Solutions to a full model for thermoviscoelastic materials

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joint work with Riccarda Rossi (Università di Brescia, Italy)
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- Handle nonlinearities $\Longrightarrow$ suitable solution notion
  - ◇ coupling **entropy inequality and total energy identity** with
  - ◇ a **generalized principle of virtual powers**
- Present other possible applications of these formulations to: phase separation, liquid crystals, immiscible fluids

The full PDE system

State variables:

- the absolute temperature θ
- the (small) displacement variables $\mathbf{u}$ ($\varepsilon_{ij}(\mathbf{u}) := (u_{i,j} + u_{j,i})/2$, $i, j = 1, 2, 3$)
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$$\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}$$

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta\chi - \eta\Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta$$

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- **Unidirectional:** $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \in (-\infty, 0]$, $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise; $\mu = 1$ in damage phenomena – $\mu = 0$ in phase transitions
- ***p*-Laplacian:** $-\Delta_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ the $p-$Laplacian ($p > d$); $\eta > 0$ in phase transitions – $\eta \geq 0$ in damage phenomena
- $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c. (e.g. $\widehat{\beta} = I_{[0,1]}$ or $W'(\chi) = \chi^3 - \chi$, etc.)

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2. a *generalization of the principle of virtual powers* inspired by:

2.1. a notion of *weak solution* introduced by [Heinemann, Kraus, WIAS, Adv. Math. Sci. Appl. (2011) and European J. Appl. Math. (2013)] for non-degenerating isothermal diffuse interface models for phase separation and damage $\Longrightarrow$ **weak** formulation of the **damage equation**

Entropic formulation: a phase transitions model

A first application of the entropic formulation: solid-liquid phase transitions

In order to show the potential power of this idea we

... give a description of the method stating more precisely the content of this recent work [E. Feireisl, H. Petzeltovà, E.R., *Existence of solutions to some models of phase changes with microscopic movements*, Math. Meth. Appl. Sci. (2009)] in which this notion of solution has been firstly applied to phase transition models

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$\Longrightarrow$ a new weaker notion of solution is needed

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Finally, couple these relations to a suitable phase dynamics

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r represents the entropy production rate. Then, in order to comply with the Clausius-Duhem inequality, we assume:

(i) r is a nonnegative measure on $\overline{Q}_T$;

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$\Rightarrow$ the total entropy is controlled by dissipation

The energy conservation and phase relation

The total energy has to be preserved. Hence

$$E(t) = E(0) \text{ for a.e. } t \in [0, T]$$

where

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Finally, the phase dynamics results as

$$\chi_t - \Delta\chi + W'(\chi) = \theta - \theta_c \quad \text{a.e. in } \Omega \times (0, T),$$

where W is a double well or double obstacle potential: $W = \widehat{\beta} + \widehat{\gamma}$ where

$\widehat{\beta} : \mathbb{R} \to [0, +\infty]$ is proper, lower semi-continuous, convex function

$\widehat{\gamma} \in C^2(\mathbb{R}),\ \widehat{\gamma}' \in C^{0,1}(\mathbb{R}) : \widehat{\gamma}''(r) \geq -K$ for all $r \in \mathbb{R}$, $W(r) \geq c_w r^2$ for all $r \in \mathrm{dom}(\widehat{\beta})$

Examples: $\widehat{\beta}(r) = r\ln(r) + (1-r)\ln(1-r)$ or $\widehat{\beta}(r) = I_{[0,1]}(r)$

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$$\chi_t-\Delta\chi+W'(\chi)=\theta-\theta_c\quad\text{a.e. in } Q_T,\qquad \chi(0)=\chi_0\quad\text{a.e. in }\Omega$$

and *the total energy conservation*

$$E(t)=E(0)\quad\text{a.e. in }[0,T],\qquad E\equiv\int_\Omega\left(\theta+W(\chi)+\frac{|\nabla\chi|^2}{2}\right)dx$$

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- **It can be suitable also in different applications** such as the ones related to SMA, liquid crystal flows, **damage phenomena** and **phase transitions in themoviscoelastic materials**

Our model [Rocca-Rossi, work in progress, 2013]

The free-energy

cf. [M. Frémond, Phase Change in Mechanics, Lecture Notes of UMI, Springer-Verlag, 2012]

$$\mathcal{F} = \int_\Omega \Big(c_v\theta(1-\log\theta) + b(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{|\nabla\chi|^2}{2} + \eta\frac{|\nabla\chi|^p}{p} + W(\chi) - \theta\chi - \rho\theta\mathrm{tr}(\varepsilon(\mathbf{u}))\Big)\,\mathrm{d}x$$

