On a hyperbolic system arising in liquid crystals modeling

E. Rocca

Università degli Studi di Pavia

Workshop on Mathematical Fluid Dynamics
Bad Boll, May 7–11, 2018

jointly with
Eduard Feireisl (Prague)-Giulio Schimperna (Pavia)-Arghir Zarnescu (Bilbao and Bucharest)

Fondazione Cariplo and Regione Lombardia Grant MEGAsTaR 2016-2019
Outline

1. Introduction of the Problem
2. The LCs modeling
3. The simplified Qian-Sheng model
4. The existence of local-in-time smooth solutions
5. Existence of global-in-time dissipative solutions
6. Weak-strong uniqueness
7. Further perspectives
Outline

1. Introduction of the Problem
2. The LCs modeling
3. The simplified Qian-Sheng model
4. The existence of local-in-time smooth solutions
5. Existence of global-in-time dissipative solutions
6. Weak-strong uniqueness
7. Further perspectives
We present here the results of the following joint work

Introduction

We present here the results of the following joint work


We consider a model of liquid crystals (LCs), based on a nonlinear hyperbolic system of differential equations, that represents an inviscid version of the model proposed in


and studied analytically in [F. de Anna, A. Zarnescu, arXiv:1608.08872]
**Introduction**

We present here the results of the following joint work


We consider a model of liquid crystals (LCs), based on a nonlinear hyperbolic system of differential equations, that represents an inviscid version of the model proposed in


and studied analytically in [F. de Anna, A. Zarnescu, arXiv:1608.08872]

Here is the plan of the talk:
Introduction

We present here the results of the following joint work


We consider a model of liquid crystals (LCs), based on a nonlinear hyperbolic system of differential equations, that represents an inviscid version of the model proposed in


and studied analytically in [F. de Anna, A. Zarnescu, arXiv:1608.08872]

Here is the plan of the talk:

- Introduce the Liquid Crystals modeling and $\mathbb{Q}$-tensors formalism
- Present a simplification of the Qian-Sheng (QS-model) and the resulting PDEs
- Propose the concept of dissipative solution, for which a global-in-time existence theorem is shown
- The main advantages and the potential future perspectives
The motivation

- Liquid crystals are a state of matter that have properties between those of a conventional liquid and those of a solid crystal. A liquid crystal may flow like a liquid, but its molecules may be oriented in a crystal-like way.

- Theoretical studies of these types of materials are motivated by real-world applications: proper functioning of many practical devices relies on optical properties of certain liquid crystalline substances in the presence or absence of an electric field: a multi-billion dollar industry.

- At the molecular level, what marks the difference between a liquid crystal and an ordinary, isotropic fluid is that, while the centers of mass of LC molecules do not exhibit any long-range correlation, molecular orientations do exhibit orientational correlations.
To the present state of knowledge, three main types of liquid crystals are distinguished, termed *smectic*, *nematic* and *cholesteric*

http://www.laynetworks.com/Molecular-Orientation-in-Liquid-Crystal-Phases.htm

**The smectic** phase forms well-defined layers that can slide one over another in a manner very similar to that of a soap

**The nematic** phase: the molecules have long-range orientational order, but no tendency to the formation of layers. Their center of mass positions all point in the **same direction** (within each specific domain)

**Crystals in the cholesteric** phase exhibit a twisting of the molecules perpendicular to the director, with the molecular axis parallel to the director
Our main aim

We consider the range of temperatures typical for the nematic phase. The nematic liquid crystals are composed of rod-like molecules, with the long axes of neighboring molecules aligned. Most mathematical work has been done on the Oseen-Frank theory, in which the mean orientation of the rod-like molecules is described by a vector field. However, more popular among physicists is the Landau-de Gennes theory, in which the order parameter describing the orientation of molecules is a matrix, the so-called $Q$-tensor. The flow velocity $u$ evidently disturbs the alignment of the molecules and also the converse is true: a change in the alignment will produce a perturbation of the velocity field $u$. Moreover, also changes of the temperature or effects of magnetic and electric fields should be considered (in the future maybe).
Our main aim

- We consider the range of temperatures typical for the nematic phase

Our main aim

- We consider the range of temperatures typical for the nematic phase.


- The nematic liquid crystals are composed of rod-like molecules, with the long axes of neighboring molecules aligned.
Our main aim

- We consider the range of temperatures typical for the nematic phase
  

- The nematic liquid crystals are composed of rod-like molecules, with the long axes of neighboring molecules aligned

- Most mathematical work has been done on the Oseen-Frank theory, in which the mean orientation of the rod-like molecules is described by a vector field $d$. However, more popular among physicists is the Landau-de Gennes theory, in which the order parameter describing the orientation of molecules is a matrix, the so-called $Q$-tensor
Our main aim

- We consider the range of temperatures typical for the nematic phase.


- The nematic liquid crystals are composed of rod-like molecules, with the long axes of neighboring molecules aligned.

- Most mathematical work has been done on the Oseen-Frank theory, in which the mean orientation of the rod-like molecules is described by a vector field $d$. However, more popular among physicists is the Landau-de Gennes theory, in which the order parameter describing the orientation of molecules is a matrix, the so-called $Q$-tensor.

- The flow velocity $u$ evidently disturbs the alignment of the molecules and also the converse is true: a change in the alignment will produce a perturbation of the velocity field $u$. 
Our main aim

- We consider the range of temperatures typical for the nematic phase


- The nematic liquid crystals are composed of rod-like molecules, with the long axes of neighboring molecules aligned

- Most mathematical work has been done on the Oseen-Frank theory, in which the mean orientation of the rod-like molecules is described by a vector field $\mathbf{d}$. However, more popular among physicists is the Landau-de Gennes theory, in which the order parameter describing the orientation of molecules is a matrix, the so-called $\mathbf{Q}$-tensor

- The flow velocity $\mathbf{u}$ evidently disturbs the alignment of the molecules and also the converse is true: a change in the alignment will produce a perturbation of the velocity field $\mathbf{u}$

- Moreover, also changes of the temperature or effects of magnetic and electric fields should be considered (in the future maybe)
Outline

1 Introduction of the Problem

2 The LCs modeling

3 The simplified Qian-Sheng model

4 The existence of local-in-time smooth solutions

5 Existence of global-in-time dissipative solutions

6 Weak-strong uniqueness

7 Further perspectives
The Landau-de Gennes theory: the molecular orientation

- Consider a nematic liquid crystal filling a bounded connected container $\Omega$ in $\mathbb{R}^3$ with "regular" boundary
The Landau-de Gennes theory: the molecular orientation

- Consider a nematic liquid crystal filling a bounded connected container $\Omega$ in $\mathbb{R}^3$ with “regular” boundary.

- The distribution of molecular orientations in a ball $B(x_0, \delta)$, $x_0 \in \Omega$ can be represented as a probability measure $\mu$ on the unit sphere $S^2$ satisfying $\mu(E) = \mu(-E)$ for $E \subset S^2$.

- For a continuously distributed measure we have $d\mu(p) = \rho(p)dp$ where $dp$ is an element of the surface area on $S^2$ and $\rho \geq 0$, $\int_{S^2} \rho(p)dp = 1$, $\rho(p) = \rho(-p)$.
The Landau-de Gennes theory: the $Q$-tensor

- The first moment $\int_{S^2} p \, d\mu(p) = 0$, the second moment $M = \int_{S^2} p \otimes p \, d\mu(p)$ is a symmetric non-negative $3 \times 3$ matrix (for every $v \in S^2$, $v \cdot M \cdot v = \int_{S^2} (v \cdot p)^2 \, d\mu(p) = < \cos^2 \theta >$, where $\theta$ is the angle between $p$ and $v$) satisfying $\text{tr}(M) = 1$.
The Landau-de Gennes theory: the $\mathcal{Q}$-tensor

The first moment $\int_{S^2} p \, d\mu(p) = 0$, the second moment $M = \int_{S^2} p \otimes p \, d\mu(p)$ is a symmetric non-negative $3 \times 3$ matrix (for every $v \in S^2$, $v \cdot M \cdot v = \int_{S^2} (v \cdot p)^2 \, d\mu(p) = \langle \cos^2 \theta \rangle$, where $\theta$ is the angle between $p$ and $v$) satisfying $\text{tr}(M) = 1$

If the orientation of molecules is equally distributed in all directions (the distribution is isotropic) and then $\mu = \mu_0$, where $d\mu_0(p) = \frac{1}{4\pi} \, dS$. In this case the second moment tensor is $M_0 = \frac{1}{4\pi} \int_{S^2} p \otimes p \, dS = \frac{1}{3} \, 1$, because $\int_{S^2} p_1 p_2 \, dS = 0$, $\int_{S^2} p_1^2 \, dS = \int_{S^2} p_2^2 \, dS$, etc., and $\text{tr}(M_0) = 1$
The Landau-de Gennes theory: the $Q$-tensor

- The first moment $\int_{S^2} p \, d\mu(p) = 0$, the second moment $M = \int_{S^2} p \otimes p \, d\mu(p)$ is a symmetric non-negative $3 \times 3$ matrix (for every $v \in S^2$, $v \cdot M \cdot v = \int_{S^2} (v \cdot p)^2 \, d\mu(p) = \langle \cos^2 \theta \rangle$, where $\theta$ is the angle between $p$ and $v$) satisfying $\text{tr}(M) = 1$

- If the orientation of molecules is equally distributed in all directions (the distribution is isotropic) and then $\mu = \mu_0$, where $d\mu_0(p) = \frac{1}{4\pi} \, dS$. In this case the second moment tensor is $M_0 = \frac{1}{4\pi} \int_{S^2} p \otimes p \, dS = \frac{1}{3} 1$, because $\int_{S^2} p_1 p_2 \, dS = 0$, $\int_{S^2} p_1^2 \, dS = \int_{S^2} p_2^2 \, dS$, etc., and $\text{tr}(M_0) = 1$

- **The de Gennes $Q$-tensor** measures the deviation of $M$ from its isotropic value

\[
Q = M - M_0 = \int_{S^2} \left( p \otimes p - \frac{1}{3} 1 \right) \, d\mu(p)
\]

- Note that (cf. [Ball, Majumdar, Molecular Crystals and Liquid Crystals (2010)])
  1. $Q = Q^T$
  2. $\text{tr}(Q) = 0$
  3. $Q \geq -\frac{1}{3} 1$
The Landau-de Gennes theory: the $Q$-tensor

- The first moment $\int_{S^2} p \, d\mu(p) = 0$, the second moment $M = \int_{S^2} p \otimes p \, d\mu(p)$ is a symmetric non-negative $3 \times 3$ matrix (for every $v \in S^2$, $v \cdot M \cdot v = \int_{S^2} (v \cdot p)^2 \, d\mu(p) = \langle \cos^2 \theta \rangle$, where $\theta$ is the angle between $p$ and $v$) satisfying $\text{tr}(M) = 1$

- If the orientation of molecules is equally distributed in all directions (the distribution is isotropic) and then $\mu = \mu_0$, where $d\mu_0(p) = \frac{1}{4\pi} dS$. In this case the second moment tensor is $M_0 = \frac{1}{4\pi} \int_{S^2} p \otimes p \, dS = \frac{1}{3} 1$, because $\int_{S^2} p_1 p_2 \, dS = 0$, $\int_{S^2} p_1^2 \, dS = \int_{S^2} p_2^2 \, dS$, etc., and $\text{tr}(M_0) = 1$

The de Gennes $Q$-tensor measures the deviation of $M$ from its isotropic value

$$Q = M - M_0 = \int_{S^2} \left(p \otimes p - \frac{1}{3} 1\right) \, d\mu(p)$$

Note that (cf. [Ball, Majumdar, Molecular Crystals and Liquid Crystals (2010)])

1. $Q = Q^T$
2. $\text{tr}(Q) = 0$
3. $Q \geq -\frac{1}{3} 1$

1.+2. implies $Q = \lambda_1 n_1 \otimes n_1 + \lambda_2 n_2 \otimes n_2 + \lambda_3 n_3 \otimes n_3$, where $\{n_i\}$ is an othonormal basis of eigenvectors of $Q$ with corresponding eigenvalues $\lambda_i$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$

2.+3. implies $-\frac{1}{3} \leq \lambda_i \leq \frac{2}{3}$

$$Q = 0$$ does not imply $\mu = \mu_0$ (e.g. $\mu = \frac{1}{6} \sum_{i=1}^3 (\delta e_i + \delta -e_i)$)
The reduction to the Oseen-Frank model

If the eigenvalues of $Q$ are all distinct then $Q$ is said to be biaxial (biaxiality implies the existence of more than one preferred direction of molecular alignment).

If two $\lambda_i$ are equal then $Q$ is said to be uniaxial (liquid crystal materials with a single preferred direction of molecular alignment).

Reduction to the Oseen-Frank (1925, 1952) model (Ericksen model, 1991): the uniaxial case:

$\lambda_1 = \lambda_2 = -s_3$, $\lambda_3 = 2s_3$, setting $n_3 = d$ where $n_i$ is an orthonormal basis of eigenvectors of $Q$ corresponding to $\lambda_i$, we have

$Q = -s_3 (1 - d \otimes d) + 2s_3 d \otimes d = s_3 (d \otimes d - \frac{1}{3} 1_{3 \times 3})$,

where $-\frac{1}{2} \leq s \leq 1$.

