

$$\Lambda_2 = \Lambda_2 (\|\nabla \mathbf{u}_{01}\|, \|\varphi_{01}\|_{H^2(\Omega)}, \|\mathbf{v}_1\|_{\mathcal{V}}, \|\nabla \mathbf{u}_{02}\|, \|\varphi_{02}\|_{H^2(\Omega)}, \|\mathbf{v}_2\|_{\mathcal{V}})$$

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$$\Lambda_2 = \Lambda_2 (\|\nabla \mathbf{u}_{01}\|, \|\varphi_{01}\|_{H^2(\Omega)}, \|\mathbf{v}_1\|_{\mathcal{V}}, \|\nabla \mathbf{u}_{02}\|, \|\varphi_{02}\|_{H^2(\Omega)}, \|\mathbf{v}_2\|_{\mathcal{V}})$$

**Sketch of the proof of differentiability of  $S : \mathcal{V} \rightarrow \mathcal{Z}$ .** Let  $\bar{\mathbf{v}} \in \mathcal{V}$  be fixed,  $\bar{\mathbf{y}} := [\bar{\mathbf{u}}, \bar{\varphi}] = S(\bar{\mathbf{v}})$ , and consider a perturbation  $\mathbf{h} \in \mathcal{V}$ . Set

$$\mathbf{y}^h := [\mathbf{u}^h, \varphi^h] := S(\bar{\mathbf{v}} + \mathbf{h})$$

$$\mathbf{p}^h := \mathbf{u}^h - \bar{\mathbf{u}} - \xi^h, \quad \mathbf{q}^h := \varphi^h - \bar{\varphi} - \eta^h$$

■ Then,  $\mathbf{p}^h, q^h$  solve

$$\begin{aligned}
 & \mathbf{p}_t - \nu \Delta \mathbf{p} + (\mathbf{p} \cdot \nabla) \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{p} + ((\mathbf{u}^h - \bar{\mathbf{u}}) \cdot \nabla)(\mathbf{u}^h - \bar{\mathbf{u}}) + \nabla \pi^h \\
 & = a(\varphi^h - \bar{\varphi}) \nabla(\varphi^h - \bar{\varphi}) - (J * (\varphi^h - \bar{\varphi})) \nabla(\varphi^h - \bar{\varphi}) + (aq - J * q) \nabla \bar{\varphi} \\
 & + (a\bar{\varphi} - J * \bar{\varphi}) \nabla q + (F'(\varphi^h) - F'(\bar{\varphi})) \nabla(\varphi^h - \bar{\varphi}) + F'(\bar{\varphi}) \nabla q \\
 & + (F'(\varphi^h) - F'(\bar{\varphi}) - F''(\bar{\varphi}) \eta^h) \nabla \bar{\varphi} \tag{0.1}
 \end{aligned}$$

$$\begin{aligned}
 & q_t + (\mathbf{u}^h - \bar{\mathbf{u}}) \cdot \nabla(\varphi^h - \bar{\varphi}) + \mathbf{p} \cdot \nabla \bar{\varphi} + \bar{\mathbf{u}} \cdot \nabla q \\
 & = \Delta (aq - J * q + F'(\varphi^h) - F'(\bar{\varphi}) - F''(\bar{\varphi}) \eta^h) \tag{0.2}
 \end{aligned}$$

- Then,  $\mathbf{p}^h, q^h$  solve

$$\begin{aligned}
 & \mathbf{p}_t - \nu \Delta \mathbf{p} + (\mathbf{p} \cdot \nabla) \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla) \mathbf{p} + ((\mathbf{u}^h - \bar{\mathbf{u}}) \cdot \nabla)(\mathbf{u}^h - \bar{\mathbf{u}}) + \nabla \pi^h \\
 & = a(\varphi^h - \bar{\varphi}) \nabla(\varphi^h - \bar{\varphi}) - (J * (\varphi^h - \bar{\varphi})) \nabla(\varphi^h - \bar{\varphi}) + (aq - J * q) \nabla \bar{\varphi} \\
 & + (a\bar{\varphi} - J * \bar{\varphi}) \nabla q + (F'(\varphi^h) - F'(\bar{\varphi})) \nabla(\varphi^h - \bar{\varphi}) + F'(\bar{\varphi}) \nabla q \\
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 \end{aligned} \tag{0.1}$$