- **damage:** $b(\chi)=\chi$**; the stiffness of the material decreases as** $\chi \searrow 0$
- **phase transitions:** $b(\chi)=1-\chi$**; elastic effects are not present in the fluid**
- $W=\widehat{\beta}+\widehat{\gamma}$, $\widehat{\gamma}\in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c.
- $\eta>0$ and $p>d$ in phase transitions, $\eta\geq 0$ in damage
- $c_v>0$, take it $=1$ for simplicity

The free-energy

cf. [M. Frémond, Phase Change in Mechanics, Lecture Notes of UMI, Springer-Verlag, 2012]

$$\mathcal{F} = \int_\Omega \Big(c_v\theta(1-\log\theta) + b(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{|\nabla\chi|^2}{2} + \eta\frac{|\nabla\chi|^p}{p} + W(\chi) - \theta\chi - \rho\theta\mathrm{tr}(\varepsilon(\mathbf{u}))\Big)\,\mathrm{d}x$$

- **damage: $b(\chi) = \chi$; the stiffness of the material decreases as $\chi \searrow 0$**
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- $\eta > 0$ and $p > d$ in phase transitions, $\eta \geq 0$ in damage
- $c_v > 0$, take it $= 1$ for simplicity

The pseudo-potential

$$\mathcal{P} = \frac{\mathrm{K}(\theta)}{2}|\nabla\theta|^2 + \frac{1}{2}|\chi_t|^2 + a(\chi)\frac{|\varepsilon(\mathbf{u}_t)|^2}{2} + \mu I_{(-\infty,0]}(\chi_t)$$

- $a(\chi) = \chi$: no viscosty in solid phase or when the material is completely damaged
- K is the heat conductivity, $\mathrm{K}(\theta) \geq c_1(1+\nu\theta^k)$ for some c_1, $\nu > 0$, $k > 1$
- $\mu \neq 0$ **in damage**: $I_{(-\infty,0]}(\chi_t) = 0$ if $\chi_t \in (-\infty,0]$, $I_{(-\infty,0]}(\chi_t) = +\infty$ otherwise (**irreversibility** of the damage)

The modelling

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \mathbf{f} \quad \left(\sigma = \sigma^d + \sigma^{nd} = \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)} + \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} \right) \quad \text{becomes}$$

$$\boxed{\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}}$$

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The "standard" principle of virtual powers

$$B - \operatorname{div} \mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{P}}{\partial \chi_t} + \frac{\partial \mathcal{F}}{\partial \chi}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi}\right) \quad \text{becomes}$$

$$\boxed{\chi_t + \mu \partial I_{(-\infty,0]}(\chi_t) - \Delta\chi - \eta\Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta}$$

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The internal energy balance

$$e_t + \operatorname{div}\mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H}\cdot\nabla\chi_t \quad \left(e = \mathcal{F} - \theta\frac{\partial \mathcal{F}}{\partial \theta}, \quad \mathbf{q} = \frac{\partial \mathcal{P}}{\partial \nabla\theta}\right)$$

becomes

$$\boxed{\theta_t + \chi_t\theta + \rho\theta\operatorname{div}\mathbf{u}_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta) = g + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2}$$

Our previous results (cf. [MathProSpeM2012 – Rome, April 16–20, 2012])

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Note: in both these results we assumed χ_0 separated from the thresholds 0 and 1 and we prove (via coercivity condition on W at the thresholds 0 and 1) that the solution χ of

$$\chi_t - \Delta\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta$$

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[The last result] [E.R., R. Rossi, preprint arXiv:1205.3578v2 (2012)]: global existence result in 3D using a suitable notion of solution and without enforcing the separation property, i.e. **allowing for degeneracy** with $\mu = 1$, $\eta > 0$, but always within the **small perturbations assumption**

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- the high order dissipative terms in the θ-equation
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We consider separately the cases:

- of **irreversible damage processes** $\mu = 1$, $\eta \geq 0$
- of **reversible model of phase transitions** $\mu = 0$, $\eta > 0$

The irreversible damage process: $\mu = 1$ and $\eta = 0$

$$\theta_t + \chi_t\theta + \rho\theta \operatorname{div} \mathbf{u}_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta)) = g + (a(\chi) + \delta)|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2$$

$$\mathbf{u}_{tt} - \operatorname{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + (b(\chi) + \delta)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f}$$

$$\chi_t + \underbrace{\mu\partial I_{(-\infty,0]}(\chi_t)}_{=\partial I_{(-\infty,0]}(\chi_t)} - \Delta\chi \underbrace{-\eta\Delta_p\chi}_{=0} + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta$$

Hypothesis (I).