Here $s \in \mathbb{R}$ is a real scalar order parameter that measures the degree of orientational ordering and $d$ is a vector representing the direction of preferred molecular alignment: the director field.
The reduction to the Oseen-Frank model

- If the eigenvalues of $\mathbb{Q}$ are all distinct then $\mathbb{Q}$ is said to be \textit{biaxial} (biaxiality implies the existence of more than one preferred direction of molecular alignment).
- If two $\lambda_i$ are equal then $\mathbb{Q}$ is said to be \textit{uniaxial} (liquid crystal materials with a single preferred direction of molecular alignment).
The reduction to the Oseen-Frank model

- If the eigenvalues of $Q$ are all distinct then $Q$ is said to be \textit{biaxial} (biaxiality implies the existence of more than one preferred direction of molecular alignment)

- If two $\lambda_i$ are equal then $Q$ is said to be \textit{uniaxial} (liquid crystal materials with a single preferred direction of molecular alignment)

Reduction to the Oseen-Frank (1925, 1952) model (Ericksen model, 1991): the \textit{uniaxial case}:

$\lambda_1 = \lambda_2 = -\frac{s}{3}$, $\lambda_3 = \frac{2s}{3}$, setting $n_3 = d$ where $n_i$ is an orthonormal basis of eigenvectors of $Q$ corresponding to $\lambda_i$, we have

$$Q = -\frac{s}{3} (1 - d \otimes d) + \frac{2s}{3} d \otimes d = s \left( d \otimes d - \frac{1}{3} 1 \right),$$

where $-\frac{1}{2} \leq s \leq 1$.

Here $s \in \mathbb{R}$ is a real scalar order parameter that measures the degree of orientational ordering and $d$ is a vector representing the direction of preferred molecular alignment: the director field.
The Landau-de Gennes free energy

Suppose (for the moment) that the material is incompressible, homogeneous and at a constant temperature $T$ in $\Omega$. At each $x \in \Omega$ we have an order parameter tensor $Q(x)$ and the Landau-de Gennes free energy (defined in the space of traceless symmetric $3 \times 3$ matrixes) is

$$F_{LG}(Q) = \int_{\Omega} \left( L^2 |\nabla Q(x)|^2 + f_B(Q(x)) \right) dx,$$

where $|\nabla Q|^2 = \sum_{i,j,k=1}^{3} Q_{ij}, Q_{jk}$ is the elastic energy density that penalizes spatial inhomogeneities and $L > 0$ is a material-dependent elastic constant. $f_B(Q)$ is the bulk free energy density, e.g., (following [de Gennes, Prost (1995)])

$$f_B(Q) = \alpha (T - T^*)^2 \text{tr}(Q^2) - b 3 \text{tr}(Q^3) + c 4 \left( \text{tr}(Q^2) \right)^2$$

where $\alpha$, $b$, $c$ are material-dependent positive constants, $T$ is the absolute temperature and $T^*$ is a characteristic liquid crystal temperature. Call $a = \alpha (T - T^*)$. 
The Landau-de Gennes free energy

Suppose (for the moment) that the material is incompressible, homogeneous and at a constant temperature $T$ in $\Omega$. At each $x \in \Omega$ we have an order parameter tensor $Q(x)$ and the Landau-de Gennes free energy (defined in the space of traceless symmetric $3 \times 3$ matrixes) is

$$\mathcal{F}_{LG}(Q) = \int_{\Omega} \left( \frac{L}{2} |\nabla Q(x)|^2 + f_B(Q(x)) \right) \, dx ,$$

where $|\nabla Q(x)|^2 = \sum_{i,j,k=1}^3 Q_{ij,k}Q_{ij,k}$ is the elastic energy density that penalizes spatial inhomogeneities and $L > 0$ is a material-dependent elastic constant $f_B(Q(x))$ is the bulk free energy density, e.g., following [de Gennes, Prost (1995)]

$$f_B(Q(x)) = \alpha \left( T - T^* \right)^2 \text{tr} \left( Q^2 \right) - b \text{tr} \left( Q^3 \right) + c \left( \text{tr} \left( Q^2 \right) \right)^2 \text{where} \ \alpha, b, c \text{are material-dependent positive constants,} \ T \text{is the absolute temperature and} \ T^* \text{is a characteristic liquid crystal temperature.}$$
The Landau-de Gennes free energy

Suppose (for the moment) that the material is incompressible, homogeneous and at a constant temperature $T$ in $\Omega$. At each $x \in \Omega$ we have an order parameter tensor $Q(x)$ and the Landau-de Gennes free energy (defined in the space of traceless symmetric $3 \times 3$ matrixes) is

$$F_{LG}(Q) = \int_{\Omega} \left( \frac{L}{2} |\nabla Q(x)|^2 + f_B(Q(x)) \right) \, dx,$$

where

- $|\nabla Q|^2 = \sum_{i,j,k=1}^{3} Q_{ij,k} Q_{ij,k}$ is the elastic energy density that penalizes spatial inhomogeneities and $L > 0$ is a material-dependent elastic constant
- $f_B(Q)$ is the bulk free energy density, e.g., (following [de Gennes, Prost (1995)])

$$f_B(Q) = \frac{\alpha(T - T^*)}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} (\text{tr}(Q^2))^2$$

where $\alpha, b, c$ are material-dependent positive constants, $T$ is the absolute temperature and $T^*$ is a characteristic liquid crystal temperature. Call $a = \alpha(T - T^*)$
The Oseen-Frank free energy

It can be shown (cf. [Majumdar, Zarnescu, ARMA (2010)]) that, if $L$ is small in

$$F_{LG}(Q) = \int_{\Omega} \left( L^2 |\nabla Q(x)|^2 + f_B(Q(x)) \right) dx,$$

it is reasonable to consider a theory where $Q$ is required to be uniaxial with constant scalar order parameter $s > 0$, i.e.

$$Q = s (d \otimes d - \frac{1}{3} I).$$

Here $d = d(x) \in S^2$ represents the preferred direction of molecular alignment. In this case $f_B$ is constant and we can consider only the elastic energy and calculating it in terms of $d$ we obtain the simplest form of the Oseen-Frank free energy (1925, 1958)

$$F_{OF} = L s^2 \int_{\Omega} |\nabla d(x)|^2 dx.$$
The Oseen-Frank free energy

- It can be shown (cf. [Majumdar, Zarnescu, ARMA (2010)]) that, if $L$ is small in

$$
\mathcal{F}_{LG}(Q) = \int_{\Omega} \left( \frac{L}{2} |\nabla Q(x)|^2 + f_B(Q(x)) \right) \, dx,
$$

it is reasonable to consider a theory where $Q$ is required to be uniaxial with constant scalar order parameter $s > 0$, i.e.

$$
Q = s \left( d \otimes d - \frac{1}{3} 1 \right).
$$

Here $d = d(x) \in S^2$ represents the preferred direction of molecular alignment.
The Oseen-Frank free energy

- It can be shown (cf. [Majumdar, Zarnescu, ARMA (2010)]) that, if $L$ is small in

$$\mathcal{F}_{LG}(Q) = \int_{\Omega} \left( \frac{L}{2} |\nabla Q(x)|^2 + f_B(Q(x)) \right) \, dx,$$

it is reasonable to consider a theory where $Q$ is required to be uniaxial with constant scalar order parameter $s > 0$, i.e.

$$Q = s \left( d \otimes d - \frac{1}{3} 1 \right).$$

Here $d = d(x) \in S^2$ represents the preferred direction of molecular alignment.

- In this case $f_B$ is constant and we can consider only the elastic energy and calculating it in terms of $d$ we obtain the simplest form of the Oseen-Frank free energy (1925, 1958)

$$\mathcal{F}_{OF} = Ls^2 \int_{\Omega} |\nabla d(x)|^2 \, dx$$
The Ball-Majumdar singular potential

In the Landau-de Gennes free energy there is no a-priori bound on the eigenvalues in order to naturally enforce the physical constraints in the eigenvalues of the symmetric, traceless tensors $Q$. Ball and Majumdar have recently introduced in [Ball, Majumdar, Molecular Crystals and Liquid Crystals (2010)] a singular component

$$f(Q) = \begin{cases} \inf_{\rho \in A} \int_{S^2} \rho(p) \log(\rho(p)) \, dp & \text{if } \lambda_i[Q] \in (-\frac{1}{3}, \frac{2}{3}), \ i = 1, 2, 3, \\ \infty & \text{otherwise}, \end{cases}$$

where $A_Q = \{ \rho : S^2 \to [0, \infty) \ | \ \int_{S^2} \rho(p) \, dp = 1 \}$.

This component is added to the bulk free-energy $f_B$ enforcing the eigenvalues to stay in the interval $(-\frac{1}{3}, \frac{2}{3})$. E. Rocca (Università degli Studi di Pavia) Hyperbolic system in LCs modeling May 7–11, 2018 14 / 48
The Ball-Majumdar singular potential

- In the Landau-de Gennes free energy there is no a-priori bound on the eigenvalues
The Ball-Majumdar singular potential

- In the Landau-de Gennes free energy there is no a-priori bound on the eigenvalues.
- In order to naturally enforce the physical constraints in the eigenvalues of the symmetric, traceless tensors $Q$, Ball and Majumdar have recently introduced in [Ball, Majumdar, Molecular Crystals and Liquid Crystals (2010)] a singular component

$$f(Q) = \begin{cases} 
\inf_{\rho \in A_Q} \int_{S^2} \rho(p) \log(\rho(p)) \, dp & \text{if } \lambda_i[Q] \in (-1/3, 2/3), \; i = 1, 2, 3, \\
\infty & \text{otherwise,} 
\end{cases}$$

$$A_Q = \left\{ \rho : S^2 \to [0, \infty) \mid \int_{S^2} \rho(p) \, dp = 1; \; Q = \int_{S^2} \left( p \otimes p - \frac{1}{3} I \right) \rho(p) \, dp \right\}.$$ 

to the bulk free-energy $f_B$ enforcing the eigenvalues to stay in the interval $(-\frac{1}{3}, \frac{2}{3})$. 

The hydrodynamic theories in the isothermal case

The hydrodynamic theory corresponding to the Oseen-Frank free energy has been developed by Ericksen (1961) and Leslie (1968) (the celebrated Leslie-Ericksen model).

The Lin-Liu model (1995) is obtained by replacing the unit-vector constraint on $d$ with a Ginzburg-Landau penalization $W(d) = \frac{1}{4} \varepsilon^2 (|d|^2 - 1)^2$, on the director field $d$, which should formally converge to the Leslie-Ericksen model when $\varepsilon \to 0$, but this is an important open issue.

For the Landau-de Gennes free energy with "regular" potential, the hydrodynamic theory has been developed in [Paicu, Zarnescu, SIAM (2011) and ARMA (2012)].
The hydrodynamic theories in the isothermal case

The hydrodynamic theory corresponding to the Oseen-Frank free energy has been developed by Ericksen (1961) and Leslie (1968) (the celebrated Leslie-Ericksen model).

The Lin-Liu model (1995) is obtained by replacing the unit-vector constraint on $d$ with a Ginzburg-Landau penalization $W(d) = \frac{1}{4} \varepsilon^2 (|d|^2 - 1)^2$, on the director field $d$, which should formally converge to the Leslie-Ericksen model when $\varepsilon \to 0$, but this is an important open issue.

For the Landau-de Gennes free energy with "regular" potential, the hydrodynamic theory has been developed in [Paicu, Zarnescu, SIAM (2011) and ARMA (2012)].
The hydrodynamic theories in the isothermal case

⇒ The hydrodynamic theory corresponding to the Oseen-Frank free energy has been developed by Ericksen (1961) and Leslie (1968) (the celebrated Leslie-Ericksen model)

⇒ The Lin-Liu model (1995) is obtained by replacing the unit-vector constraint on \( d \) with a Ginzburg-Landau penalization \( W(d) = \frac{1}{4\varepsilon^2}(|d|^2 - 1)^2 \), on the director field \( d \), which should formally converge to the Leslie-Ericksen model when \( \varepsilon \to 0 \), but this is an important open issue.
The hydrodynamic theories in the isothermal case

⇒ The hydrodynamic theory corresponding to the Oseen-Frank free energy has been developed by Ericksen (1961) and Leslie (1968) (the celebrated Leslie-Ericksen model)

⇒ The Lin-Liu model (1995) is obtained by replacing the unit-vector constraint on $d$ with a Ginzburg-Landau penalization $W(d) = \frac{1}{4\varepsilon^2}(|d|^2 - 1)^2$, on the director field $d$, which should formally converge to the Leslie-Ericksen model when $\varepsilon \to 0$, but this is an important open issue

⇒ For the Landau-de Gennes free energy with “regular” potential, the hydrodynamic theory has been developed in [Paicu, Zarnescu, SIAM (2011) and ARMA (2012)]
Outline

1 Introduction of the Problem

2 The LCs modeling

3 The simplified Qian-Sheng model

4 The existence of local-in-time smooth solutions

5 Existence of global-in-time dissipative solutions

6 Weak-strong uniqueness

7 Further perspectives
Our main aims

We present here a system modeling the hydrodynamics of nematic liquid crystals in the $Q$-tensor framework.
Our main aims

We present here a system modeling the hydrodynamics of nematic liquid crystals in the $Q$-tensor framework.

The system is an inviscid version of the QS-model proposed in

Our main aims

We present here a system modeling the hydrodynamics of nematic liquid crystals in the $Q$-tensor framework.

The system is an inviscid version of the QS-model proposed in


as a simplification of the QS-model, that captures its essential features and exhibits a number of interesting conservation and geometric properties that could be relevant in particular to describing defect patterns, thanks to the (presumptively) singular character of the equations.
Our main aims

We present here a system modeling the hydrodynamics of nematic liquid crystals in the $Q$-tensor framework.

The system is an inviscid version of the QS-model proposed in


as a simplification of the QS-model, that captures its essential features and exhibits a number of interesting conservation and geometric properties that could be relevant in particular to describing defect patterns, thanks to the (presumptively) singular character of the equations.

The most characteristic specific feature of this model is the presence of an inertial term that appears as a second-order material derivative.
Our main aims

We present here a system modeling the hydrodynamics of nematic liquid crystals in the $\mathbb{Q}$-tensor framework.

The system is an inviscid version of the QS-model proposed in


as a simplification of the QS-model, that captures its essential features and exhibits a number of interesting conservation and geometric properties that could be relevant in particular to describing defect patterns, thanks to the (presumptively) singular character of the equations.

The most characteristic specific feature of this model is the presence of an inertial term that appears as a second-order material derivative.

This term provides a hyperbolic character to the equations and is the main source of difficulties in the analysis.
Our main aims

We present here a system modeling the hydrodynamics of nematic liquid crystals in the $Q$-tensor framework.