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 \end{aligned} \tag{0.2}$$

- Let us test (0.1) by  $\mathbf{p}$  in  $L^2_{div}(\Omega)^2$  and (0.2) by  $q$  in  $L^2(\Omega)$ . After some technical arguments we are led to

$$\frac{d}{dt} (\|\mathbf{p}^h\|^2 + \|q^h\|^2) + \nu \|\nabla \mathbf{p}^h\|^2 + c_0 \|\nabla q^h\|^2 \leq \alpha(t) \|\mathbf{p}^h\|^2 + \bar{\Gamma} \|q^h\|^2 + \beta_h(t)$$

$$\bar{\Gamma} = \bar{\Gamma}(\|\nabla \mathbf{u}_0\|, \|\varphi_0\|_{H^2(\Omega)}, \|\bar{\mathbf{v}}\|_\nu)$$

and  $\alpha, \beta_h \in L^1(0, T)$  given by

$$\begin{aligned}\alpha &:= \bar{\Gamma} (1 + \|\bar{\mathbf{u}}\|_{H^2(\Omega)^2}^2) \\ \beta_h &:= \bar{\Gamma} (\|\mathbf{u}^h - \bar{\mathbf{u}}\|^2 \|\nabla(\mathbf{u}^h - \bar{\mathbf{u}})\|^2 + \|\varphi^h - \bar{\varphi}\|^2 \|\varphi^h - \bar{\varphi}\|_{H^1(\Omega)}^2 \\ &\quad + \|\nabla(\mathbf{u}^h - \bar{\mathbf{u}})\|^2 \|\nabla(\varphi^h - \bar{\varphi})\|^2 + \|\varphi^h - \bar{\varphi}\|_{H^1(\Omega)}^4 + \|\varphi^h - \bar{\varphi}\|_{H^1(\Omega)}^2 \|\varphi^h - \bar{\varphi}\|_{H^2(\Omega)}^2)\end{aligned}$$

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■ Thanks to Stability estimate II we have

$$\int_0^T \beta_h(t) dt \leq \bar{\Gamma} \|\mathbf{h}\|_{\mathcal{V}}^4$$

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and so by Gronwall lemma ( $\mathbf{p}^h(0) = \mathbf{q}^h(0) = 0$ )

$$\begin{aligned}\|\mathbf{p}^h\|_{L^\infty(0, T; L^2_{div}(\Omega)^2)}^2 + \nu \|\mathbf{p}^h\|_{L^2(0, T; H^1_{div}(\Omega)^2)}^2 + \|\mathbf{q}^h\|_{L^\infty(0, T; L^2(\Omega))}^2 \\ + c_0 \|\mathbf{q}^h\|_{L^2(0, T; H^1(\Omega))}^2 \leq \bar{\Gamma} \|\mathbf{h}\|_{\mathcal{V}}^4\end{aligned}$$

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$$\implies \frac{\|S(\bar{\mathbf{v}} + \mathbf{h}) - S(\bar{\mathbf{v}}) - [\boldsymbol{\xi}^h, \boldsymbol{\eta}^h]\|_{\mathcal{Z}}}{\|\mathbf{h}\|_{\mathcal{V}}} \leq \bar{\Gamma} \|\mathbf{h}\|_{\mathcal{V}} \rightarrow 0 \quad \text{as } \mathbf{h} \rightarrow 0 \text{ in } \mathcal{V}$$



**Remark.** The weaker differentiability property of the control-to-state map from  $\mathcal{V}$  with values in

$$[C([0, T]; L^2_{div}(\Omega)^2) \cap L^2(0, T; H^1_{div}(\Omega)^2)] \times [C([0, T]; H^1(\Omega)') \cap L^2(0, T; L^2(\Omega))]$$

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**Remark.** The weaker differentiability property of the control-to-state map from  $\mathcal{V}$  with values in  $[C([0, T]; L^2_{div}(\Omega)^2) \cap L^2(0, T; H^1_{div}(\Omega)^2)] \times [C([0, T]; H^1(\Omega)') \cap L^2(0, T; L^2(\Omega))]$  easier to establish: test (0.2) by  $(-\Delta)^{-1}q$  and use only Stability estimate I

Nevertheless, with this weaker differentiability we get necessary conditions for existence of the optimal control for the control problem associated to the "incomplete" cost functional

$$J(y, \mathbf{v}) := \frac{\beta_1}{2} \|\mathbf{u} - \mathbf{u}_Q\|_{L^2(Q)^2}^2 + \frac{\beta_2}{2} \|\varphi - \varphi_Q\|_{L^2(Q)}^2 + \frac{\beta_3}{2} \|\mathbf{u}(T) - \mathbf{u}_\Omega\|^2 + \frac{\gamma}{2} \|\mathbf{v}\|_{L^2(Q)^2}^2$$