The function $\mathsf{K} : [0, +\infty) \to (0, +\infty)$ is continuous and

$$\exists c_1,\, \nu > 0,\, k > 1 \; : \quad \forall \theta \in [0, +\infty) \quad \mathsf{K}(\theta) \geq c_1(1 + \nu\theta^k)\,.$$

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Hypothesis (II).

$a \in \mathrm{C}^1(\mathbb{R})$, $b \in \mathrm{C}^2(\mathbb{R})$ are such that $a(x)$, $b(x) \geq 0$, $b'(x) \geq 0$ for all $x \in \mathbb{R}$.

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Hypothesis (III). $W = \widehat{\beta} + \widehat{\gamma}$, where

$\widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to \mathbb{R}$ is proper, l.s.c., convex;, $\mathrm{dom}(\widehat{\beta}) \subseteq [0, +\infty)$ is bounded,

$$\widehat{\gamma} \in \mathrm{C}^2(\mathbb{R}), \quad \exists \, c_w > 0 : \quad W(r) \geq c_w r^2 \quad \forall r \in \mathrm{dom}(\widehat{\beta}) .$$

Hereafter, we shall denote by $\beta = \partial\widehat{\beta}$ the subdifferential of $\widehat{\beta}$, and set $\gamma := \widehat{\gamma}'$.

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Hypothesis (IV).

$$\mathbf{f} \in L^2(0, T; L^2(\Omega)),$$

$$g \in L^1(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega)'), \quad g \geq 0 \quad \text{a.e. in } \Omega \times (0, T) \, ,$$

and that the initial data comply with

$$\theta_0 \in L^1(\Omega), \quad \exists \, \theta_* > 0 \; : \quad \min_{\Omega} \theta_0 \geq \theta_* > 0 \, , \quad \log \theta_0 \in L^1(\Omega),$$

$$\mathbf{u}_0 \in H^1_0(\Omega), \quad \mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^d), \quad \chi_0 \in H^1(\Omega), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega).$$

Existence of weak solutions

Given $\delta > 0$, $\mu = 1$, $\eta = 0$, there exists (measurable) functions

$$\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$
$$\mathbf{u} \in H^1(0,T;H^1_0(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)) \cap H^2(0,T;H^{-1}(\Omega;\mathbb{R}^d)),$$
$$\chi \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \text{for a.e. } x \in \Omega$$
$$\chi(0,x) = \chi_0(x) \qquad \text{for a.e. } x \in \Omega$$

together with

the *entropy inequality*

the *total energy inequality*

the weak momentum equation (in $H^{-1}(\Omega)$)

the *generalized principle of virtual powers*

The entropy inequality + total energy inequality

The entropy inequality + total energy inequality

The *entropy inequality*

$$\int_0^T \int_\Omega (\log(\theta) + \chi)\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \rho \int_0^T \int_\Omega \mathrm{div}(\mathbf{u}_t)\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \mathsf{K}(\theta)\nabla \log(\theta) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq - \int_0^T \int_\Omega \mathsf{K}(\theta)\frac{\varphi}{\theta}\nabla \log(\theta) \cdot \nabla\theta \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \left((a(\chi) + \delta)|\varepsilon(\mathbf{u}_t)|^2 + g + |\chi_t|^2\right) \frac{\varphi}{\theta} \,\mathrm{d}x\,\mathrm{d}t$$

for all $\varphi \in \mathcal{D}(\overline{\Omega} \times [0, T])$ with $\varphi \geq 0$;