The system is an inviscid version of the QS-model proposed in


as a simplification of the QS-model, that captures its essential features and exhibits a number of interesting conservation and geometric properties that could be relevant in particular to describing defect patterns, thanks to the (presumptively) singular character of the equations.

The most characteristic specific feature of this model is the presence of an inertial term that appears as a second-order material derivative.

This term provides a hyperbolic character to the equations and is the main source of difficulties in the analysis.

It should be noted that this second material derivative also appears in the more commonly used Ericksen-Leslie model of liquid crystals, but there it generates even worse effects due to the additional presence of the unit-length constraint.
The PDEs

The system we are studying is:

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \odot \nabla_x Q) \\
\partial_t Q + \mathbf{v} \cdot \nabla_x Q &= P \\
\partial_t P + \mathbf{v} \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda I
\end{align*}
\]

For the sake of simplicity, we restrict ourselves to the periodic boundary conditions, for which the underlying spatial domain may be identified with the flat torus:

\[
T^3 = ([-\pi, \pi]\{-\pi, \pi\})^3
\]

The system is

- the **standard Euler system** for the fluid velocity \( \mathbf{v} = \mathbf{v}(t, x) \in \mathbb{R}^3 \), coupled via a nonlinear forcing term with
The PDEs

The system we are studying is:

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x \mathbf{Q} \circ \nabla_x \mathbf{Q}) \\
\partial_t \mathbf{Q} + \mathbf{v} \cdot \nabla_x \mathbf{Q} &= P \\
\partial_t P + \mathbf{v} \cdot \nabla_x P &= -\frac{\partial \mathcal{F}}{\partial \mathbf{Q}} + \Delta \mathbf{Q} - \lambda \mathbf{I}
\end{align*}
\]

For the sake of simplicity, we restrict ourselves to the periodic boundary conditions, for which the underlying spatial domain may be identified with the flat torus:

\[
\mathcal{T}^3 = ([\pi, \pi] \times \{-\pi, \pi\})^3
\]

The system is

- the **standard Euler system** for the fluid velocity \( \mathbf{v} = \mathbf{v}(t, x) \in \mathbb{R}^3 \), coupled via a nonlinear forcing term with
- a **wave-like equation** governing the time evolution of the \( \mathbf{Q} \)-tensor \( \mathbf{Q} = \mathbf{Q}(t, x) \in \mathbb{R}^{3 \times 3}_{0, \text{sym}} - \text{a symmetric traceless matrix} \)
The PDEs

The system we are studying is:

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x \mathbb{Q} \odot \nabla_x \mathbb{Q}) \\
\partial_t \mathbb{Q} + \mathbf{v} \cdot \nabla_x \mathbb{Q} &= \mathbb{P} \\
\partial_t \mathbb{P} + \mathbf{v} \cdot \nabla_x \mathbb{P} &= -\frac{\partial F}{\partial \mathbb{Q}} + \Delta \mathbb{Q} - \lambda \mathbb{I}
\end{align*}
\]

For the sake of simplicity, we restrict ourselves to the periodic boundary conditions, for which the underlying spatial domain may be identified with the flat torus:

\[\mathcal{T}^3 = ([−\pi, \pi]|{−\pi, \pi})^3\]

The system is

- the standard Euler system for the fluid velocity \( \mathbf{v} = \mathbf{v}(t, x) \in \mathbb{R}^3 \), coupled via a nonlinear forcing term with
- a wave-like equation governing the time evolution of the \( \mathbb{Q} \)-tensor \( \mathbb{Q} = \mathbb{Q}(t, x) \in \mathbb{R}^{3 \times 3}_{0, \text{sym}} \) - a symmetric traceless matrix
- The pressure \( \Pi \) and the factor \( \lambda \mathbb{I} \) may be seen as Lagrange multipliers compensating the deviation of the motion from the divergenceless and zero-trace state, respectively
Energy functional

The problem admits a natural energy functional

\[ \mathcal{E}[\mathbf{v}, \mathbf{P}, \mathbf{Q}] = \int_{\Omega} \left[ \frac{1}{2} |\mathbf{v}|^2 + \frac{1}{2} |\mathbf{P}|^2 + \frac{1}{2} |\nabla \mathbf{Q}|^2 + \mathcal{F}(\mathbf{Q}) \right] \, dx \]

It is easy to check, by multiplying by \( \mathbf{v} \) the velocity-equation:

\[ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \Pi = - \text{div} (\nabla \mathbf{Q} \otimes \nabla \mathbf{Q}) \]

by \( \mathbf{P}(= \partial_t \mathbf{Q} + \mathbf{v} \cdot \nabla \mathbf{Q}) \) the equation:

\[ \partial_t \mathbf{P} + \mathbf{v} \cdot \nabla \mathbf{P} = - \frac{\partial \mathcal{F}}{\partial \mathbf{Q}} + \Delta \mathbf{Q} - \lambda \mathbf{I} \]

and integrating the resulting sum by parts, that the total energy is conserved

\[ \frac{d}{dt} \mathcal{E}[\mathbf{v}, \mathbf{P}, \mathbf{Q}] = 0 \]

for any smooth solution of our system.
Qualitative features

The system can be regarded as an extended Euler system and this was the main motivation for proposing it in the Gay-Balmaz-Tronci paper.
Qualitative features

The system can be regarded as an extended Euler system and this was the main motivation for proposing it in the Gay-Balmaz-Tronci paper.

Indeed, if one defines the extended circulation vector, where $P_{ij} \nabla_x Q_{ij}$ represents the circulation associated with the fluid interaction with the nematic order parameter field (here and in the following we assume Einstein summation convention, of summation over repeated indices)

$$C_{QS} := \mathbf{v} + P_{ij} \nabla_x Q_{ij}$$

and the extended vorticity

$$\tilde{\omega}_{QS} := \nabla_x \times C_{QS}$$

then we have the Euler-like equation

$$\partial_t \tilde{\omega}_{QS} + \nabla_x \times (\mathbf{v} \times \tilde{\omega}_{QS}) = 0$$
Qualitative features

The system can be regarded as an extended Euler system and this was the main motivation for proposing it in the Gay-Balmaz-Tronci paper.

Indeed, if one defines the extended circulation vector, where $P_{ij} \nabla_x Q_{ij}$ represents the circulation associated with the fluid interaction with the nematic order parameter field (here and in the following we assume Einstein summation convention, of summation over repeated indices)

$$C_{QS} := \mathbf{v} + P_{ij} \nabla_x Q_{ij}$$

and the extended vorticity

$$\tilde{\omega}_{QS} := \nabla_x \times C_{QS}$$

then we have the Euler-like equation

$$\partial_t \tilde{\omega}_{QS} + \nabla_x \times (\mathbf{v} \times \tilde{\omega}_{QS}) = 0$$

In addition there hold a circulation theorem

$$\frac{d}{dt} \int_{\Gamma(t)} \mathbf{v} + P_{ij} \nabla_x Q_{ij} \, d\sigma = 0$$

where $\Gamma(t)$ is a closed path moving with velocity $\mathbf{v}$ and the helicity conservation

$$\frac{d}{dt} \int_{\Omega} C_{QS} \cdot \tilde{\omega}_{QS} \, dx = 0$$

as a direct consequence of the Euler-like equation.
Our goals and main difficulties

Our goal is to study the existence of **global-in-time** solutions to the problem

\[
\begin{aligned}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \circ \nabla_x Q) \\
\partial_t Q + \mathbf{v} \cdot \nabla_x Q &= P \\
\partial_t P + \mathbf{v} \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda I
\end{aligned}
\]

This may seem a rather ambitious task as the problem is highly non-linear involving the incompressible Euler system for which the existence of physically admissible solutions is an open problem even in the class of weak solutions (see [Wiedemann, Ann. Inst. H. Poincaré Anal. Non Linéaire (2011)] and [Székelyhidi–Wiedemann, Arch. Rational Mech. Anal. (2012)])

To circumvent these well-known difficulties, we introduce a new class of dissipative solutions inspired by a similar concept introduced by P.L. Lions in the context of Euler flow

The dissipative solutions enjoy the following properties:

Any classical solution of problem above is a dissipative solution

Any (sufficiently) smooth dissipative solution is a classical solution

A dissipative solution coincides with the classical solution emanating from the same initial data as long as the latter exists
Our goals and main difficulties

Our goal is to study the existence of global-in-time solutions to the problem

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \circ \nabla_x Q) \\
\partial_t Q + \mathbf{v} \cdot \nabla_x Q &= P \\
\partial_t P + \mathbf{v} \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda I
\end{align*}
\]

This may seem a rather ambitious task as the problem is highly non-linear involving the incompressible Euler system for which the existence of physically admissible solutions is an open problem even in the class of weak solutions (see [Wiedemann, Ann. Inst. H. Poincaré Anal. Non Linéaire (2011)] and [Székelyhidi–Wiedemann, Arch. Rational Mech. Anal. (2012)])
Our goals and main difficulties

Our goal is to study the existence of global-in-time solutions to the problem:

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \circ \nabla_x Q) \\
\partial_t Q + \mathbf{v} \cdot \nabla_x Q &= P \\
\partial_t P + \mathbf{v} \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda I
\end{align*}
\]

This may seem a rather ambitious task as the problem is highly non-linear involving the incompressible Euler system for which the existence of physically admissible solutions is an open problem even in the class of weak solutions (see [Wiedemann, Ann. Inst. H. Poincaré Anal. Non Linéaire (2011)] and [Székelyhidi–Wiedemann, Arch. Rational Mech. Anal. (2012)])

To circumvent these well-known difficulties, we introduce a new class of dissipative solutions inspired by a similar concept introduced by P.L. Lions in the context of Euler flow.
Our goals and main difficulties

Our goal is to study the existence of global-in-time solutions to the problem

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \odot \nabla_x Q) \\
\partial_t Q + \mathbf{v} \cdot \nabla_x Q &= P \\
\partial_t P + \mathbf{v} \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda \mathbb{I}
\end{align*}
\]

This may seem a rather ambitious task as the problem is highly non-linear involving the incompressible Euler system for which the existence of physically admissible solutions is an open problem even in the class of weak solutions (see [Wiedemann, Ann. Inst. H. Poincaré Anal. Non Linéaire (2011)] and [Székelyhidi–Wiedemann, Arch. Rational Mech. Anal. (2012)])

To circumvent these well-known difficulties, we introduce a new class of dissipative solutions inspired by a similar concept introduced by P.L. Lions in the context of Euler flow

The dissipative solutions enjoy the following properties:

- Any classical solution of problem above is a dissipative solution
- Any (sufficiently) smooth dissipative solution is a classical solution
- A dissipative solution coincides with the classical solution emanating from the same initial data as long as the latter exists
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish existence of smooth solutions of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces. This will be done in an entirely standard way by the energy method well developed in the theory of hyperbolic conservation laws.

2. Next, we introduce the concept of dissipative solution which will satisfy:
   - the system of equations in the sense of distributions, where the right-hand sides will contain two extra terms playing the role of defect measures.
   - the energy inequality:
     \[ E[v, p, q](\tau) + D(\tau) \leq E[v, p, q](0) \text{ for a.a. } \tau > 0, \]
     with a dissipation defect \( D \) dominating, in a certain sense specified later on, the defect measure in the equations.

3. Next, we derive a relative energy inequality playing the role of a "distance" between a dissipative solution and any sufficiently smooth process.

4. Similarly to [E. Feireisl, A. Novotný, and Y. Sun, Indiana Univ. Math. J. (2011)], we use the relative energy to show the weak-strong uniqueness property for the class of dissipative solutions.

5. Finally, we observe that the same procedure used in the construction of local smooth solutions gives rise to a dissipative solution.
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish **existence of smooth solutions** of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces.
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish existence of smooth solutions of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces. This will be done in an entirely standard way by the energy method well developed in the theory of hyperbolic conservation laws.

2. Next, we introduce the concept of dissipative solution which will satisfy:

   ▶ the system of equations in the sense of distributions, where the right-hand sides will contain two extra terms playing the role of defect measures
   ▶ the energy inequality: 
     \[ E[v, P, Q](\tau) + D(\tau) \leq E[v, P, Q](0) \text{ for a.a. } \tau > 0, \]

   with a dissipation defect dominating, in a certain sense specified later on, the defect measure in the equations.

3. Next, we derive a relative energy inequality playing the role of a “distance” between a dissipative solution and any sufficiently smooth process.

4. Similarly to [E. Feireisl, A. Novotný, and Y. Sun, Indiana Univ. Math. J. (2011)], we use the relative energy to show the weak-strong uniqueness property for the class of dissipative solutions.

5. Finally, we observe that the same procedure used in the construction of local smooth solutions gives rise to a dissipative solution.
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish **existence of smooth solutions** of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces. This will be done in an entirely standard way by the energy method well developed in the theory of hyperbolic conservation laws.

2. Next, we introduce the concept of dissipative solution which will satisfy:

   ▶ the system of equations in the sense of distributions, where the right-hand sides will contain two extra terms playing the role of defect measures;

   ▶ the energy inequality:

     $E[v, P, Q](\tau) + D(\tau) \leq E[v, P, Q](0)$ for a.a. $\tau > 0$,

     with a dissipation defect $D$ dominating, in a certain sense specified later on, the defect measure in the equations.

3. Next, we derive a relative energy inequality playing the role of a “distance” between a dissipative solution and any sufficiently smooth process.

4. Similarly to [E. Feireisl, A. Novotný, and Y. Sun, Indiana Univ. Math. J. (2011)], we use the relative energy to show the weak-strong uniqueness property for the class of dissipative solutions.

5. Finally, we observe that the same procedure used in the construction of local smooth solutions gives rise to a dissipative solution.
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish existence of smooth solutions of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces. This will be done in an entirely standard way by the energy method well developed in the theory of hyperbolic conservation laws.

