Weierstrass Institute for
Applied Analysis and Stochastics

On a non-isothermal diffuse interface model for two phase flows of incompressible fluids

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Outline

- The motivation
- The PDEs (equations and inequalities)
- The modelling
- The analytical results in 3D
- The expected improvements in 2D
- Some open related problems

- A **non-isothermal** model for the flow of a **mixture of two**
  - viscous
  - incompressible
  - Newtonian fluids
  - of equal density

The motivation

- A **non-isothermal** model for the flow of a **mixture of two**
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  - incompressible
  - Newtonian fluids
  - of equal density
- Avoid problems related to interface singularities

  $\Longrightarrow$ use a **diffuse interface model**

  $\Longrightarrow$ the classical sharp interface replaced by a **thin interfacial region**
- A partial mixing of the macroscopically immiscible fluids is allowed

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- The original idea of diffuse interface model for fluids: HOHENBERG and HALPERIN, '77

  $\Longrightarrow$ **H-model**

  Later, GURTIN ET AL., '96: continuum mechanical derivation based on microforces
- Models of two-phase or two-component fluids are receiving growing attention (e.g., ABELS, BOYER, GARCKE, GRÜN, GRASSELLI, LOWENGRUB, TRUSKINOVSKI, ...)

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- **Including temperature dependence is a widely open issue**
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- Our idea: a weak formulation of the system as a combination of *total energy balance* plus *entropy production inequality* $\Longrightarrow$ **"Entropic formulation"**

- This method has been recently proposed by [BULÍČEK-MÁLEK-FEIREISL, '09] for the Navier-Stokes-Fourier system and has been proved to be effective to study e.g.
  - nonisothermal models for **phase transitions** ([FEIREISL-PETZELTOVÁ-R., '09]) and
  - the evolution of **nematic liquid crystals** ([FRÉMOND, FEIREISL, R., SCHIMPERNA, ZARNESCU, '12,'13])

- We want to describe the behavior of a mixture of two incompressible fluids of the same density in terms of the following state variables
  - $\mathbf{u}$: macroscopic **velocity** (Navier-Stokes),
  - p: **pressure** (Navier-Stokes),
  - φ: **order parameter** (Cahn-Hilliard),
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- We do not neglect convection and capillarity effects. We assume constant mobility and smooth configuration potential in Cahn-Hilliard. We take temperature dependent coefficients wherever possible. We assume the system being insulated from the exterior.

- a **weak form of the momentum balance**

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x p = \operatorname{div}(\nu(\theta) D\mathbf{u}) - \operatorname{div}(\nabla_x \varphi \otimes \nabla_x \varphi), \quad \operatorname{div} \mathbf{u} = 0;$$

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- the **Cahn-Hilliard system** in $H^1(\Omega)'$

$$\varphi_t + \mathbf{u} \cdot \nabla_x \varphi = \Delta \mu, \quad \mu = -\varepsilon \Delta \varphi + \frac{1}{\varepsilon} F'(\varphi) - \theta;$$

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- a **weak form of the total energy balance**

$$\partial_t\left(\frac{1}{2}|\mathbf{u}|^2 + e\right) + \mathbf{u}\cdot\nabla_x\left(\frac{1}{2}|\mathbf{u}|^2 + e\right) + \operatorname{div}\left(p\mathbf{u} + \mathbf{q} - \mathbb{S}\mathbf{u}\right)$$

$$-\operatorname{div}\left(\varepsilon\varphi_t\nabla_x\varphi + \mu\nabla_x\mu\right) = 0 \quad \text{where} \quad e = \frac{1}{\varepsilon}F(\varphi) + \frac{\varepsilon}{2}|\nabla_x\varphi|^2 + \int_1^\theta c_v(s)\,\mathrm{d}s;$$

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- the weak form of the **entropy production inequality**

$$(\Lambda(\theta) + \varphi)_t + \mathbf{u}\cdot\nabla_x(\Lambda(\theta)) + \mathbf{u}\cdot\nabla_x\varphi - \operatorname{div}\left(\frac{\kappa(\theta)\nabla_x\theta}{\theta}\right)$$

$$\geq \frac{\nu(\theta)}{\theta}|D\mathbf{u}|^2 + \frac{1}{\theta}|\nabla_x\mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x\theta|^2, \quad \text{where} \quad \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s}\,\mathrm{d}s.$$