The entropy inequality + total energy inequality

The *entropy inequality*

$$\int_0^T \int_\Omega (\log(\theta) + \chi)\varphi_t \, dx \, dt + \rho \int_0^T \int_\Omega \mathrm{div}(\mathbf{u}_t)\varphi \, dx \, dt - \int_0^T \int_\Omega \mathsf{K}(\theta)\nabla \log(\theta) \cdot \nabla\varphi \, dx \, dt$$

$$\leq - \int_0^T \int_\Omega \mathsf{K}(\theta)\frac{\varphi}{\theta}\nabla \log(\theta) \cdot \nabla\theta \, dx \, dt - \int_0^T \int_\Omega \left((a(\chi) + \delta)|\varepsilon(\mathbf{u}_t)|^2 + g + |\chi_t|^2\right) \frac{\varphi}{\theta} \, dx \, dt$$

for all $\varphi \in \mathcal{D}(\overline{\Omega} \times [0, T])$ with $\varphi \geq 0$;

The *total energy inequality* for almost all $t \in (0, T)$

$$\mathscr{E}(\theta(t), \mathbf{u}(t), \mathbf{u}_t(t), \chi(t)) \leq \mathscr{E}(\theta(0), \mathbf{u}(0), \mathbf{u}_t(0), \chi(0)) + \int_0^t \int_\Omega g \, dx \, ds \int_0^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t \, dx \, ds$$

where

$$\mathscr{E}(\theta, \mathbf{u}, \mathbf{u}_t, \chi) := \int_\Omega \theta \, dx + \frac{1}{2}\int_\Omega |\mathbf{u}_t|^2 \, dx + \frac{1}{2}(b(\chi(t)) + \delta)|\varepsilon(\mathbf{u})|^2(t) + \frac{1}{2}\int_\Omega |\nabla\chi|^2 \, dx + \int_\Omega W(\chi) \, dx$$

The entropy inequality + total energy inequality

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In case we **add the $p-$laplacian (i.e. $\eta > 0$) we obtain the total energy identity**

$$\mathscr{E}_p(\theta(t),\mathbf{u}(t),\mathbf{u}_t(t),\chi(t)) = \mathscr{E}_p(\theta(0),\mathbf{u}(0),\mathbf{u}_t(0),\chi(0)) + \int_0^t \int_\Omega g \,\mathrm{d}x\,\mathrm{d}s \int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t \,\mathrm{d}x\,\mathrm{d}s$$

where

$$\mathscr{E}_p(\theta,\mathbf{u},\mathbf{u}_t,\chi) := \mathscr{E}(\theta,\mathbf{u},\mathbf{u}_t,\chi) + \frac{1}{p}\int_\Omega |\nabla\chi|^p \,\mathrm{d}x$$

The generalized principle of virtual powers

The relations: $\chi_t(x,t) \leq 0$ for almost all $(x,t) \in \Omega \times (0,T)$, as well as

$$\int_\Omega \left(\chi_t(t)\varphi + \nabla\chi(t)\cdot\nabla\varphi + \xi(t)\varphi + \gamma(\chi(t))\varphi + b'(\chi(t))\frac{|\varepsilon(\mathbf{u}(t))|^2}{2}\varphi - \theta(t)\varphi \right) \mathrm{d}x \geq 0$$

$$\text{for all } \varphi \in W^{1,2}_-(\Omega), \quad \text{for a.a. } t \in (0,T)$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

$$\xi \in L^1(0,T;L^1(\Omega)) \qquad \text{and} \qquad \langle \xi(t), \varphi - \chi(t) \rangle_{W^{1,2}(\Omega)} \leq 0 \ \ \forall \varphi \in W^{1,2}_+(\Omega), \text{ for a.a. } t \in (0,T)$$

and the energy inequality for all $t \in (0,T]$, for $s = 0$ (in case we **omit the p-laplacian**), and for almost all $0 < s \leq t$ (in case we **add the p-laplacian**):

$$\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \left(\frac{1}{2}|\nabla\chi(t)|^2 + W(\chi(t)) \right) \mathrm{d}x$$

$$\leq \int_\Omega \left(\frac{1}{2}|\nabla\chi(s)|^2 + W(\chi(s)) \right) \mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta \right) \mathrm{d}x\,\mathrm{d}r$$

where

$$W^{1,2}_+(\Omega) := \{ \zeta \in W^{1,2}(\Omega) \,:\, \zeta(x) \geq 0 \quad \text{for a.a. } x \in \Omega \} \qquad \text{and analogously for } W^{1,2}_-(\Omega)$$