If  $\bar{\mathbf{v}} \in \mathcal{V}_{ad}$  is an optimal control for Problem **(CP)**, then

$$f'(\bar{\mathbf{v}})(\mathbf{v} - \bar{\mathbf{v}}) \geq 0 \quad \forall \mathbf{v} \in \mathcal{V}_{ad}$$

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But

$$f'(\mathbf{v}) = J'_y(S(\mathbf{v}), \mathbf{v})S'(\mathbf{v}) + J'_v(S(\mathbf{v}), \mathbf{v})$$

and hence the Frechét differentiability result for  $S : \mathcal{V} \rightarrow \mathcal{Z}$  yields

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### Corollary

Let  $\bar{\mathbf{v}} \in \mathcal{V}_{ad}$  be an optimal control for Problem **(CP)** with associated state

$\bar{\mathbf{y}} = [\bar{\mathbf{u}}, \bar{\varphi}] := S(\bar{\mathbf{v}})$ . Then

$$\begin{aligned} & \beta_1 \int_0^T \int_{\Omega} (\bar{\mathbf{u}} - \mathbf{u}_Q) \cdot \boldsymbol{\xi}^h + \beta_2 \int_0^T \int_{\Omega} (\bar{\varphi} - \varphi_Q) \eta^h + \beta_3 \int_{\Omega} (\bar{\mathbf{u}}(T) - \mathbf{u}_{\Omega}) \cdot \boldsymbol{\xi}^h(T) \\ & + \beta_4 \int_{\Omega} (\bar{\varphi}(T) - \varphi_{\Omega}) \eta^h(T) + \gamma \int_0^T \int_{\Omega} \bar{\mathbf{v}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) \geq 0 \quad \forall \mathbf{v} \in \mathcal{V}_{ad} \end{aligned}$$

where  $[\boldsymbol{\xi}^h, \eta^h]$  is the unique sol to the linearized system corresponding to  $\mathbf{h} = \mathbf{v} - \bar{\mathbf{v}}$

**Aim:** eliminate  $\xi^h, \eta^h$  from the previous inequality. Hence, introduce the **adjoint system**

$$\tilde{\mathbf{p}}_t = -\nu \Delta \tilde{\mathbf{p}} - (\bar{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{p}} + (\tilde{\mathbf{p}} \cdot \nabla^T) \bar{\mathbf{u}} + \tilde{q} \nabla \bar{\varphi} - \beta_1 (\bar{\mathbf{u}} - \mathbf{u}_Q)$$

$$\begin{aligned} \tilde{q}_t = & - (a \Delta \tilde{q} + \nabla J * \nabla \tilde{q} + F''(\bar{\varphi}) \Delta \tilde{q}) - \bar{\mathbf{u}} \cdot \nabla \tilde{q} \\ & - (a \tilde{\mathbf{p}} \cdot \nabla \bar{\varphi} - J * (\tilde{\mathbf{p}} \cdot \nabla \bar{\varphi}) + F'''(\bar{\varphi}) \tilde{\mathbf{p}} \cdot \nabla \bar{\varphi}) + \tilde{\mathbf{p}} \cdot \nabla \bar{\mu} - \beta_2 (\bar{\varphi} - \varphi_Q) \end{aligned}$$

$$\operatorname{div}(\tilde{\mathbf{p}}) = 0$$

$$\tilde{\mathbf{p}} = 0, \quad \frac{\partial \tilde{q}}{\partial \mathbf{n}} = 0 \quad \text{on } \Sigma$$

$$\tilde{\mathbf{p}}(T) = \beta_3 (\bar{\mathbf{u}}(T) - \mathbf{u}_\Omega), \quad \tilde{q}(T) = \beta_4 (\bar{\varphi}(T) - \varphi_\Omega)$$

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$$\tilde{q}_t = - (a \Delta \tilde{q} + \nabla J * \nabla \tilde{q} + F''(\bar{\varphi}) \Delta \tilde{q}) - \bar{\mathbf{u}} \cdot \nabla \tilde{q}$$

$$- (a \tilde{\mathbf{p}} \cdot \nabla \bar{\varphi} - J * (\tilde{\mathbf{p}} \cdot \nabla \bar{\varphi}) + F''(\bar{\varphi}) \tilde{\mathbf{p}} \cdot \nabla \bar{\varphi}) + \tilde{\mathbf{p}} \cdot \nabla \bar{\mu} - \beta_2 (\bar{\varphi} - \varphi_Q)$$

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$$\tilde{\mathbf{p}}(T) = \beta_3 (\bar{\mathbf{u}}(T) - \mathbf{u}_\Omega), \quad \tilde{q}(T) = \beta_4 (\bar{\varphi}(T) - \varphi_\Omega)$$