- We start by specifying two functionals:
  - the **free energy** Ψ, related to the equilibrium state of the material, and
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- Then we impose the balances of **momentum**, **configuration energy**, and both of **internal energy** and of **entropy**, in terms of these functionals
- The **thermodynamical consistency** of the model is then a direct consequence of the solution notion

The **total free energy** is given as a function of the state variables $E = (\theta, \varphi, \nabla_x\varphi)$

$$\Psi(E) = \int_\Omega \psi(E)\,\mathrm{d}x, \quad \psi(E) = f(\theta) - \theta\varphi + \frac{\varepsilon}{2}|\nabla_x\varphi|^2 + \frac{1}{\varepsilon}F(\varphi)$$

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- $f(\theta)$ is related to the specific heat $c_v(\theta) = Q'(\theta)$ by $Q(\theta) = f(\theta) - \theta f'(\theta)$. In our case we need $c_v(\theta) \sim c_\delta \theta^\delta$ for some $\delta \in (1/2, 1)$

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- $\varepsilon > 0$ is related to the interfacial thickness
- we need $F(\varphi)$ to be the classical smooth double well potential $F(\varphi) \sim \frac{1}{4}(\varphi^2 - 1)^2$

The **dissipation potential** is taken as function of $\delta E = (D\mathbf{u}, \frac{D\varphi}{Dt}, \nabla_x \theta)$ and E

$$\Phi(\delta E, E) = \int_\Omega \phi(D\mathbf{u}, \nabla_x\theta)\,\mathrm{d}x + \left\langle \frac{D\varphi}{Dt}, J^{-1}\frac{D\varphi}{Dt} \right\rangle$$

$$= \int_\Omega \left(\frac{\nu(\theta)}{2}|D\mathbf{u}|^2 + I_{\{0\}}(\operatorname{div}\mathbf{u}) + \frac{\kappa(\theta)}{2\theta}|\nabla_x\theta|^2 \right)\,\mathrm{d}x + \left\| \frac{D\varphi}{Dt} \right\|^2_{H^1_\#(\Omega)'}$$

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&= \int_\Omega \left(\frac{\nu(\theta)}{2}|D\mathbf{u}|^2 + I_{\{0\}}(\operatorname{div}\mathbf{u}) + \frac{\kappa(\theta)}{2\theta}|\nabla_x\theta|^2 \right) \mathrm{d}x + \left\| \frac{D\varphi}{Dt} \right\|^2_{H^1_\#(\Omega)'}
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- $D\mathbf{u} = (\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u})/2$ the symmetric gradient
- $\frac{D(\cdot)}{Dt} = (\cdot)_t + \mathbf{u}\cdot\nabla_x(\cdot)$ the material derivative
- $J : H^1_\#(\Omega) \to H^1_\#(\Omega)'$ the **Riesz isomorphism**
  $\langle Ju, v\rangle := ((u,v))_{H^1_\#(\Omega)} := \int_\Omega \nabla_x u \cdot \nabla_x v\,\mathrm{d}x,$
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- $\nu = \nu(\theta) > 0$ the viscosity coefficient, $\kappa = \kappa(\theta) > 0$ the heat conductivity
- **Incompressibility**: I_0 the indicator function of $\{0\}$: $I_0 = 0$ if $\operatorname{div}\mathbf{u} = 0$, $+\infty$ otherwise)

- The **dissipation potential** was taken as

$$\Phi = \Phi\left(D\mathbf{u}, \frac{D\varphi}{Dt}, \nabla_x\theta\right) = \int_\Omega \phi(D\mathbf{u}, \nabla_x\theta)\,\mathrm{d}x + \left\langle \frac{D\varphi}{Dt}, J^{-1}\frac{D\varphi}{Dt}\right\rangle$$

- If a time-dependent set of variables is given such that
  - a.e. in $(0,T)$, Ψ and Φ are finite
  - $\mathbf{u}$ is such that $\mathbf{u}\cdot\mathbf{n} = 0$ on Γ
  - φ satisfies the **mass conservation constraint** $\varphi(t,x) = \varphi(0,x) = \varphi_0(x)$ a.e.