2. Next, we introduce the concept of dissipative solution which will satisfy:
   - the system of equations in the sense of distributions, where the right-hand sides will contain two extra terms playing the role of defect measures.
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish **existence of smooth solutions** of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces. This will be done in an entirely standard way by the energy method well developed in the theory of hyperbolic conservation laws

2. Next, we introduce the concept of dissipative solution which will satisfy:
   - the system of equations in the sense of distributions, where the right-hand sides will contain two extra terms playing the role of **defect measures**
   - the energy inequality:
     \[ \mathcal{E}[\mathbf{v}, P, Q](\tau) + D(\tau) \leq \mathcal{E}[\mathbf{v}, P, Q](0) \text{ for a.a. } \tau > 0, \]
     with a **dissipation defect** \( D \) dominating, in a certain sense specified later on, the defect measure in the equations
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish existence of smooth solutions of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces. This will be done in an entirely standard way by the energy method well developed in the theory of hyperbolic conservation laws.

2. Next, we introduce the concept of dissipative solution which will satisfy:
   - the system of equations in the sense of distributions, where the right-hand sides will contain two extra terms playing the role of defect measures,
   - the energy inequality:
     \[ E[v, P, Q](\tau) + D(\tau) \leq E[v, P, Q](0) \quad \text{for a.a.} \quad \tau > 0, \]
     with a dissipation defect \( D \) dominating, in a certain sense specified later on, the defect measure in the equations.

3. Next, we derive a relative energy inequality playing the role of a “distance” between a dissipative solution and any sufficiently smooth process.
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish existence of smooth solutions of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces. This will be done in an entirely standard way by the energy method well developed in the theory of hyperbolic conservation laws.

2. Next, we introduce the concept of dissipative solution which will satisfy:
   ▶ the system of equations in the sense of distributions, where the right-hand sides will contain two extra terms playing the role of defect measures
   ▶ the energy inequality:
     \[ \mathcal{E}[\mathbf{v}, P, Q](\tau) + D(\tau) \leq \mathcal{E}[\mathbf{v}, P, Q](0) \text{ for a.a. } \tau > 0, \]
     with a dissipation defect \( D \) dominating, in a certain sense specified later on, the defect measure in the equations

3. Next, we derive a relative energy inequality playing the role of a “distance” between a dissipative solution and any sufficiently smooth process.

4. Similarly to [E. Feireisl, A. Novotný, and Y. Sun, Indiana Univ. Math. J. (2011)], we use the relative energy to show the weak-strong uniqueness property for the class of dissipative solutions.
An idea of the proof

Our strategy is based on the following steps:

1. First, we establish existence of smooth solutions of our system defined on a possibly short time interval the length of which depends on the norm of the initial data in certain Sobolev spaces. This will be done in an entirely standard way by the energy method well developed in the theory of hyperbolic conservation laws.

2. Next, we introduce the concept of dissipative solution which will satisfy:
   - the system of equations in the sense of distributions, where the right-hand sides will contain two extra terms playing the role of defect measures.
   - the energy inequality:
     \[ E[v, P, Q](\tau) + D(\tau) \leq E[v, P, Q](0) \text{ for a.a. } \tau > 0, \]
     with a **dissipation defect** $D$ dominating, in a certain sense specified later on, the defect measure in the equations.

3. Next, we derive a **relative energy inequality** playing the role of a “distance” between a dissipative solution and any sufficiently smooth process.

4. Similarly to [E. Feireisl, A. Novotný, and Y. Sun, Indiana Univ. Math. J. (2011)], we use the relative energy to show the **weak-strong uniqueness** property for the class of dissipative solutions.

5. Finally, we observe that the same procedure used in the construction of local smooth solutions gives rise to a **dissipative solution**.
Notations

- For $A, B$ two $3 \times 3$ matrices we denote the inner product on the space of matrices as $A : B = \text{tr}(AB)$.

- The product $\nabla A \otimes \nabla B$ is a matrix with $ij$ component $\partial_i A : \partial_j B$.

- If $M(x)$ is a $3 \times 3$-matrix, then $|M|$ denotes the Frobenius norm of the matrix, i.e. $|M| = \sqrt{M : M^t}$. Furthermore $\nabla \cdot M$ stands for the vector field $\left( \sum_{j=1}^{3} \frac{\partial M_{ij}}{\partial x_j} \right)_{i=1, \ldots, 3}$.

- If $v$ is a 3-dimensional vector and $Q$ is a $3 \times 3$ matrix then $v \otimes Q$ is a third order tensor with components $v_i Q_{kl}$ with $i, k, l \in \{1, 2, 3\}$.

- If $P$ is a $3 \times 3$ then $\nabla_x P$ is a third-order tensor and we denote by $v \otimes Q : \nabla_x P$ the scalar $\sum_{i,j,k=1}^{3} v_k Q_{ij} \frac{\partial P_{ij}}{\partial x_k}$.
Outline

1. Introduction of the Problem
2. The LCs modeling
3. The simplified Qian-Sheng model
4. The existence of local-in-time smooth solutions
5. Existence of global-in-time dissipative solutions
6. Weak-strong uniqueness
7. Further perspectives
Assumptions on the potential

(A1) The function $F : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is isotropic, i.e., $F(Q) = F(RQR^t)$, $\forall R \in O(3)$.

(A2) $F$ is $\Lambda$-convex: there exists $\Lambda \geq 0$ such that $F(Q) + \Lambda|Q|^2$ is a strictly convex non-negative function. Hence we will set $G(Q) := F(Q) + \Lambda|Q|^2$.

(A3) There exists a constant $\bar{C} > 0$ such that $|\partial F(Q)| \leq \bar{C}(1 + |Q|^q)$, for some exponent $q < 5$.

Remarks. Assumption (A3) could be generalized and include also potentials with faster (polynomial) growth at infinity, the only complication being the possible appearance of a further defect term in the energy balance. The function $G(Q)$ can be controlled only in $L^1$ in that case.

Exponential growth at infinity or "singular" potentials (i.e., $F$ being identically infinity outside a bounded set, like the Ball-Majumdar potential) is more delicate: a further measure-valued term (the limit of $\partial F(Q)$) would then occur.

Examples of functions that satisfy (A1)-(A3) are:

$$F(Q) = a^2|Q|^2 + b^3 \text{tr}(Q^3) + c^4|Q|^4$$

where $a, b \in \mathbb{R}$ and $c > 0$. 

E. Rocca (Università degli Studi di Pavia)
Assumptions on the potential

(A1) The function $\mathcal{F} : R_{0,\text{sym}}^{3 \times 3} \rightarrow R$ is \textit{isotropic}, i.e.

$$\mathcal{F}(Q) = \mathcal{F}(RQR^t), \forall R \in O(3).$$
Assumptions on the potential

(A1) The function $\mathcal{F}: R_{0,\text{sym}}^{3 \times 3} \to R$ is isotropic, i.e.

$$\mathcal{F}(Q) = \mathcal{F}(RQR^t), \ \forall R \in O(3).$$

(A2) $\mathcal{F}$ is $\Lambda$-convex: there exists $\Lambda \geq 0$ such that $\mathcal{F} + \Lambda |Q|^2$ is a strictly convex non-negative function. Hence we will set

$$G(Q) := \mathcal{F}(Q) + \Lambda |Q|^2.$$
Assumptions on the potential

(A1) The function $\mathcal{F} : R_{0,\text{sym}}^{3 \times 3} \to R$ is isotropic, i.e.

$$\mathcal{F}(Q) = \mathcal{F}(RQR^t), \forall R \in O(3).$$

(A2) $\mathcal{F}$ is $\Lambda$-convex: there exists $\Lambda \geq 0$ such that $\mathcal{F} + \Lambda |Q|^2$ is a strictly convex non-negative function. Hence we will set

$$\mathcal{G}(Q) := \mathcal{F}(Q) + \Lambda |Q|^2.$$

(A3) There exists a constant $\bar{C} > 0$ such that

$$|\partial \mathcal{F}(Q)| \leq \bar{C}(1 + |Q|^q), \text{ for some exponent } q < 5.$$
Assumptions on the potential

(A1) The function $\mathcal{F} : R_{0,\text{sym}}^{3 \times 3} \to \mathbb{R}$ is isotropic, i.e.

$$\mathcal{F}(Q) = \mathcal{F}(RQR^t), \ \forall R \in O(3).$$

(A2) $\mathcal{F}$ is $\Lambda$-convex: there exists $\Lambda \geq 0$ such that $\mathcal{F} + \Lambda |Q|^2$ is a strictly convex non-negative function. Hence we will set

$$G(Q) := \mathcal{F}(Q) + \Lambda |Q|^2.$$

(A3) There exists a constant $\bar{C} > 0$ such that

$$|\partial \mathcal{F}(Q)| \leq \bar{C}(1 + |Q|^q), \ \text{for some exponent } q < 5.$$

Remarks.
Assumptions on the potential

(A1) The function $\mathcal{F} : \mathbb{R}^{3 \times 3}_{0, \text{sym}} \to \mathbb{R}$ is isotropic, i.e.

$$\mathcal{F}(Q) = \mathcal{F}(RQR^t), \forall R \in O(3).$$

(A2) $\mathcal{F}$ is $\Lambda$-convex: there exists $\Lambda \geq 0$ such that $\mathcal{F} + \Lambda |Q|^2$ is a strictly convex non-negative function. Hence we will set

$$G(Q) := \mathcal{F}(Q) + \Lambda |Q|^2.$$

(A3) There exists a constant $\bar{C} > 0$ such that

$$|\partial \mathcal{F}(Q)| \leq \bar{C}(1 + |Q|^q), \text{ for some exponent } q < 5.$$
Theorem 1: local strong solutions

Let $s \geq 3$ and $F \in C^{s+1}_s(R^3 \times R^3_0, \text{sym}; R)$ satisfying (A1)-(A3). Consider the initial data $v(0, \cdot) = v_0 \in W^{s, 2}(T^3; R^3)$, $P(0, \cdot) = P_0 \in W^{s, 2}(T^3; R^3 \times R^3_0, \text{sym})$, $Q(0, \cdot) = Q_0 \in W^{s+1, 2}(T^3; R^3 \times R^3_0, \text{sym})$ such that $\text{div} x v_0 = 0$.

Then there exists $T_0 > 0$ depending solely on the norm of the initial data such that our problem admits a strong solution in $[0, T_0] \times T^3$, unique in the class $v \in C([0, T_0]; W^{s, 2}(T^3; R^3))$, $P \in C([0, T_0]; W^{s, 2}(T^3; R^3 \times R^3_0, \text{sym}))$, $Q \in C([0, T_0]; W^{s+1, 2}(T^3; R^3 \times R^3_0, \text{sym}))$. 
Theorem 1: local strong solutions

Let $s \geq 3$ and $\mathcal{F} \in C^{s+1}(R_{0,\text{sym}}^3; R)$ satisfying (A1)-(A3). Consider the initial data

$$
\nu(0, \cdot) = \nu_0 \in W^{s,2}(T^3; R^3), \quad \mathbb{P}(0, \cdot) = \mathbb{P}_0 \in W^{s,2}(T^3; R_{0,\text{sym}}^3),
$$

$$
\mathbb{Q}(0, \cdot) = \mathbb{Q}_0 \in W^{s+1,2}(T^3; R_{0,\text{sym}}^3)
$$

such that $\text{div}_x \nu_0 = 0$. 
Theorem 1: local strong solutions

Let $s \geq 3$ and $\mathcal{F} \in C^{s+1}(R_0^{3 \times 3}; R)$ satisfying (A1)-(A3). Consider the initial data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 \in W^{s,2}(T^3; R^3), \; P(0, \cdot) = P_0 \in W^{s,2}(T^3; R_0^{3 \times 3}),$$

$$Q(0, \cdot) = Q_0 \in W^{s+1,2}(T^3; R_0^{3 \times 3})$$

such that $\text{div}_x \mathbf{v}_0 = 0$.

Then there exists $T_0 > 0$ depending solely on the norm of the initial data such that our problem

$\text{div}_x \mathbf{v} = 0$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = -\text{div}_x (\nabla_x Q \circ \nabla_x Q)$$

$$\partial_t Q + \mathbf{v} \cdot \nabla_x Q = P$$

$$\partial_t P + \mathbf{v} \cdot \nabla_x P = -\frac{\partial \mathcal{F}}{\partial Q} + \Delta Q - \lambda I$$

admits a strong solution in $[0, T_0] \times T^3$, unique in the class

$$\mathbf{v} \in C([0, T_0]; W^{s,2}(T^3; R^3)),$$

$$P \in C([0, T_0]; W^{s,2}(T^3; R_0^{3 \times 3})), \; Q \in C([0, T_0]; W^{s+1,2}(T^3; R_0^{3 \times 3})).$$
Outline