Generalized principle of virtual powers vs classical phase inclusion

Generalized principle of virtual powers vs classical phase inclusion

- If $(w, \mathbf{u}, \chi)$ are "more regular" and satisfy the notion of *weak solution*: the one-sided inequality $(\forall \varphi \in L^2(0,T; W^{1,2}_{-}(\Omega)) \cap L^\infty(Q))$:

$$\int_0^T \int_\Omega \chi_t \varphi + \nabla\chi\nabla\varphi + \xi\varphi + \gamma(\chi)\varphi + b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \theta\varphi \geq 0 \qquad \text{(one-sided)}$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ and the energy inequality:

$$\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}|\nabla\chi(t)|^2 + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\leq \frac{1}{2}|\nabla\chi(s)|^2 + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta\right)\mathrm{d}x\,\mathrm{d}r \qquad \text{(energy)}$$

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  the one-sided inequality $(\forall \varphi \in L^2(0,T; W^{1,2}_{-}(\Omega)) \cap L^\infty(Q))$:

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  with $\xi \in \partial I_{[0,+\infty)}(\chi)$ and the energy inequality:

$$\begin{aligned}&\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}|\nabla \chi(t)|^2 + \int_\Omega W(\chi(t))\,\mathrm{d}x \\ &\leq \frac{1}{2}|\nabla \chi(s)|^2 + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta\right)\mathrm{d}x\,\mathrm{d}r\end{aligned} \qquad \text{(energy)}$$

- "Differentiating in time" the energy inequality (energy) and using the chain rule, we conclude that $(w, \mathbf{u}, \chi, \xi)$ comply with

$$\left\langle \chi_t(t) - \Delta\chi(t) + \xi(t) + \gamma(\chi(t)) + b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} - \theta(t), \chi_t(t)\right\rangle_{W^{1,2}(\Omega)} \leq 0 \text{ for a.e. } t \quad \text{(ineq)}$$

  (one-sided) − (ineq) + "$\chi_t \leq 0$ a.e." are equivalent to the usual phase inclusion

$$\chi_t - \Delta\chi + \xi + \gamma(\chi) + b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} - \theta \in -\partial I_{(-\infty,0]}(\chi_t) \text{ in } W^{1,2}(\Omega)^*$$

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- In case $\eta > 0$, we also get the strong convergence of χ_n to χ in $L^p(0,T;W^{1,p}(\Omega))$, using the compact embedding of $W^{1,p}(\Omega)$ in $C^0(\bar{\Omega})$ for $p > d \Longrightarrow$ total energy **identity**

$$\mathscr{E}_p(\theta(t),\mathbf{u}(t),\mathbf{u}_t(t),\chi(t)) = \mathscr{E}_p(\theta(0),\mathbf{u}(0),\mathbf{u}_t(0),\chi(0)) + \int_0^t \int_\Omega g \,\mathrm{d}x\,\mathrm{d}s \int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t \,\mathrm{d}x\,\mathrm{d}s$$

and the energy inequality for χ for all $t \in (0,T]$ and **for almost all** $0 < s \leq t$:

$$\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \left(\frac{1}{2}|\nabla\chi(t)|^2 + \frac{1}{p}|\nabla\chi(t)|^p + W(\chi(t))\right)\mathrm{d}x$$
$$\leq \int_\Omega \left(\frac{1}{2}|\nabla\chi(s)|^2 + \frac{1}{p}|\nabla\chi(s)|^p + W(\chi(s))\right)\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta\right)\mathrm{d}x\,\mathrm{d}r$$

Positivity of θ

From the θ-equation

$$\theta_t + \chi_t\theta + \rho\theta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(\mathsf{K}(\theta)\nabla\theta)) = g + (a(\chi)+\delta)|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2$$

we get

$$\theta_t - \mathrm{div}(\mathsf{K}(\theta)\nabla\theta) \geq -\frac{1}{2}\theta^2$$

and so the function $h(t)$ solving

$$h_t = -\frac{1}{2}h^2, \quad h(0) = \theta_* > 0$$

is a subsolution of the θ-equation. Hence, we get

$$\theta(t,\cdot) \geq h(t) > \theta_* > 0 \quad \text{for all } t \in [0,T]$$

A priori estimates

$$\theta_t + \chi_t\theta + \rho\theta \operatorname{div} \mathbf{u}_t - \operatorname{div}(\mathsf{K}(\theta)\nabla\theta)) = g + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2 \quad (1)$$

$$\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f} \quad (2)$$

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta \quad (3)$$

Energy estimate. $\int_0^t((1)\times 1 + (\text{momentum})\times \mathbf{u}_t + (3)\times\chi_t)\,\mathrm{d}s$ gives an estimate for