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then $\mathbf{u}$ is divergence-free and we get

$$\int_\Omega \frac{D\varphi}{Dt} = \int_\Omega (\varphi_t + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x = 0$$

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- Then we can set $\mu_\# := -J^{-1}\frac{D\varphi}{Dt}$, so that $\frac{D\varphi}{Dt} = -J\mu_\# = \Delta\mu_\#$ and we get

$$\Phi(\delta E, E) = \int_\Omega \widetilde{\phi}(\delta E, E)\,\mathrm{d}x, \quad \text{where } \widetilde{\phi}(\delta E, E) = \phi(\delta E, E) + \frac{1}{2}|\nabla_x\mu_\#|^2$$

- It is obtained (at least for no-flux b.c.'s) as the following **gradient-flow problem**

$$\partial_{L^2_\#(\Omega),\frac{D\varphi}{Dt}}\Phi + \partial_{L^2_\#(\Omega),\varphi_\#}\Psi = 0$$

where $L^2_\#(\Omega) = \{\xi \in L^2(\Omega) \,:\, \overline{\xi} := |\Omega|^{-1}\int_\Omega \xi \,\mathrm{d}x = 0\}$, $\varphi_\# = \varphi - \overline{\varphi_0}$

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- Combining the previous relations we then get

$$J^{-1}\left(\frac{D\varphi}{Dt}\right) = \varepsilon\Delta\varphi - \frac{1}{\varepsilon}\left(F'(\varphi) - \overline{F'(\varphi)}\right) + \theta - \overline{\theta}, \quad \frac{\partial\varphi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma, \;\; \overline{\varphi}(t) = \overline{\varphi_0}$$

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- Applying the distributional Laplace operator to both hand sides and noting that $-\Delta J^{-1}v = v$ for any $v \in L^2_\#(\Omega)$, we then arrive at **the Cahn-Hilliard system with Neumann hom. b.c. for μ and φ**

$$\frac{D\varphi}{Dt} = \Delta\mu, \quad \mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta, \quad \frac{\partial\varphi}{\partial\mathbf{n}} = \frac{\partial\mu}{\partial\mathbf{n}} = 0 \text{ on } \Gamma \qquad \text{(CahnHill)}$$

where the auxiliary variable μ takes the name of *chemical potential*

The **Navier-Stokes system** is obtained as a momentum balance by setting

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where the stress σ is split into its

- **dissipative part**

  $$\sigma^d := \frac{\partial \phi}{\partial D\mathbf{u}} = \nu(\theta) D\mathbf{u} - p\mathbb{I}, \quad \operatorname{div} \mathbf{u} = 0,$$

  representing kinetic energy which **dissipates** (i.e. is transformed into heat) due to viscosity, and its

- **non-dissipative part** σ^{nd} to be determined later in agreement with Thermodynamics

The balance of internal energy takes the form

$$\frac{De}{Dt} + \operatorname{div}\mathbf{q} = \nu(\theta)|D\mathbf{u}|^2 + \sigma^{nd} : D\mathbf{u} + B\frac{D\varphi}{Dt} + \frac{\partial\psi}{\partial\nabla_x\varphi}\cdot\nabla_x\frac{D\varphi}{Dt} + \boxed{N}$$

where $e = \psi - \theta\psi_\theta$, $B = B^{nd} + B^d$ and

$$B^{nd} = \frac{\partial\psi}{\partial\varphi} = \frac{1}{\varepsilon}F'(\varphi) - \theta, \quad B^d = \partial_{L^2_\#(\Omega),\frac{D\varphi}{Dt}}\Phi = J^{-1}\left(\frac{D\varphi}{Dt}\right)$$

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On the right hand side there appears a new (with respect to the standard theory of [Frémond, '02]) term $\boxed{N}$ balancing the **nonlocal dependence of the last term in the pseudopotential of dissipation** Φ

$$\Phi = \Phi\left(D\mathbf{u}, \frac{D\varphi}{Dt}\right) = \int_\Omega \phi\,\mathrm{d}x + \left\langle \frac{D\varphi}{Dt}, J^{-1}\frac{D\varphi}{Dt}\right\rangle$$

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It will result from the Second Principle of Thermodynamics that $\int_\Omega N(x)\,\mathrm{d}x = 0$, in agreement with natural expectations