1. Introduction of the Problem
2. The LCs modeling
3. The simplified Qian-Sheng model
4. The existence of local-in-time smooth solutions
5. Existence of global-in-time dissipative solutions
6. Weak-strong uniqueness
7. Further perspectives
We name by dissipative solutions the trio of functions \([v, P, Q]\) such that
We name by dissipative solutions the trio of functions \([\mathbf{v}, P, Q]\) such that

\[
\mathbf{v} \in C_{\text{weak}}([0, T]; L^2(T^3; R^3)), \quad P \in C_{\text{weak}}([0, T]; L^2(T^3; R^{3\times 3})), \\
Q \in C_{\text{weak}}([0, T]; W^{1,2}(T^3; R^{3\times 3})) \cap C([0, T]; L^2(T^3; R^{3\times 3})),
\]

they satisfy the relations:

\[
\int_{\Omega} \mathbf{v}(\tau, \cdot) \cdot \nabla_x \varphi \, dx = 0 \text{ for any } \varphi \in C^\infty_c(T^3),
\]

\[
\left[ \int_{\Omega} \mathbf{v} \cdot \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \mathbf{v} \cdot \partial_t \varphi + (\mathbf{v} \otimes \mathbf{v}) : \nabla_x \varphi + (\nabla_x Q \odot \nabla_x Q) : \nabla_x \varphi \, dx + \left\langle \mathbb{R}^1; \nabla_x \varphi \right\rangle \, dt
\]

for any \(\varphi \in C^\infty_c([0, T] \times T^3; R^3), \) \(\text{div}_x \varphi = 0;\)

\[
\left[ \int_{\Omega} Q : \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} [Q : \partial_t \varphi + (\mathbf{v} \otimes Q) : \nabla_x \varphi + \mathbb{P} : \varphi] \, dx \, dt \text{ for any } \varphi \in C^\infty_c([0, T] \times T^3; R^{3\times 3});
\]

\[
\left[ \int_{\Omega} \mathbb{P} : \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \left[ \mathbb{P} : \partial_t \varphi + (\mathbf{v} \otimes \mathbb{P}) : \nabla_x \varphi - \frac{\partial F(Q)}{\partial Q} : \varphi - \nabla_x Q : \nabla_x \varphi \right] \, dx + \left\langle \mathbb{R}^2; \nabla_x \varphi \right\rangle \, dt
\]

for any \(\varphi \in C^\infty_c([0, T] \times T^3; R^{3\times 3});\)

for certain \(\mathbb{R}^1, \mathbb{R}^2,\)
We name by dissipative solutions the trio of functions \([v, P, Q]\) such that

\[
v \in C_{\text{weak}}([0, T]; L^2(T^3; R^3)), \quad P \in C_{\text{weak}}([0, T]; L^2(T^3; R^3 \times R^3)), \quad Q \in C_{\text{weak}}([0, T]; W^{1,2}(T^3; R^3 \times R^3) \cap C([0, T]; L^2(T^3; R^3 \times R^3)),
\]

they satisfy the relations:

\[
\int_{\Omega} v(\tau, \cdot) \cdot \nabla_x \varphi \, dx = 0 \text{ for any } \varphi \in C^\infty_c(T^3),
\]

\[
\left[ \int_{\Omega} v \cdot \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} v \cdot \partial_t \varphi + (v \otimes v) : \nabla_x \varphi + (\nabla_x Q \otimes \nabla_x Q) : \nabla_x \varphi \, dx + \left\langle \mathbb{R}^1; \nabla_x \varphi \right\rangle \, dt
\]

for any \(\varphi \in C^\infty_c([0, T] \times T^3; R^3)\), \(\text{div}_x \varphi = 0\);

\[
\left[ \int_{\Omega} Q : \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} [Q : \partial_t \varphi + (v \otimes Q) : \nabla_x \varphi + P : \varphi] \, dx \, dt \text{ for any } \varphi \in C^\infty_c([0, T] \times T^3; R^{3 \times 3}_{0,\text{sym}}),
\]

\[
\left[ \int_{\Omega} P : \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \left[ P : \partial_t \varphi + (v \otimes P) : \nabla_x \varphi - \frac{\partial F(Q)}{\partial Q} : \varphi - \nabla_x Q : \nabla_x \varphi \right] \, dx + \left\langle \mathbb{R}^2; \nabla_x \varphi \right\rangle \, dt
\]

for any \(\varphi \in C^\infty_c([0, T] \times T^3; R^{3 \times 3}_{0,\text{sym}})\);

for certain \(\mathbb{R}^1, \mathbb{R}^2\), and the energy balance for any \(\tau \in [0, T]\):

\[
\left[ \int_{\Omega} \left[ \frac{1}{2} |v|^2 + \frac{1}{2} |P|^2 + \frac{1}{2} |\nabla_x Q|^2 + G(Q) \right] \, dx \right]_{t=0}^{t=\tau} + \mathcal{D}(\tau) = 2\Lambda \int_0^\tau \int_{\Omega} Q : P \, dx \, dt
\]

for certain \(\mathcal{D} \in L^\infty(0, T) \) (dissipation defect), where

\[
\int_0^\tau \left[ \| R^1(t, \cdot) \|_{\mathcal{M}(T^3)} + \| R^2(t, \cdot) \|_{\mathcal{M}(T^3)} \right] \, dt \leq c \int_0^\tau \mathcal{D}(t) \, dt, \quad \tau \in [0, T]
\]
Theorem 2: dissipative solutions

Let $F \in C^2(R^3 \times R^3_0, \text{sym}; R)$ satisfy assumptions (A1)-(A3). Then our problem

$$\begin{align*}
\text{div}_x v &= 0 \\
\partial_t v + v \cdot \nabla_x v + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \circ \nabla_x Q) \\
\partial_t Q + v \cdot \nabla_x Q &= P \\
\partial_t P + v \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda I
\end{align*}$$

admits a dissipative solution $[v, P, Q]$ in $(0, T) \times T^3$ for any initial data $v_0 \in L^2(T^3; R^3)$, $\text{div}_x v_0 = 0$, $P_0 \in L^2(T^3; R^3 \times R^3_0, \text{sym})$, $Q_0 \in W^{1,2}(T^3; R^3 \times R^3_0, \text{sym})$. 

E. Rocca (Università degli Studi di Pavia)
Theorem 2: dissipative solutions

Let $\mathcal{F} \in C^2(\mathbb{R}^{3 \times 3}; \mathbb{R})$ satisfy assumptions (A1)-(A3).
Theorem 2: dissipative solutions

Let $\mathcal{F} \in C^2(R_0^{3\times3};\mathbb{R})$ satisfy assumptions (A1)-(A3).

Then our problem

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \odot \nabla_x Q) \\
\partial_t Q + \mathbf{v} \cdot \nabla_x Q &= P \\
\partial_t P + \mathbf{v} \cdot \nabla_x P &= -\frac{\partial \mathcal{F}}{\partial Q} + \Delta Q - \lambda I
\end{align*}
\]
Theorem 2: dissipative solutions

Let $\mathcal{F} \in C^2(R_{0,\text{sym}}^3; \mathbb{R})$ satisfy assumptions (A1)-(A3).

Then our problem

\[
\text{div}_x v = 0 \\
\partial_t v + v \cdot \nabla_x v + \nabla_x \Pi = -\text{div}_x (\nabla_x Q \odot \nabla_x Q) \\
\partial_t Q + v \cdot \nabla_x Q = P \\
\partial_t P + v \cdot \nabla_x P = -\frac{\partial \mathcal{F}}{\partial Q} + \Delta Q - \lambda I
\]

admits a dissipative solution $[v, P, Q]$ in $(0, T) \times T^3$ for any initial data

\[
v_0 \in L^2(T^3; R^3), \quad \text{div}_x v_0 = 0, \quad P_0 \in L^2(T^3; R_{0,\text{sym}}^3), \quad Q_0 \in W^{1,2}(T^3; R_{0,\text{sym}}^3).
\]
Remark on the correctors

Apparently no information about the specific form of the correctors $R_1, R_2$ in
\[ \int_\Omega v \cdot \phi \, dx \] \[ \tau \quad \tau = 0 \]
but, actually:

$R_1$ is the tensor of measures whose $(i,j)$-entry is $R_{i,j,1} + R_{i,j,1,2}$ with $v_i n v_j n - v_i v_j \rightarrow R_{i,j,1}$ weakly-$(\ast)$ in $L^\infty(0, T; M(T^3))$ and

\[ \sum_{i,j} \|R_{i,j,1}\|_{M(T^3)} \leq 3 \int_\Omega (|v|_2^2 - |v|_2^2) \, dx. \]

$R_2$ is the tensor of measures whose $(i,j,k)$-entry is $R_{i,j,k,2}$ with $v_i n P_k n - v_i P_k n \rightarrow R_{i,j,k,2}$ weakly-$(\ast)$ in $L^\infty(0, T; M(T^3))$ and

\[ \sum_{i,j,k} \int_0^\tau \|R_{i,j,k,2}\|_{M(T^3)} \, dt \leq c \int_0^\tau \int_\Omega (|\nabla_x Q|_2^2 - |\nabla_x Q|_2^2) \, dx \, dt + c \int_0^\tau \int_\Omega (|P|_2^2 - |P|_2^2) \, dx \, dt. \]
Remark on the correctors

Apparently no information about the specific form of the correctors $\mathbb{R}^1, \mathbb{R}^2$ in

$$\left[ \int_\Omega \mathbf{v} \cdot \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \mathbf{v} \cdot \partial_t \varphi + (\mathbf{v} \otimes \mathbf{v}) : \nabla_x \varphi + (\nabla_x \mathcal{Q} \otimes \nabla_x \mathcal{Q}) : \nabla_x \varphi \, dx + \left\langle \mathbb{R}^1; \nabla_x \varphi \right\rangle \, dt$$

$$\left[ \int_\Omega \mathcal{P} : \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \mathcal{P} : \partial_t \varphi + (\mathbf{v} \otimes \mathcal{P}) : \nabla_x \varphi - \frac{\partial \mathcal{F}(\mathcal{Q})}{\partial \mathcal{Q}} : \varphi - \nabla_x \mathcal{Q} : \nabla_x \varphi \right] \, dx + \left\langle \mathbb{R}^2; \nabla_x \varphi \right\rangle \, dt$$
Remark on the correctors

Apparently no information about the specific form of the correctors $\mathbb{R}^1$, $\mathbb{R}^2$ in

$$
\left[ \int_\Omega \mathbf{v} \cdot \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \mathbf{v} \cdot \partial_t \varphi + (\mathbf{v} \otimes \mathbf{v}) : \nabla_x \varphi + (\nabla_x \mathbf{Q} \otimes \nabla_x \mathbf{Q}) : \nabla_x \varphi \, dx + \left\langle \mathbb{R}^1; \nabla_x \varphi \right\rangle \, dt
$$

$$
\left[ \int_\Omega \mathbb{P} : \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \mathbb{P} : \partial_t \varphi + (\mathbf{v} \otimes \mathbb{P}) : \nabla_x \varphi - \frac{\partial \mathcal{F}(\mathbf{Q})}{\partial \mathbf{Q}} : \varphi - \nabla_x \mathbf{Q} : \nabla_x \varphi \right] \, dx + \left\langle \mathbb{R}^2; \nabla_x \varphi \right\rangle \, dt
$$

but, actually:

- $\mathbb{R}^1$ is the tensor of measures whose $(i, j)$-entry is $\mathcal{R}^{i,j}_{1,1} + \mathcal{R}^{i,j}_{1,2}$ with

  - $v^i_n v^j_n - v^i v^j \to \mathcal{R}^{i,j}_{1,1}$ weakly-(*) in $L^\infty(0, T; \mathcal{M}(T^3))$ and

  $$
  \sum_{i,j} \| \mathcal{R}^{i,j}_{1,1} \|_{\mathcal{M}(T^3)} \leq 3 \int_\Omega (|\mathbf{v}|^2 - |\mathbf{v}|) \, dx.
  $$

  - $\sum_{\alpha, \beta} \left( \partial_i Q^\alpha_n \beta \partial_j Q^\alpha_n \beta - \partial_i Q^\alpha \beta \partial_j Q^\alpha \beta \right) \to \mathcal{R}^{i,j}_{1,2}$ weakly-(*) in $L^\infty(0, T; \mathcal{M}(T^3))$ and

  $$
  \sum_{i,j} \| \mathcal{R}^{i,j}_{1,2} \|_{\mathcal{M}(T^3)} \leq c \int_\Omega (|\nabla_x \mathbf{Q}|^2 - |\nabla_x \mathbf{Q}|^2) \, dx,
  $$

- $\mathbb{R}^2$ is the tensor of measures whose $(i, j, k)$-entry is $\mathcal{R}^{i,j,k}_{2}$ with

  $$
  v^i_n \mathbf{P}^k_n - v^i \mathbf{P}^k_j \to \mathcal{R}^{i,j,k}_{2} \text{ weakly-*(*) in } L^\infty(0, T; \mathcal{M}(T^3))
  $$

and

$$
\sum_{i,j} \int_0^T \| \mathcal{R}^{i,j,k}_{2} \|_{\mathcal{M}(T^3)} \, dt \leq c \int_0^T \int_\Omega (|\mathbf{v}|^2 - |\mathbf{v}|) \, dx \, dt + c \int_0^T \int_\Omega \left( |\mathbf{P}|^2 - |\mathbf{P}|^2 \right) \, dx \, dt.
$$
Remark on the dissipation

The nonnegative function $D \in L^\infty(0, T)$ in the energy balance:

$$\int_\Omega \left[ \frac{1}{2} |v|^2 + \frac{1}{2} |P|^2 + \frac{1}{2} |\nabla x Q|^2 + G(Q) \right] dx dt = \tau_t \big|_{t=0} + D(\tau) = 2\Lambda \int_0^\tau \int_\Omega Q : P dx dt$$

is obtained as the limit of the difference in square brackets in $E[v(\tau), P(\tau), Q(\tau)] + [E[v_n(\tau), P_n(\tau), Q_n(\tau)] - E[v(\tau), P(\tau), Q(\tau)]] = E[v_n(0), P_n(0), Q_n(0)] + 2\Lambda \int_0^\tau \int_\Omega Q_n : P_n dx dt$.