$\|\theta\|_{L^\infty(0,T;L^1(\Omega))}$, $\|\mathbf{u}\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))}$, $\|(b(\chi)+\delta)^{1/2}\varepsilon(\mathbf{u})\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}$, $\|\chi\|_{L^\infty(0,T;H^1(\Omega))}$

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$$\|\theta\|_{L^m(0,T;L^m(\Omega))} \le C \qquad \text{for all } \frac{7}{6} \le m < \frac{5}{3}$$

and using the Hyp. on K ($\mathrm{K}(\theta) \ge c_1(1+\nu\theta^k)$):

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Fifth estimate. By comparison, we get (for some $\alpha > 1$ depending on d (the dimension of Ω))

$$\|(\log\theta)_t\|_{L^1(0,T;(W^{2,\alpha}(\Omega))^*)} + \|\mathbf{u}_{tt}\|_{L^2(0,T;H^{-1}(\Omega))} \le C$$

Weak sequential stability

$$\theta_n \to \theta \quad \text{weakly star in } L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{1}$$

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly star in } H^2(0,T;H^{-1}(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)), \tag{2}$$

$$\partial_t \mathbf{u}_n \to \partial_t \mathbf{u} \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{3}$$

$$\chi_n \to \chi \quad \text{weakly star in } H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)), \tag{4}$$

$$\log(\theta_n) \to v \quad \text{strongly in } L^2(0,T;L^s(\Omega)), \tag{5}$$

for some $s \in (1,6)$ for $d=3$ whence $\log(\theta_n) \to v$ a.e. and so $v = \log\theta$ and $\theta_n \to \theta$ a.e. and we have

$$\theta_n \to \theta \quad \text{strongly in } L^h(\Omega \times (0,T)), \text{ for every } h \in [1, 8/3) \text{ for } d=3$$
$$\chi_n \to \chi \quad \text{strongly in } L^q(\Omega \times (0,T)) \quad \forall q \in [1,+\infty)$$

Test the approximated $\mathbf{u}$-equation by $\partial_t(\mathbf{u}_n - \mathbf{u})$, where $\mathbf{u}$ is the limit of $\mathbf{u}_n$ obtained in the previous convergence. Hence, we finally get

$$\|(\mathbf{u}_n - \mathbf{u})_t(t)\|^2_{L^2(\Omega)} + \int_0^t (a(\chi_n)+\delta)|(\varepsilon(\mathbf{u})_n - \varepsilon(\mathbf{u}))_t|^2 \, ds + (b(\chi_n(t)+\delta)|(\varepsilon(\mathbf{u})_n - \varepsilon(\mathbf{u}))(t)|^2 \to 0$$

as $n \nearrow \infty$, which entails

$$\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in } W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega))$$

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but via weak convergences and lower semicontinuity we obtain

– The *entropy inequality* (for all $\varphi \in \mathcal{D}(\overline{\Omega} \times [0, T])$ with $\varphi \geq 0$)

$$\int_0^T \int_\Omega (\log(\theta) + \chi)\varphi_t \, \mathrm{d}x \, \mathrm{d}t + \rho \int_0^T \int_\Omega \operatorname{div}(\mathbf{u}_t)\varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \mathsf{K}(\theta)\nabla \log(\theta) \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq - \int_0^T \int_\Omega \mathsf{K}(\theta)\frac{\varphi}{\theta}\nabla \log(\theta) \cdot \nabla\theta \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \left((a(\chi) + \delta)|\varepsilon(\mathbf{u}_t)|^2 + g + |\chi_t|^2\right) \frac{\varphi}{\theta} \, \mathrm{d}x \, \mathrm{d}t$$

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– The *total energy inequality* for almost all $t \in (0, T)$

$$\mathscr{E}(\theta(t), \mathbf{u}(t), \mathbf{u}_t(t), \chi(t)) \leq \mathscr{E}(\theta(0), \mathbf{u}(0), \mathbf{u}_t(0), \chi(0)) + \int_0^t \int_\Omega g \,\mathrm{d}x\,\mathrm{d}s \int_0^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t \,\mathrm{d}x\,\mathrm{d}s$$