To deduce the expressions for σ^{nd} and N, we impose validity of the **Clausius-Duhem inequality** in the form

$$\theta\Big(\frac{Ds}{Dt} + \operatorname{div}\Big(\frac{\mathbf{q}}{\theta}\Big)\Big) \geq 0$$

where $e = \psi + \theta s$, being $s = -\psi_\theta$ the entropy density and we get

$$\sigma^{nd} = -\varepsilon \nabla_x \varphi \otimes \nabla_x \varphi, \quad N = \frac{1}{2}\Delta(\mu - \overline{\mu})^2$$

and the internal energy balance can be rewritten as

$$(Q(\theta))_t + \mathbf{u}\cdot\nabla_x Q(\theta) + \theta\frac{D\varphi}{Dt} - \operatorname{div}(\kappa(\theta)\nabla_x\theta) = \nu(\theta)|D\mathbf{u}|^2 + |\nabla_x\mu|^2$$

where $Q(\theta) = f(\theta) - \theta f'(\theta)$ and $Q'(\theta) =: c_v(\theta)$

To deduce the expressions for σ^{nd} and N, we impose validity of the **Clausius-Duhem inequality** in the form

$$\theta\Big(\frac{Ds}{Dt} + \operatorname{div}\Big(\frac{\mathbf{q}}{\theta}\Big)\Big) \geq 0$$

where $e = \psi + \theta s$, being $s = -\psi_\theta$ the entropy density and we get

$$\sigma^{nd} = -\varepsilon \nabla_x \varphi \otimes \nabla_x \varphi, \quad N = \frac{1}{2}\Delta(\mu - \overline{\mu})^2$$

and the internal energy balance can be rewritten as

$$(Q(\theta))_t + \mathbf{u} \cdot \nabla_x Q(\theta) + \theta \frac{D\varphi}{Dt} - \operatorname{div}(\kappa(\theta)\nabla_x \theta) = \nu(\theta)|D\mathbf{u}|^2 + |\nabla_x \mu|^2$$

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The **dissipation** terms on the right hand side are in perfect agreement with Φ

$$\Phi = \int_\Omega \widetilde{\phi}\, \mathrm{d}x, \quad \text{where } \widetilde{\phi} = \phi + \frac{1}{2}|\nabla_x \mu|^2$$

Following [BULÍČEK, FEIREISL, & MÁLEK], we replace the pointwise internal energy balance by the **total energy balance**

$$\left(\partial_t + \mathbf{u}\cdot\nabla_x\right)\left(\frac{|\mathbf{u}|^2}{2} + e\right) + \mathrm{div}\left(p\mathbf{u} - \kappa(\theta)\nabla_x\theta - (\nu(\theta)D\mathbf{u})\mathbf{u}\right)$$
$$= \mathrm{div}\left(\varphi_t\nabla_x\varphi + \mu\nabla_x\mu\right) \qquad \text{(energy)}$$

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$$e = F(\varphi) + \frac{1}{2}|\nabla_x\varphi|^2 + Q(\theta) \quad Q'(\theta) = c_v(\theta)$$

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and the **entropy inequality**

$$(\Lambda(\theta) + \varphi)_t + \mathbf{u} \cdot \nabla_x (\Lambda(\theta) + \varphi) - \operatorname{div} \left(\frac{\kappa(\theta)\nabla_x \theta}{\theta} \right) \qquad \text{(entropy)}$$
$$\geq \frac{\nu(\theta)}{\theta}|D\mathbf{u}|^2 + \frac{1}{\theta}|\nabla_x \mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x \theta|^2, \quad \text{where} \quad \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s}\, \mathrm{d}s \sim \theta^\delta$$

- a **weak form of the momentum balance** (in distributional sense)

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} + \nabla_x p = \operatorname{div}(\nu(\theta) D\mathbf{u}) - \operatorname{div}(\nabla_x \varphi \otimes \nabla_x \varphi), \quad \operatorname{div} \mathbf{u} = 0;$$