Notice that it actually represents a dissipation defect.
Remark on the dissipation

The nonnegative function $D \in L^\infty(0, T)$ in the energy balance:

$$\left[ \int_\Omega \left[ \frac{1}{2} |\mathbf{v}|^2 + \frac{1}{2} |\mathbf{P}|^2 + \frac{1}{2} |\nabla x Q|^2 + G(Q) \right] \, dx \right]_{t=0}^{t=\tau} + D(\tau) = 2\Lambda \int_0^T \int_\Omega Q : \mathbf{P} \, dx \, dt$$

is obtained as the limit of the difference in square brackets in

$$E[\mathbf{v}(\tau), \mathbf{P}(\tau), Q(\tau)] + \left[ E[\mathbf{v}_n(\tau), \mathbf{P}_n(\tau), Q_n(\tau)] - E[\mathbf{v}(\tau), \mathbf{P}(\tau), Q(\tau)] \right]$$

$$= E[\mathbf{v}_n(0), \mathbf{P}_n(0), Q_n(0)] + 2\Lambda \int_0^t \int_\Omega Q_n : \mathbf{P}_n \, dx \, dt$$

where the modified energy functional is:

$$E[\mathbf{v}, \mathbf{P}, Q] := \int_\Omega \left[ \frac{1}{2} |\mathbf{v}|^2 + \frac{1}{2} |\mathbf{P}|^2 + \frac{1}{2} |\nabla x Q|^2 + G(Q) \right] \, dx$$
Remark on the dissipation

The nonnegative function $\mathcal{D} \in L^\infty(0, T)$ in the energy balance:

$$\left[ \int_\Omega \left[ \frac{1}{2} |v|^2 + \frac{1}{2} |P|^2 + \frac{1}{2} |\nabla_x Q|^2 + G(Q) \right] \, dx \right]_{t=0}^{t=\tau} + \mathcal{D}(\tau) = 2\Lambda \int_0^\tau \int_\Omega Q : P \, dx \, dt$$

is obtained as the limit of the difference in square brackets in

$$E[v(\tau), P(\tau), Q(\tau)] + \left[ E[v_n(\tau), P_n(\tau), Q_n(\tau)] - E[v(\tau), P(\tau), Q(\tau)] \right]$$

$$= E[v_n(0), P_n(0), Q_n(0)] + 2\Lambda \int_0^t \int_\Omega Q_n : P_n \, dx \, dt$$

where the modified energy functional is:

$$E[v, P, Q] := \int_\Omega \left[ \frac{1}{2} |v|^2 + \frac{1}{2} |P|^2 + \frac{1}{2} |\nabla_x Q|^2 + G(Q) \right] \, dx$$

Notice that it actually represents a dissipation defect
Relative energy

The dissipative solutions may seem rather weak as we have apparently no information about the specific form of neither the dissipation defect $D$ nor the correctors $R_1, R_2$. Nevertheless, we show that a dissipative solution coincides with the strong solution emanating from the same initial data as long as the latter exists. We consider the modified energy functional

$$E(v, P, Q) = \int_{\Omega} \left[ \frac{1}{2} |v|^2 + \frac{1}{2} |p|^2 + \frac{1}{2} |\nabla x Q|^2 + G(Q) \right] \, dx$$

along with the associated relative energy functional

$$E(v, P, Q \mid \tilde{v}, \tilde{P}, \tilde{Q}) = \int_{\Omega} \left[ |v - \tilde{v}|^2 + |p - \tilde{p}|^2 + |\nabla x Q - \nabla x \tilde{Q}|^2 \right] + G(Q) - \partial G(\tilde{Q}) : (Q - \tilde{Q}) - G(\tilde{Q}) \, dx$$

$$= E(v, P, Q) + E(\tilde{v}, \tilde{P}, \tilde{Q}) - \int_{\Omega} \left[ v \cdot \tilde{v} + p : \tilde{P} + \nabla x Q : \nabla x \tilde{Q} \right] - \left[ \partial G(\tilde{Q}) : (Q - \tilde{Q}) + 2G(\tilde{Q}) \right] \, dx$$

defined for any trio of smooth function $[\tilde{v}, \tilde{P}, \tilde{Q}]$. The functional $E$ plays a role of a “distance” between a solution $[v, P, Q]$ and a generic triplet $[\tilde{v}, \tilde{P}, \tilde{Q}]$. 
Relative energy

The dissipative solutions may seem rather weak as we have apparently no information about the specific form of neither the dissipation defect $\mathcal{D}$ nor the correctors $R^1, R^2$. Nevertheless, we show that a dissipative solution coincides with the strong solution emanating from the same initial data as long as the latter exists. We consider the modified energy functional

$$E(v, P, Q) = \int_\Omega \left[ \frac{1}{2} |v|^2 + \frac{1}{2} |P|^2 + \frac{1}{2} |\nabla_x Q|^2 + G(Q) \right] \, dx$$

along with the associated *relative energy functional*

$$\mathcal{E} \left( v, P, Q \mid \tilde{v}, \tilde{P}, \tilde{Q} \right)$$

$$= \frac{1}{2} \int_\Omega \left[ |v - \tilde{v}|^2 + |P - \tilde{P}|^2 + |\nabla_x Q - \nabla_x \tilde{Q}|^2 \right] + G(Q) - \partial G(Q) : (Q - \tilde{Q}) - G(\tilde{Q}) \, dx$$

$$= E(v, P, Q) + E(\tilde{v}, \tilde{P}, \tilde{Q}) - \int_\Omega \left[ v \cdot \tilde{v} + P : \tilde{P} + \nabla_x Q : \nabla_x \tilde{Q} \right] - \left[ \partial G(\tilde{Q}) : (Q - \tilde{Q}) + 2G(\tilde{Q}) \right] \, dx$$

defined for any trio of smooth function $[\tilde{v}, \tilde{P}, \tilde{Q}]$. The functional $\mathcal{E}$ plays a role of a “distance” between a solution $[v, P, Q]$ and a generic triplet $[\tilde{v}, \tilde{P}, \tilde{Q}]$.

Our goal is to derive the relative entropy inequality - an explicit formula for

$$\left[ \mathcal{E} \left( v, P, Q \mid \tilde{v}, \tilde{P}, \tilde{Q} \right) \right]_{t=0}^{t=\tau}$$
Outline

1 Introduction of the Problem
2 The LCs modeling
3 The simplified Qian-Sheng model
4 The existence of local-in-time smooth solutions
5 Existence of global-in-time dissipative solutions
6 Weak-strong uniqueness
7 Further perspectives
Theorem 3: Weak-strong uniqueness

Our ultimate goal is to show that any dissipative solution necessarily coincides with a strong solution originating from the same initial data on the existence interval of the latter. A simple idea is to take the strong solution \([\tilde{v}, \tilde{P}, \tilde{Q}]\) as "test functions" in the relative energy inequality and to use a Gronwall-type argument.

Under the hypotheses of Theorem 2 let the initial data enjoy the regularity properties:

\[
\begin{align*}
    v(0, \cdot) &= v_0 \in W^{s,2}(T_3; \mathbb{R}^3), \\
    P(0, \cdot) &= P_0 \in W^{s,2}(T_3; \mathbb{R}^3 \times \mathbb{R}^3_{0, \text{sym}}), \\
    Q(0, \cdot) &= Q_0 \in W^{s+1,2}(T_3; \mathbb{R}^3 \times \mathbb{R}^3_{0, \text{sym}}),
\end{align*}
\]

such that \(\text{div}_x v_0 = 0\).

Let \([v, P, Q]\) be a dissipative solution of our problem

\[
\begin{align*}
    \text{div}_x v &= 0 \\
    \partial_t v + v \cdot \nabla_x v + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \otimes \nabla_x Q) \\
    \partial_t Q + v \cdot \nabla_x Q &= P \\
    \partial_t P + v \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda I
\end{align*}
\]

and let \([\tilde{v}, \tilde{P}, \tilde{Q}]\) a the strong solution of the same problem belonging to the regularity class specified in Theorem 1 in the space-time cylinder \((0, T) \times T^3\).

Then we have

\[
\begin{align*}
    v &= \tilde{v}, \\
    P &= \tilde{P}, \\
    Q &= \tilde{Q} \text{ a.a. in } (0, T) \times T^3.
\end{align*}
\]
Theorem 3: Weak-strong uniqueness

Our ultimate goal is to show that any dissipative solution necessarily coincides with a strong solution originating from the same initial data on the existence interval of the latter. A simple idea is to take the strong solution $[\tilde{v}, \tilde{P}, \tilde{Q}]$ as “test functions” in the relative energy inequality and to use a Gronwall-type argument.

Under the hypotheses of Theorem 2 let the initial data enjoy the regularity properties:

\[ \mathbf{v}(0, \cdot) = \mathbf{v}_0 \in W^{s,2}(T^3; \mathbb{R}^3), \quad P(0, \cdot) = P_0 \in W^{s,2}(T^3; \mathbb{R}^{3 \times 3}_{0, \text{sym}}), \]
\[ Q(0, \cdot) = Q_0 \in W^{s+1,2}(T^3; \mathbb{R}^{3 \times 3}_{0, \text{sym}}) \]

such that $\text{div}_x \mathbf{v}_0 = 0$. 

Theorem 3: Weak-strong uniqueness

Our ultimate goal is to show that any dissipative solution necessarily coincides with a strong solution originating from the same initial data on the existence interval of the latter. A simple idea is to take the strong solution \([\tilde{v}, \tilde{P}, \tilde{Q}]\) as “test functions” in the relative energy inequality and to use a Gronwall-type argument.

Under the hypotheses of Theorem 2 let the initial data enjoy the regularity properties:

\[
\begin{align*}
\mathbf{v}(0, \cdot) &= \mathbf{v}_0 \in W^{s,2}(T^3; R^3), \\
P(0, \cdot) &= P_0 \in W^{s,2}(T^3; R^{3 \times 3}_{0,\text{sym}}), \\
Q(0, \cdot) &= Q_0 \in W^{s+1,2}(T^3; R^{3 \times 3}_{0,\text{sym}})
\end{align*}
\]

such that \(\text{div}_x \mathbf{v}_0 = 0\). Let \([\mathbf{v}, P, Q]\) be a dissipative solution of our problem

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \odot \nabla_x Q) \\
\partial_t Q + \mathbf{v} \cdot \nabla_x Q &= P \\
\partial_t P + \mathbf{v} \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda I
\end{align*}
\]

and let \([\tilde{v}, \tilde{P}, \tilde{Q}]\) a the strong solution of the same problem belonging to the regularity class specified in Theorem 1 in the space-time cylinder \((0, T) \times T^3\).
Theorem 3: Weak-strong uniqueness

Our ultimate goal is to show that any dissipative solution necessarily coincides with a strong solution originating from the same initial data on the existence interval of the latter. A simple idea is to take the strong solution \([\tilde{v}, \tilde{P}, \tilde{Q}]\) as “test functions” in the relative energy inequality and to use a Gronwall-type argument.

Under the hypotheses of Theorem 2 let the initial data enjoy the regularity properties:

\[
\begin{align*}
\mathbf{v}(0, \cdot) &= \mathbf{v}_0 \in W^{s,2}(T^3; R^3), & \mathbf{P}(0, \cdot) &= \mathbf{P}_0 \in W^{s,2}(T^3; R^{3 \times 3}_{0,\text{sym}}), \\
\mathbf{Q}(0, \cdot) &= \mathbf{Q}_0 \in W^{s+1,2}(T^3; R^{3 \times 3}_{0,\text{sym}})
\end{align*}
\]

such that \(\text{div}_x \mathbf{v}_0 = 0\). Let \([\mathbf{v}, \mathbf{P}, \mathbf{Q}]\) be a dissipative solution of our problem

\[
\begin{align*}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x \mathbf{Q} \odot \nabla_x \mathbf{Q}) \\
\partial_t \mathbf{Q} + \mathbf{v} \cdot \nabla_x \mathbf{Q} &= \mathbf{P} \\
\partial_t \mathbf{P} + \mathbf{v} \cdot \nabla_x \mathbf{P} &= -\frac{\partial F}{\partial \mathbf{Q}} + \Delta \mathbf{Q} - \lambda I
\end{align*}
\]

and let \([\tilde{v}, \tilde{P}, \tilde{Q}]\) a the strong solution of the same problem belonging to the regularity class specified in Theorem 1 in the space-time cylinder \((0, T) \times T^3\). Then we have

\[
\begin{align*}
\mathbf{v} &= \tilde{v}, & \mathbf{P} &= \tilde{P}, & \mathbf{Q} &= \tilde{Q} \text{ a.a. in } (0, T) \times T^3.
\end{align*}
\]
An idea of the proof

Using the weak formulation, the energy balance, and the bounds for the correctors and taking now as test functions the strong solutions $\tilde{v}, \tilde{P}, \tilde{Q}$, the relative energy inequality takes the final form

$$
\mathcal{E}(v, P, Q | \tilde{v}, \tilde{P}, \tilde{Q}) (\tau) + D(\tau) \leq 2\Lambda \int_{\tau_0}^{\tau} \int_{\Omega} (Q - \tilde{Q}) : (P - \tilde{P}) \, dx \, dt - \int_{\tau_0}^{\tau} \int_{\Omega} (v - \tilde{v}) \cdot \nabla x \tilde{v} \cdot (v - \tilde{v}) \, dx \, dt + \int_{\tau_0}^{\tau} \int_{\Omega} (\tilde{v} - v) \cdot \nabla x \tilde{P} \cdot (P - \tilde{P}) \, dx \, dt + \int_{\tau_0}^{\tau} \int_{\Omega} (\tilde{v} - v) \cdot \nabla x \partial G(\tilde{Q}) \cdot (Q - \tilde{Q}) \, dx \, dt + \int_{\tau_0}^{\tau} \int_{\Omega} \tilde{P} : (\partial G(Q) - \partial^2 G(\tilde{Q})(Q - \tilde{Q}) - \partial G(\tilde{Q})) \, dx \, dt + \int_{\tau_0}^{\tau} \int_{\Omega} \left[ (\nabla x \tilde{Q} - \nabla x Q) \cdot \Delta x \tilde{Q} \cdot (v - \tilde{v}) - (\nabla x Q - \nabla x \tilde{Q}) \cdot \nabla x \tilde{v} \cdot (\nabla x Q - \nabla x \tilde{Q}) \right] \, dx \, dt + c \int_{\tau_0}^{\tau} (\| \nabla x \tilde{P} \|_{C(T^3)} + \| \nabla x \tilde{v} \|_{C(T^3)}) \, dt.
$$