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– The generalized principle of virtual powers for all $\varphi \in W^{1,2}_-(\Omega)$, for a.a. $t \in (0, T)$

$$\int_\Omega \left(\chi_t(t)\varphi + \nabla\chi(t) \cdot \nabla\varphi + \xi(t)\varphi + \gamma(\chi(t))\varphi + b'(\chi(t))\frac{|\varepsilon(\mathbf{u}(t))|^2}{2}\varphi - \theta(t)\varphi\right) \mathrm{d}x \geq 0\,;$$

$$\int_0^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \int_\Omega \left(\frac{1}{2}|\nabla\chi(t)|^2 + W(\chi(t))\right) \mathrm{d}x \leq \int_\Omega \left(\frac{1}{2}|\nabla\chi_0|^2 + W(\chi_0)\right) \mathrm{d}x$$

$$+ \int_0^t \int_\Omega \chi_t \left(-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta\right) \mathrm{d}x\,\mathrm{d}r\,, \text{ with } \xi \in \partial I_{[0,+\infty)}(\chi) \text{ and for all } t \in (0, T]$$

The reversible phase transitions in thermoviscoelastic materials: $\mu = 0$ and $\eta > 0$

$$\theta_t + \chi_t\theta + \rho\theta \,\mathrm{div}\, \mathbf{u}_t - \mathrm{div}(\mathsf{K}(\theta)\nabla\theta)) = g + (a(\chi) + \delta)|\varepsilon(\mathbf{u}_t)|^2 + |\chi_t|^2$$

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Hypothesis on W

Beside, **Hypothesis (III).** We suppose that the potential W is given by $W = \widehat{\beta} + \widehat{\gamma}$, where

$$\widehat{\beta} : \mathrm{dom}(\widehat{\beta}) \to \mathbb{R} \text{ is proper, l.s.c., convex;}$$

$$\widehat{\gamma} \in \mathrm{C}^2(\mathbb{R}), \quad \exists\, c_w > 0 : \quad W(r) \geq c_w r^2 \quad \forall r \in \mathrm{dom}(\widehat{\beta}) .$$

Hereafter, we shall denote by $\beta = \partial\widehat{\beta}$ the subdifferential of $\widehat{\beta}$, and set $\gamma := \widehat{\gamma}'$

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Hereafter, we shall denote by $\beta = \partial\widehat{\beta}$ the subdifferential of $\widehat{\beta}$, and set $\gamma := \widehat{\gamma}'$

Assume that $\widehat{\beta}$ is a regular potential satisfying a suitable growth condition (depending on d):
Hypothesis (IV). $W \in C^1(\mathbb{R})$ and there exist $C_w > 0$ and $p \in (1, 6)$ if $d = 3$, $p \in (1, +\infty)$ if $d = 2$ such that

$$|W'(r)| \leq C_w(1 + |r|^p) \quad \forall r \in \mathbb{R}$$

Existence of weak solutions

Given $\delta > 0$, $\mu = 0$, $\eta > 0$, there exists functions

$$\theta \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$
$$\mathbf{u} \in H^1(0,T;H_0^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H_0^1(\Omega)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d)),$$
$$\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega)),$$

fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \text{for a.e. } x \in \Omega,$$
$$\chi(0,x) = \chi_0(x) \qquad \text{for a.e. } x \in \Omega,$$

- the *entropy inequality*
- the *total energy identity*
- the weak momentum (in $H^{-1}(\Omega)$) and phase (in $(W^{1,p}(\Omega))^*$) - equations

The further estimate in case $\eta > 0$

- Being $\eta > 0$ in the χ-equation

$$\chi_t - \Delta\chi - \eta\Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \theta \qquad \text{(phase)}$$

we can test the **u**-equation

$$\mathbf{u}_{tt} - \operatorname{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + (b(\chi) + \delta)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f} \qquad \text{(momentum)}$$

by $-\operatorname{div}(\varepsilon(\mathbf{u}_t))$, using the $L^\infty(0, T; W^{1,p}(\Omega))$-regularity of χ

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$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi)+\delta)\varepsilon(\mathbf{u}_t) + (b(\chi)+\delta)\varepsilon(\mathbf{u}) - \rho\theta\mathbf{1}) = \mathbf{f} \qquad \text{(momentum)}$$

by $-\mathrm{div}(\varepsilon(\mathbf{u}_t))$, using the $L^\infty(0,T;W^{1,p}(\Omega))$-regularity of χ

- This allows us to obtain an estimate for **u** in

$$H^1(0,T;H^2_0(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$$

which allows us to pass to the limit in the quadratic term $|\varepsilon(\mathbf{u})|^2$ in (phase)