### Proposition

The adjoint system has a unique weak sol  $\tilde{\mathbf{p}}, \tilde{q}$  satisfying

$$\tilde{\mathbf{p}} \in C([0, T]; L^2_{div}(\Omega)^2) \cap L^2(0, T; H^1_{div}(\Omega)^2), \quad \tilde{q} \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

**Theorem**

Let  $\bar{\mathbf{v}} \in \mathcal{V}_{ad}$  be an optimal control for Problem **(CP)** with associated state  $\bar{\mathbf{y}} = [\bar{\mathbf{u}}, \bar{\varphi}] = S(\bar{\mathbf{v}})$  and adjoint state  $[\tilde{\mathbf{p}}, \tilde{\mathbf{q}}]$ . Then

$$\gamma \int_0^T \int_{\Omega} \bar{\mathbf{v}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) + \int_0^T \int_{\Omega} \tilde{\mathbf{p}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) \geq 0 \quad \forall \mathbf{v} \in \mathcal{V}_{ad}$$



**Theorem**

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- The system **(nloc CHNS)**, written for  $[\bar{\mathbf{u}}, \bar{\varphi}]$ , the adjoint system and the above variational inequality form together the first order necessary optimality conditions

## Theorem

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$$\gamma \int_0^T \int_{\Omega} \bar{\mathbf{v}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) + \int_0^T \int_{\Omega} \tilde{\mathbf{p}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) \geq 0 \quad \forall \mathbf{v} \in \mathcal{V}_{ad}$$

- The system **(nloc CHNS)**, written for  $[\bar{\mathbf{u}}, \bar{\varphi}]$ , the adjoint system and the above variational inequality form together the first order necessary optimality conditions
- Since  $\mathcal{V}_{ad}$  is a nonempty, closed and convex subset of  $L^2(Q)^2$ , then the above variational inequality with  $\gamma > 0$  is equivalent to

$$\bar{\mathbf{v}} = P_{\mathcal{V}_{ad}} \left( \left\{ -\frac{\tilde{\mathbf{p}}}{\gamma} \right\} \right)$$

where  $P_{\mathcal{V}_{ad}}$  is the orthogonal projector in  $L^2(Q)^2$  onto  $\mathcal{V}_{ad}$

(In progress, jointly with E. Rocca & J. Sprekels)

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- Minimize the cost functional  $J(y, \mathbf{v})$  where  $y := [\mathbf{u}, \varphi]$  solves

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{v}$$

$$\varphi_t + \mathbf{u} \cdot \nabla \varphi = \operatorname{div}(m(\varphi) \nabla \mu)$$

$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

$$\operatorname{div}(\mathbf{u}) = 0$$

$$\partial_n \mu = 0 \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \varphi(0) = \varphi_0$$

and the control  $\mathbf{v}$  belongs to a suitable closed, bounded and convex subset of the space of controls  $\mathcal{V} := L^2(0, T; L^2_{div}(\Omega)^2)$

(In progress, jointly with E. Rocca & J. Sprekels)

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$$\varphi_t + \mathbf{u} \cdot \nabla \varphi = \operatorname{div}(m(\varphi) \nabla \mu)$$

$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

$$\operatorname{div}(\mathbf{u}) = 0$$

$$\partial_n \mu = 0 \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \varphi(0) = \varphi_0$$

and the control  $\mathbf{v}$  belongs to a suitable closed, bounded and convex subset of the space of controls  $\mathcal{V} := L^2(0, T; L^2_{div}(\Omega)^2)$

- Aim: first order necessary conditions for existence of optimal control

(In progress, jointly with E. Rocca & J. Sprekels)

- Minimize the cost functional  $J(y, \mathbf{v})$  where  $y := [\mathbf{u}, \varphi]$  solves

$$\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{v}$$

$$\varphi_t + \mathbf{u} \cdot \nabla \varphi = \operatorname{div}(m(\varphi) \nabla \mu)$$

$$\mu = a\varphi - J * \varphi + F'(\varphi)$$

$$\operatorname{div}(\mathbf{u}) = 0$$

$$\partial_n \mu = 0 \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega$$

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- Aim: first order necessary conditions for existence of optimal control
- First prove existence of strong solutions in 2D (i.e., extend the regularity result in F., Grasselli & Krejčí '13 to the case of the nonlocal CHNS system in 2D with deg. mob.+sing. pot.)

- compressible models

- **non-isothermal model(s)**

(Eleuteri, Rocca & Schimperna, Discrete Contin. Dyn. Syst. '15 for the local CHNS)

- multicomponent models