Applying Gronwall’s lemma we get the desired conclusion.
An idea of the proof

Using the weak formulation, the energy balance, and the bounds for the correctors and taking now as test functions the strong solutions \([\tilde{v}, \tilde{P}, \tilde{Q}]\), the relative energy inequality takes the final form

\[
\mathcal{E} \left( v, P, Q \bigg| \tilde{v}, \tilde{P}, \tilde{Q} \right) (\tau) + D(\tau) \\
\leq 2\Lambda \int_0^T \int_\Omega (Q - \tilde{Q}) : (P - \tilde{P}) \, dx \, dt - \int_0^T \int_\Omega (v - \tilde{v}) \cdot \nabla_x \tilde{v} \cdot (v - \tilde{v}) \, dx \, dt \\
+ \int_0^T \int_\Omega (\tilde{v} - v) \cdot \nabla_x \tilde{P} \cdot (P - \tilde{P}) \, dx \, dt + \int_0^T \int_\Omega (\tilde{v} - v) \cdot \nabla_x \partial G(\tilde{Q}) \cdot (Q - \tilde{Q}) \, dx \, dt \\
+ \int_0^T \int_\Omega \tilde{P} : \left( \partial G(Q) - \partial^2 G(\tilde{Q})(Q - \tilde{Q}) - \partial G(\tilde{Q}) \right) \, dx \, dt \\
+ \int_0^T \int_\Omega \left[ (\nabla_x \tilde{Q} - \nabla_x Q) \cdot \Delta_x \tilde{Q} \cdot (v - \tilde{v}) - (\nabla_x Q - \nabla_x \tilde{Q}) \cdot \nabla_x \tilde{v} \cdot (\nabla_x Q - \nabla_x \tilde{Q}) \right] \, dx \, dt \\
+ c \int_0^T \left( \|\nabla_x \tilde{P}\|_{C(T^3)} + \|\nabla_x \tilde{v}\|_{C(T^3)} \right) D(\cdot) \, dt.
\]

Applying Gronwall’s lemma we get the desired conclusion.
Combining Theorem 3 with the local existence result established in Theorem 2 we immediately get the following corollary:
Combining Theorem 3 with the local existence result established in Theorem 2 we immediately get the following corollary: Let \([v, P, Q]\) be a dissipative solution of our problem

\[
\begin{align*}
\text{div}_x v &= 0 \\
\partial_t v + v \cdot \nabla_x v + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \circ \nabla_x Q) \\
\partial_t Q + v \cdot \nabla_x Q &= P \\
\partial_t P + v \cdot \nabla_x P &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda I
\end{align*}
\]

in \((0, T) \times T^3\) enjoying the regularity specified in Theorem 1.
Combining Theorem 3 with the local existence result established in Theorem 2 we immediately get the following corollary: Let \([\mathbf{v}, \mathbb{P}, Q]\) be a dissipative solution of our problem

\[
\begin{aligned}
\text{div}_x \mathbf{v} &= 0 \\
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi &= -\text{div}_x (\nabla_x Q \odot \nabla_x Q) \\
\partial_t Q + \mathbf{v} \cdot \nabla_x Q &= \mathbb{P} \\
\partial_t \mathbb{P} + \mathbf{v} \cdot \nabla_x \mathbb{P} &= -\frac{\partial F}{\partial Q} + \Delta Q - \lambda \mathbb{I}
\end{aligned}
\]

in \((0, T) \times \mathbb{T}^3\) enjoying the regularity specified in Theorem 1.

Then \([\mathbf{v}, \mathbb{P}, Q]\) is a strong solution, in particular, the dissipation defect \(\mathcal{D}\) as well as the defect measures \(\mathbb{R}^1, \mathbb{R}^2\) vanish identically in \([0, T] \times \mathbb{T}^3\).
Outline

1. Introduction of the Problem
2. The LCs modeling
3. The simplified Qian-Sheng model
4. The existence of local-in-time smooth solutions
5. Existence of global-in-time dissipative solutions
6. Weak-strong uniqueness
7. Further perspectives
The study of more refined models

We should include more effects in the model, like:

1. changes of the temperature (we have results for the Landau-de Gennes nematic liquid crystals flows)
2. effects of magnetic and electric fields (this is only a work in progress): LCs behave differently depending on what direction electric or magnetic fields are applied relative to the director and so the introduction of the dependence on the magnetic and electric field in the mathematical models and analysis would be particularly challenging and useful in view of applications.
The study of more refined models

We should include more effects in the model, like:

1. changes of the *temperature* (we have results for the Landau-de Gennes nematic liquid crystals flows)

2. effects of *magnetic and electric fields* (this is only a work in progress):
   LCs behave differently depending on what direction electric or magnetic fields are applied relative to the director and so the introduction of the dependence on the magnetic and electric field in the mathematical models and analysis would be particularly challenging and useful in view of applications
Point 1. Our main contributions in the Landau-de Gennes case

We study the **non-isothermal** evolutionary system for nematic liquid crystals within the recent Ball-Majumdar $Q$-tensorial model preserving the physical eigenvalue constraint on the **traceless and symmetric matrices** $Q$:
Point 1. Our main contributions in the Landau-de Gennes case

We study the non-isothermal evolutionary system for nematic liquid crystals within the recent Ball-Majumdar $Q$-tensorial model preserving the physical eigenvalue constraint on the traceless and symmetric matrices $Q$:

Point 1. Our main contributions in the Landau-de Gennes case

We study the non-isothermal evolutionary system for nematic liquid crystals within the recent Ball-Majumdar $\mathcal{Q}$-tensorial model preserving the physical eigenvalue constraint on the traceless and symmetric matrices $\mathcal{Q}$:


2. [E. Feireisl, E. R., G. Schimperna, A. Zarnescu], *Nonisothermal nematic liquid crystal flows with the Ball-Majumdar free energy*, Annali di Matematica, 194 (2015)
Point 1. Our main contributions in the Landau-de Gennes case

We study the non-isothermal evolutionary system for nematic liquid crystals within the recent Ball-Majumdar $Q$-tensorial model preserving the physical eigenvalue constraint on the traceless and symmetric matrices $Q$:


2. [E. Feireisl, E. R., G. Schimperna, A. Zarnescu], *Nonisothermal nematic liquid crystal flows with the Ball-Majumdar free energy*, Annali di Matematica, 194 (2015)

We still work in the three-dimensional torus $\Omega \subset \mathbb{R}^3$ in order to avoid complications connected with boundary conditions. We consider the evolution of the following variables:

- the mean velocity field $\mathbf{v}$
- the tensor field $Q$, representing preferred (local) orientation of the crystals
- the absolute temperature $\theta$
The free energy and the $\mathbb{Q}$-tensor equation

The free energy density takes the form

$$\mathcal{F} = \frac{1}{2} |\nabla \mathbb{Q}|^2 + f_B(\theta, \mathbb{Q}) - \theta \log \theta - a\theta^m$$

where

- $f_B(\theta, \mathbb{Q}) = \theta f(\mathbb{Q}) + G(\mathbb{Q})$ is bulk the configuration potential
- $f$ is the convex l.s.c. and singular Ball-Majumdar potential, $G$ is a smooth function of $\mathbb{Q}$
- $a\theta^m$ prescribes a power-like specific heat
The free energy and the $\mathcal{Q}$-tensor equation

The free energy density takes the form

$$ F = \frac{1}{2} |\nabla \mathcal{Q}|^2 + f_B(\theta, \mathcal{Q}) - \theta \log \theta - a\theta^m $$

where

- $f_B(\theta, \mathcal{Q}) = \theta f(Q) + G(Q)$ is bulk the configuration potential
- $f$ is the convex l.s.c. and singular Ball-Majumdar potential, $G$ is a smooth function of $\mathcal{Q}$
- $a\theta^m$ prescribes a power-like specific heat

We assume that the driving force governing the dynamics of the director $\mathcal{Q}$ is of “gradient type”

$$ \partial_{\mathcal{Q}} F : $$

$$ \partial_t \mathcal{Q} + \mathbf{v} \cdot \nabla \mathcal{Q} - S(\nabla \mathbf{v}, \mathcal{Q}) = \Gamma(\theta) \mathcal{H}, $$

(eq-Q)

- The left hand side is the “generalized material derivative” $D_t \mathcal{Q} = \partial_t \mathcal{Q} + \mathbf{v} \cdot \nabla \mathcal{Q} - S(\nabla \mathbf{v}, \mathcal{Q})$
- $S$ represents deformation and stretching effects of the crystal director along the flow
The free energy and the $Q$-tensor equation

The free energy density takes the form

$$\mathcal{F} = \frac{1}{2} |\nabla Q|^2 + f_B(\theta, Q) - \theta \log \theta - a\theta^m$$

where

- $f_B(\theta, Q) = \theta f(Q) + G(Q)$ is bulk the configuration potential
- $f$ is the convex l.s.c. and singular Ball-Majumdar potential, $G$ is a smooth function of $Q$
- $a\theta^m$ prescribes a power-like specific heat

We assume that the driving force governing the dynamics of the director $Q$ is of “gradient type”

$$\partial_Q \mathcal{F}$$:

$$\partial_t Q + \mathbf{v} \cdot \nabla Q - S(\nabla \mathbf{v}, Q) = \Gamma(\theta) \mathbb{H}, \quad (eq-Q)$$

- The left hand side is the “generalized material derivative” $D_t Q = \partial_t Q + \mathbf{v} \cdot \nabla Q - S(\nabla \mathbf{v}, Q)$
- $S$ represents deformation and stretching effects of the crystal director along the flow
- The right hand side is of “gradient type” $-\mathbb{H} = \partial_Q \mathcal{F}$, i.e.

$$\mathbb{H} = \Delta Q - \theta \frac{\partial f(Q)}{\partial Q} - \frac{\partial G(Q)}{\partial Q} = \Delta Q - \theta \frac{\partial f(Q)}{\partial Q} + \lambda Q, \; \lambda \geq 0$$
The free energy and the $Q$-tensor equation

The free energy density takes the form

$$\mathcal{F} = \frac{1}{2} |\nabla Q|^2 + f_B(\theta, Q) - \theta \log \theta - a\theta^m$$

where

- $f_B(\theta, Q) = \theta f(Q) + G(Q)$ is bulk the configuration potential
- $f$ is the convex l.s.c. and singular Ball-Majumdar potential, $G$ is a smooth function of $Q$
- $a\theta^m$ prescribes a power-like specific heat

We assume that the driving force governing the dynamics of the director $Q$ is of "gradient type"

$$\partial_Q \mathcal{F}$$

$$\partial_t Q + \mathbf{v} \cdot \nabla Q - S(\nabla \mathbf{v}, Q) = \Gamma(\theta) \mathbb{H}, \quad \text{(eq-Q)}$$

- The left hand side is the "generalized material derivative" $D_t Q = \partial_t Q + \mathbf{v} \cdot \nabla Q - S(\nabla \mathbf{v}, Q)$
- $S$ represents deformation and stretching effects of the crystal director along the flow
- The right hand side is of "gradient type" $-\mathbb{H} = \partial_Q \mathcal{F}$, i.e.

$$\mathbb{H} = \Delta Q - \theta \frac{\partial f(Q)}{\partial Q} - \frac{\partial G(Q)}{\partial Q} = \Delta Q - \theta \frac{\partial f(Q)}{\partial Q} + \lambda Q, \; \lambda \geq 0$$

- $\Gamma(\theta)$ represents a collective rotational viscosity coefficient
Equation of momentum

In the context of nematic liquid crystals, we have the incompressibility constraint
\[ \text{div} \, v = 0 \]
By virtue of Newton's second law, the balance of momentum reads
\[ \partial_t v + \text{div}(v \otimes v) = \text{div} \, \sigma + g \] (eq-v)
The stress \( \sigma \) is given by
\[ \sigma = \nu(\theta) \left( \nabla v + \nabla^T v \right) - p I + T \]
The coupling term (or "extra-stress") \( T \) depends both on \( \theta \) and \( Q \)
\[ T = 2 \xi (H:Q) (Q + \frac{1}{3} I) - \xi \left[ H (Q + \frac{1}{3} I) + (Q + \frac{1}{3} I) H \right] + (Q H - H Q) - \nabla Q \cdot \nabla Q \]
Equation of momentum

- In the context of nematic liquid crystals, we have the **incompressibility** constraint

\[
\text{div } \mathbf{v} = 0
\]
Equation of momentum

- In the context of nematic liquid crystals, we have the incompressibility constraint
  \[ \text{div} \, \mathbf{v} = 0 \]

- By virtue of Newton’s second law, the balance of momentum reads
  \[ \partial_t \mathbf{v} + \text{div} (\mathbf{v} \otimes \mathbf{v}) = \text{div} \, \sigma + \mathbf{g} \]  
  (eq-v)

The stress \( \sigma \) is given by

\[ \sigma = \nu (\theta) \left( \nabla \mathbf{v} + \nabla^T \mathbf{v} \right) - p I + T \]

The coupling term (or “extra-stress”) \( T \) depends both on \( \theta \) and \( Q \)

\[ T = 2 \xi \left( \mathbf{H} : \mathbf{Q} \right) \left( \mathbf{Q} + \frac{1}{3} I \right) - \xi \left[ \mathbf{H} \left( \mathbf{Q} + \frac{1}{3} I \right) + \left( \mathbf{Q} + \frac{1}{3} I \right) \mathbf{H} \right] + \left( \mathbf{Q} \mathbf{H} - \mathbf{H} \mathbf{Q} \right) - \nabla Q \odot \nabla Q \]
In the context of nematic liquid crystals, we have the \textit{incompressibility} constraint

\[
\text{div} \, \mathbf{v} = 0
\]

By virtue of Newton’s second law, the \textit{balance of momentum} reads

\[
\partial_t \mathbf{v} + \text{div}(\mathbf{v} \otimes \mathbf{v}) = \text{div} \, \sigma + \mathbf{g} \quad \text{(eq-v)}
\]

The stress \(\sigma\) is given by

\[
\sigma = \frac{\nu(\theta)}{2} (\nabla \mathbf{v} + \nabla^t \mathbf{v}) - p \mathbb{I} + \mathbb{T}
\]
In the context of nematic liquid crystals, we have the \textit{incompressibility} constraint

\[ \text{div} \, \mathbf{v} = 0 \]

By virtue of Newton’s second law, the balance of momentum reads

\[ \partial_t \mathbf{v} + \text{div} (\mathbf{v} \otimes \mathbf{v}) = \text{div} \, \sigma + \mathbf{g} \tag{eq-v} \]

The stress \( \sigma \) is given by

\[ \sigma = \frac{\nu(\theta)}{2} (\nabla \mathbf{v} + \nabla^t \mathbf{v}) - pI + \mathbf{T} \]