The further estimate in case $\eta > 0$

- Being $\eta > 0$ in the χ-equation

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  which allows us to pass to the limit in the quadratic term $|\varepsilon(\mathbf{u})|^2$ in (phase)

- By using the compact embedding of $W^{1,p}(\Omega)$ in $C^0(\bar{\Omega})$, we can get a strong convergence of χ_n to χ in $L^p(0,T;W^{1,p}(\Omega))$ allowing us to obtain the *total energy identity* (not only inequality)

$$\mathscr{E}(\theta(t),\mathbf{u}(t),\mathbf{u}_t(t),\chi(t)) = \mathscr{E}(\theta(0),\mathbf{u}(0),\mathbf{u}_t(0),\chi(0)) + \int_0^t\int_\Omega g\,\mathrm{d}x\,\mathrm{d}s \int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}s$$

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- The point probably is that the **generalized principle of virtual powers is taylored specifically on the irreversible** case and does not fit to the reversible one
- A different (weaker) notion of phase equation would be needed in the reversible case $\Longrightarrow$ This is still an **OPEN PROBLEM!**

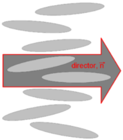
A further application to liquid crystals

- In [E. Feireisl, M. Frémond, E.R., G. Schimperna, ARMA 2012] we have coupled the incompressible Nevier-Stokes equation

$$\mathrm{div}\,\mathbf{v} = 0, \quad \mathbf{v}_t + \mathbf{v}\cdot\nabla_x\mathbf{v} + \nabla_x p = \mathrm{div}\,\mathbb{S} + \mathrm{div}\,\sigma^{nd} + \mathbf{g}$$

$$\mathbb{S} = \nu(\theta)\left(\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v}\right), \quad \sigma^{nd} = -\nabla_x\mathbf{d}\odot\nabla_x\mathbf{d} + (\partial_{\mathbf{d}}W(\mathbf{d}) - \Delta\mathbf{d})\otimes\mathbf{d}$$

where $\nabla_x\mathbf{d}\odot\nabla_x\mathbf{d}$ is the 3×3 matrix given by $\nabla_i\mathbf{d}\cdot\nabla_j\mathbf{d}$, $(\mathbf{a}\otimes\mathbf{b})_{ij} := a_i b_j$, $1\le i,j\le 3$, and the evolution of the director field $\mathbf{d}$, representing preferred orientation of molecules in a neighborhood of any point of a reference domain



$$\mathbf{d}_t + \mathbf{v}\cdot\nabla_x\mathbf{d} - \mathbf{d}\cdot\nabla_x\mathbf{v} = \Delta\mathbf{d} - \partial_{\mathbf{d}}W(\mathbf{d})$$

with an **entropic formulation** of the inernal energy balance displaying higher order nonlinearities on the right hand side

$$\theta_t + \mathbf{v}\cdot\theta + \mathrm{div}\,\mathbf{q} = \mathbb{S} : \nabla_x\mathbf{v} + |\Delta\mathbf{d} - \partial_{\mathbf{d}}W(\mathbf{d})|^2$$

- In [E. Feireisl, E.R., G. Schimperna, A. Zarnescu, Comm. Math. Sci., to appear] we have extended it to the tensorial Ball-Majumdar model for liquid crystals

A further application to two phase fluids

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$$\operatorname{div}\mathbf{v} = 0\,, \quad \partial_t\mathbf{v} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla p = \operatorname{div}\mathbb{S} - \mu\nabla_x\chi\,, \quad \mathbb{S} = \nu(\theta,\chi)\left(\nabla_x\mathbf{v} + \nabla_x^t\mathbf{v}\right)$$
$$\partial_t\theta + \lambda(\theta)\left(\chi_t + \mathbf{v}\cdot\nabla_x\chi\right) + \operatorname{div}(\theta\mathbf{v}) + \operatorname{div}\mathbf{q} = \mathbb{S} : \nabla_x\mathbf{v} + |\nabla_x\mu|^2$$
$$\partial_t\chi + \mathbf{v}\cdot\nabla_x\chi = \Delta\mu\,, \quad \mu = -\Delta\chi + W'(\chi) - \lambda(\theta)$$

Entropic notion of solution is needed in order to interpret the internal energy balance

Thanks for your attention!

cf. http://www.mat.unimi.it/users/rocca/