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- the weak form of the **entropy production inequality**

$$(\Lambda(\theta)+\varphi)_t + \mathbf{u}\cdot\nabla_x(\Lambda(\theta)) + \mathbf{u}\cdot\nabla_x\varphi - \operatorname{div}\left(\frac{\kappa(\theta)\nabla_x\theta}{\theta}\right)$$

$$\geq \frac{\nu(\theta)}{\theta}|D\mathbf{u}|^2 + \frac{1}{\theta}|\nabla_x\mu|^2 + \frac{\kappa(\theta)}{\theta^2}|\nabla_x\theta|^2, \quad \text{where} \quad \Lambda(\theta) = \int_1^\theta \frac{c_v(s)}{s}\,\mathrm{d}s.$$

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  - The **viscosity** $\nu(\theta)$ is assumed **smooth and bounded**
  - The **specific heat** $c_v(\theta) \sim \theta^\delta$, $1/2 < \delta < 1$
  - The **heat conductivity** $\kappa(\theta) \sim 1 + \theta^\beta$, $\beta \geq 2$
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- Concerning B.C.'s, our results are proved for **no-flux** conditions for θ, φ, and μ and **complete slip** conditions for $\mathbf{u}$

$$\mathbf{u} \cdot \mathbf{n}_{|\Gamma} = 0 \quad \text{(the fluid cannot exit } \Omega\text{, it can move tangentially to } \Gamma)$$

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}_{|\Gamma} = 0, \quad \text{where } \mathbb{S} = \nu(\theta) D\mathbf{u} \quad \text{(exclude friction effects with the boundary)}$$

They can be easily extended to the case of periodic B.C.'s for all unknowns

Theorem

We can prove existence of **at least** *one* **global in time weak solution** $(\mathbf{u}, \varphi, \mu, \theta)$

$$
\begin{aligned}
&\mathbf{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;\mathbf{V_n})\\
&\varphi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^1(\Omega)')\\
&\mu \in L^2(0,T;H^1(\Omega)) \cap L^{\frac{14}{5}}((0,T)\times\Omega)\\
&\theta \in L^\infty(0,T;L^{\delta+1}(\Omega)) \cap L^\beta(0,T;L^{3\beta}(\Omega)) \cap L^2(0,T;H^1(\Omega))\\
&\theta > 0 \ \text{a.e. in } (0,T)\times\Omega, \quad \log\theta \in L^2(0,T;H^1(\Omega))
\end{aligned}
$$

to system given by (momentum), (CahnHill), (entropy) and (energy), in distributional sense and for finite-energy initial data

$$
\mathbf{u}_0 \in L^2(\Omega), \ \operatorname{div}\mathbf{u}_0 = 0, \quad \varphi_0 \in H^1(\Omega), \quad \theta_0 \in L^{\delta+1}(\Omega), \quad \theta_0 > 0 \ \text{a.e.}
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- Existence proof based on a classical **a-priori estimates** – **compactness** scheme
- The basic information is contained in the **energy** and **entropy** relations
- Note that the power-like growth of the heat conductivity and of the specific heat is required in order to provide sufficient summability of the temperature

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Is this sufficient to pass to the limit?

- The **total energy balance** contains some nasty extra terms $\varphi_t \nabla_x \varphi + \mu \nabla_x \mu$. In particular, φ_t lies only in some **negative order** space (cf. (CahnHill))
- Using (CahnHill) and integrating by parts carefully the bad terms tranform into

$$- \Delta \mu^2 + \operatorname{div}\left((\mathbf{u} \cdot \nabla_x \varphi) \nabla_x \varphi \right)$$

$$+ \operatorname{div}(\nabla_x \mu \cdot \nabla_x \nabla_x \varphi) - \operatorname{div} \operatorname{div}(\nabla_x \mu \otimes \nabla_x \varphi)$$

- The above terms can be controlled by getting some **extra-integrability** of φ and μ from (CahnHill). To this aim having a **"smooth" potential** F is crucial!

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$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_x\mathbf{u} + \nabla_x p = \mathrm{div}(D\mathbf{u}) - \mathrm{div}\left(\nabla_x\varphi \otimes \nabla_x\varphi\right)$$

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- Hence, there is hope to get extra-regularity **for constant viscosity** ν (i.e., independent of temperature)
- Indeed we get
  $\mathbf{u}_t\in L^2(0,T;L^2(\Omega))$ and $\mathbf{u}\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))$

Assumptions in 2D

- Constant viscosity $\nu = 1$
- Constant specific heat $c_v = 1$ (in other words, $f(\theta) = -\theta \log \theta$)
- Power-like conductivity (for simplicity $\kappa(\theta) = \theta^2$)
- Periodic boundary conditions

Theorem

We can prove existence of at least one **"strong" solution** *to system given by*

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_x\mathbf{u} + \nabla_x p = \operatorname{div}(D\mathbf{u}) - \operatorname{div}\left(\nabla_x\varphi\otimes\nabla_x\varphi\right) \qquad \text{(mom)}$$

$$\varphi_t + \mathbf{u}\cdot\nabla_x\varphi = \Delta\mu \qquad \text{(CH1)}$$

$$\mu = -\Delta\varphi + F'(\varphi) - \theta \qquad \text{(CH2)}$$

$$\theta_t + \mathbf{u}\cdot\nabla_x\theta + \theta(\varphi_t + \mathbf{u}\cdot\nabla_x\varphi) - \Delta\theta^3 = |D\mathbf{u}|^2 + |\nabla_x\mu|^2 \qquad \text{(heat)}$$

for finite-energy initial data, namely

$$\mathbf{u}_0 \in H^1_{\mathrm{per}}(\Omega), \ \ \operatorname{div}\mathbf{u}_0 = 0,$$
$$\varphi_0 \in H^3_{\mathrm{per}}(\Omega),$$
$$\theta_0 \in H^1_{\mathrm{per}}(\Omega), \quad \theta_0 > 0 \ \textit{a.e.}, \quad \log\theta_0 \in L^1(\Omega)$$

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  - (CH1)$_t$ $\times(-\Delta)^{-1}\varphi_t$

$$\varphi_{tt} + \mathbf{u}_t \cdot \nabla_x \varphi + \mathbf{u} \cdot \nabla_x \varphi_t = \Delta \mu_t \quad \boxed{\times(-\Delta)^{-1}\varphi_t}$$

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- Only possibility, to **test** (heat) **by** φ_t

$$\theta_t + \mathbf{u} \cdot \nabla_x \theta + \theta(\varphi_t + \mathbf{u} \cdot \nabla_x \varphi) - \Delta\theta^3 = |D\mathbf{u}|^2 + |\nabla_x \mu|^2 \quad \boxed{\times\varphi_t}$$

to let it disappear

The main estimate

- Try (CH1)$_t$ $\times(-\Delta)^{-1}\varphi_t$

$$\varphi_{tt} + \mathbf{u}_t \cdot \nabla_x \varphi + \mathbf{u} \cdot \nabla_x \varphi_t = \Delta\mu_t \quad \times (-\Delta)^{-1}\varphi_t$$

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- getting

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla_x \mu\|_{L^2}^2 + \|\theta\|_{L^4}^4\right) + \|\varphi_t\|_{H^1}^2 + \|\theta^3\|_{H^1}^2 \leq c \int_\Omega |\nabla_x \mu|^2 |\varphi_t + \theta^3| \,\mathrm{d}x + \text{l.o.t.}$$

where l.o.t. can be easily handled

Having the inequality

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which is bad!

- We know, however, that

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- Interpolating and optimizing w.r.t. q, we arrive at

$$\|\xi\|_{(H^1)^*} \leq c\|\xi\|_{L^1}\left(1 + \log^{1/2}\|\xi\|_{L^2}\right) \quad \text{for all } \xi \in L^2(\Omega)$$

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Applying the above to $\xi = |\nabla_x \mu|^2$, we get a differential inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla_x\mu\|_{L^2}^2 + \|\theta\|_{L^4}^4\right) + \|\varphi_t\|_{H^1}^2 + \|\theta^3\|_{H^1}^2 \leq c\|\nabla_x\mu\|_{L^2}^2\left(\|\nabla_x\mu\|_{L^2}^2 \log\|\nabla_x\mu\|_{L^2}^2\right) + \ldots$$

Hence, we get a **global estimate** thanks to a (generalized) Gronwall lemma

Work in progress and further developments

- Uniqueness in 2D 

- Convergence to equilibria in 2D. Existence of attractors 

- Allen-Cahn-type models 

- Singular potentials in Cahn-Hilliard (or Allen-Cahn) 

- Non-isothermal nonlocal models