The coupling term (or “extra-stress”) \( \mathbf{T} \) depends both on \( \theta \) and \( \mathbf{Q} \)

\[ \mathbf{T} = 2 \xi (\mathbf{H} : \mathbf{Q}) \left( \mathbf{Q} + \frac{1}{3} \mathbf{I} \right) - \xi \left[ \mathbf{H} \left( \mathbf{Q} + \frac{1}{3} \mathbf{I} \right) + \left( \mathbf{Q} + \frac{1}{3} \mathbf{I} \right) \mathbf{H} \right] + (\mathbf{QH} - \mathbf{HQ}) - \nabla \mathbf{Q} \odot \nabla \mathbf{Q} \]

where \( \xi \) is a fixed scalar parameter
The evolution of temperature is prescribed by stating the entropy inequality

\[ s_t + v \cdot \nabla s - \text{div} \left( \kappa(\theta) \theta \nabla \theta \right) \geq 1 \theta \left( \nu(\theta) \right)^2 \left| \nabla v + \nabla v \right|_2^2 + \Gamma(\theta) |H|_2^2 + \kappa(\theta) \theta \left| \nabla \theta \right|_2^2 \]

where \( s = -\partial \theta F'' = -f(Q) + 1 + \log \theta + \text{ma} \theta m - 1 \)

The viscosity \( \nu \) is smooth and bounded - without any growth condition

\( \kappa(r) = A_0 + A_k r^k, A_0, A_k > 0, 3k + 2m > 9, m > \frac{3}{2}, m \leq 6k \)

\( \Gamma(r) = \Gamma_0 + \Gamma_1 r, \Gamma_0, \Gamma_1 > 0 \)

The "heat" balance can be recovered by (formally) multiplying by \( \theta \). Due to the quadratic terms, we can only interpret (eq-\( \theta \)) as an inequality.
Entropy inequality

The evolution of temperature is prescribed by stating the entropy inequality

\[ s_t + \mathbf{v} \cdot \nabla s - \text{div} \left( \frac{\kappa(\theta)}{\theta} \nabla \theta \right) \geq \frac{1}{\theta} \left( \frac{\nu(\theta)}{2} |\nabla \mathbf{v} + \nabla^t \mathbf{v}|^2 + \Gamma(\theta)|\mathbf{H}|^2 + \frac{\kappa(\theta)}{\theta} |\nabla \theta|^2 \right) \]

where \( s'' = -\partial_\theta F'' = -f(\mathcal{Q}) + 1 + \log \theta + ma\theta^{m-1} \)
Entropic inequality

The evolution of temperature is prescribed by stating the **entropy inequality**

\[ s_t + \mathbf{v} \cdot \nabla s - \text{div} \left( \frac{\kappa(\theta)}{\theta} \nabla \theta \right) \geq \frac{1}{\theta} \left( \frac{\nu(\theta)}{2} |\nabla \mathbf{v} + \nabla^t \mathbf{v}|^2 + \Gamma(\theta)|\mathbf{H}|^2 + \frac{\kappa(\theta)}{\theta} |\nabla \theta|^2 \right) \]

where \( s'' = -\partial_\theta \mathcal{F}'' = -f(\mathcal{Q}) + 1 + \log \theta + ma\theta^{m-1} \)

- The viscosity \( \nu \) is smooth and bounded - without any growth condition
- \( \kappa(r) = A_0 + A_k r^k, \ A_0, A_k > 0, \frac{3k+2m}{3} > 9, \frac{3}{2} < m \leq \frac{6k}{5} \)
- \( \Gamma(r) = \Gamma_0 + \Gamma_1 r, \ \Gamma_0, \Gamma_1 > 0 \)
Entropy inequality

The evolution of temperature is prescribed by stating the entropy inequality

\[ s_t + \mathbf{v} \cdot \nabla s - \operatorname{div} \left( \frac{\kappa(\theta)}{\theta} \nabla \theta \right) \geq \frac{1}{\theta} \left( \frac{\nu(\theta)}{2} |\nabla \mathbf{v} + \nabla^t \mathbf{v}|^2 + \Gamma(\theta)|\mathbf{H}|^2 + \frac{\kappa(\theta)}{\theta} |\nabla \theta|^2 \right) \]

(eq-\theta)

where \( s'' = -\partial_\theta \mathcal{F}'' = -f(Q) + 1 + \log \theta + ma\theta^{m-1} \)

- The viscosity \( \nu \) is smooth and bounded - without any growth condition
- \( \kappa(r) = A_0 + A_k r^k, A_0, A_k > 0, \frac{3k+2m}{3} > 9, \frac{3}{2} < m \leq \frac{6k}{5} \)
- \( \Gamma(r) = \Gamma_0 + \Gamma_1 r, \Gamma_0, \Gamma_1 > 0 \)
- The “heat” balance can be recovered by (formally) multiplying by \( \theta \)
- Due to the quadratic terms, we can only interpret (eq-\theta) as an inequality
Total energy balance

Following an idea by [Bulíček, Feireisl, & Málek (2009)], we can complement the system with the total energy balance:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} |v|^2 + e \right) + \text{div} \left( \left( \frac{1}{2} |v|^2 + e \right)v \right) + \text{div} q = \text{div} \left( \sigma v \right) + \text{div} \left( \Gamma(\theta) \nabla Q : \left( \Delta Q - \theta \frac{\partial f(Q)}{\partial Q} + \lambda \theta \right) \right) + g \cdot v$$

where $e = F + s\theta$ is the internal energy.

Note the explicit occurrence of the pressure $p$ "hidden" inside $\sigma = \nu(\theta) \left( \nabla v + \nabla^T v \right) - p I + T$. To control it, assuming periodic b.c.'s is essential.
Total energy balance

- Passing from the heat equation to the entropy inequality gives rise to some information loss
Total energy balance

- Passing from the heat equation to the entropy inequality gives rise to some information loss
- Following an idea by Bulíček, Feireisl, & Málek (2009), we can complement the system with the total energy balance

\[
\partial_t \left( \frac{1}{2} |v|^2 + e \right) + \text{div} \left( \left( \frac{1}{2} |v|^2 + e \right) v \right) + \text{div} q = \text{div}(\sigma v) + \text{div} \left( \Gamma(\theta) \nabla Q : \left( \Delta Q - \theta \frac{\partial f(Q)}{\partial Q} + \lambda \theta \right) \right) + g \cdot v \tag{eq-bal}
\]

where \( e = \mathcal{F} + s\theta \) is the internal energy
Total energy balance

- Passing from the heat equation to the entropy inequality gives rise to some information loss
- Following an idea by [Bulíček, Feireisl, & Málek (2009)], we can complement the system with the total energy balance

\[
\partial_t \left( \frac{1}{2} |\mathbf{v}|^2 + e \right) + \text{div} \left( \left( \frac{1}{2} |\mathbf{v}|^2 + e \right) \mathbf{v} \right) + \text{div} \mathbf{q} = \text{div} (\sigma \mathbf{v}) + \text{div} \left( \Gamma(\theta) \nabla Q : \left( \Delta Q - \theta \frac{\partial f(Q)}{\partial Q} + \lambda \theta \right) \right) + \mathbf{g} \cdot \mathbf{v}
\]

(\text{eq-bal})

where \( e = \mathcal{F} + s\theta \) is the internal energy

- Note the explicit occurrence of the pressure \( p \) “hidden” inside

\[
\sigma = \frac{\nu(\theta)}{2} \left( \nabla \mathbf{v} + \nabla^t \mathbf{v} \right) - p\mathbb{I} + \mathbb{T}
\]
Total energy balance

- Passing from the heat equation to the entropy inequality gives rise to some information loss.
- Following an idea by Bulíček, Feireisl, & Málek (2009), we can complement the system with the total energy balance:
  \[
  \partial_t \left( \frac{1}{2} |v|^2 + e \right) + \text{div} \left( \left( \frac{1}{2} |v|^2 + e \right) v \right) + \text{div} \, q \quad \text{(eq-bal)}
  \]
  \[
  = \text{div} (\sigma v) + \text{div} \left( \Gamma(\theta) \nabla Q : \left( \Delta Q - \theta \frac{\partial f(Q)}{\partial Q} + \lambda \theta \right) \right) + g \cdot v
  \]
  where \( e = F + s \theta \) is the internal energy.
- Note the explicit occurrence of the pressure \( p \) “hidden” inside
  \[
  \sigma = \frac{\nu(\theta)}{2} (\nabla v + \nabla^T v) - p \mathbb{I} + \mathbb{T}
  \]
- To control it, assuming periodic b.c.’s is essential.
Main result: the “Entropic formulation”

Theorem: existence of global in time “Entropic solutions”

We can prove existence of at least one “Entropic solution” to system \((\text{eq-v})+(\text{eq-Q})+(\text{eq-}\theta)+(\text{eq-bal})\) for finite-energy initial data, namely

\[
\theta_0 \in L^\infty(\Omega), \quad \text{essinf}_{x \in \Omega} \theta_0(x) = \underline{\theta} > 0,
\]

\[
Q_0 \in H^1(\Omega), \quad f(Q_0) \in L^1(\Omega),
\]

\[
v_0 \in L^2(\Omega), \quad \text{div } v_0 = 0.
\]
Main result: the “Entropic formulation”

**Theorem: existence of global in time “Entropic solutions”**

We can prove existence of at least one “Entropic solution” to system (eq-v)+(eq-Q)+(eq-θ)+(eq-bal) for finite-energy initial data, namely

\[
\begin{align*}
\theta_0 &\in L^\infty(\Omega), \quad \text{essinf}_{x\in\Omega} \theta_0(x) = \underline{\theta} > 0, \\
Q_0 &\in H^1(\Omega), \quad f(Q_0) \in L^1(\Omega), \\
v_0 &\in L^2(\Omega), \quad \text{div}v_0 = 0.
\end{align*}
\]

Notice that, if the solution is more regular, the entropy inequality becomes an equality and, multiplying it by \(\theta\) we just get the standard internal energy balance

\[
\dot{\vartheta} + \nu \cdot \nabla \vartheta + \text{div}q = \vartheta (\partial_t f(Q) + \nu \cdot \nabla f(Q)) + \nu(\theta)\left|\nabla v + \nabla_t^v\right|^2 + \Gamma(\vartheta)|\mathbb{H}|^2
\]
Main result: the “Entropic formulation”

Theorem: existence of global in time “Entropic solutions”

We can prove existence of at least one “Entropic solution” to system (eq-v)+(eq-Q)+(eq-θ)+(eq-bal) for finite-energy initial data, namely

\[ \theta_0 \in L^\infty(\Omega), \quad \text{essinf}_{x \in \Omega} \theta_0(x) = \underline{\theta} > 0, \]
\[ Q_0 \in H^1(\Omega), \quad f(Q_0) \in L^1(\Omega), \]
\[ v_0 \in L^2(\Omega), \quad \text{div} \, v_0 = 0. \]

- Notice that, if the solution is more regular, the entropy inequality becomes an equality and, multiplying it by \( \theta \) we just get the standard internal energy balance

\[ \vartheta_t + \mathbf{v} \cdot \nabla x \vartheta + \text{div} \, q = \vartheta (\partial_t f(Q) + \mathbf{v} \cdot \nabla x f(Q)) + \nu(\theta) \left| \nabla x \mathbf{v} + \nabla^t x \mathbf{v} \right|^2 + \Gamma(\vartheta) |\mathbb{H}|^2 \]

- However, this regularity is out of reach for this model: that is why this solution notion is significative
Open problems for the QS-model

- Include the temperature dependence
- Deal with the singular potential case
- ...
Many thanks to all of you for the attention!

http://matematica.unipv.it/rocca/
Convergences of the defect measures

Examining the difference

\[ v_i^j v_n^j - v^i v^j = (v_n^j - v^i)(v_i^j - v^j) - v^i(v^j - v_n^j) - v^j(v^i - v_n^i), \]  

(7.1)

we find out that

\[ v_i^j v_n^j - v^i v^j \rightarrow R_{1,1}^{i,j} \text{ weakly-*(*) in } L^\infty(0, T; \mathcal{M}(T^3)). \]

Let now \( \zeta \in C(T^3) \) with \( \|\zeta\|_\infty \leq 1 \). Then, testing (7.1) by \( \zeta \), we obtain

\[
\int_\Omega (v_i^j v_n^j - v^i v^j)\zeta \, dx \leq \frac{1}{2} \int_\Omega (v_i^j - v^i)^2 |\zeta| \, dx + \frac{1}{2} \int_\Omega (v_n^j - v^j)^2 |\zeta| \, dx \\
- \int_\Omega (v^i(v^j - v_n^j) + v^j(v^i - v_n^i))\zeta \, dx
\]

(7.2)

Hence, letting \( n \uparrow \infty \), we obtain

\[
\int_\Omega R_{1,1}^{i,j} \zeta \, dx \leq \frac{1}{2} \lim_{n \uparrow \infty} \int_\Omega (v_n^i - v^i)^2 |\zeta| \, dx + \frac{1}{2} \lim_{n \uparrow \infty} \int_\Omega (v_n^j - v^j)^2 |\zeta| \, dx,
\]

where the first integral has in fact to be intended as the integral of the function \( \zeta \) with respect to the measure \( R_{1,1}^{i,j} \). This convention will be extensively used also in the sequel.

Hence, passing to the supremum with respect to \( \zeta \), and summing over \( i, j \), we arrive at

\[
\sum_{i,j} \|R_{1,1}^{i,j}\|_{\mathcal{M}(T^3)} \leq 3 \int_\Omega \left( |v|^2 - |v|^2 \right) \, dx.
\]

(7.3)
The existence of local-in-time smooth solutions

We establish the existence of local-in-time solutions in the Sobolev framework $W^{s,2}(\mathcal{T}^N)$ of functions with derivatives up to order $s$ square integrable in $\mathcal{T}^N$. 

Let us start by recalling the following nowadays standard results (see e.g. [A. Majda (1984)]):

1. For $u, v \in W^{s,2} \cap L^\infty(\mathcal{T}^N)$ and $\alpha$ a multi-index with $|\alpha| \leq s$ 
   \[ \| \partial^\alpha x (uv) \|_{L^2(\mathcal{T}^N)} \leq c_s \left( \| u \|_{L^\infty(\mathcal{T}^N)} \| \nabla^s x v \|_{L^2(\mathcal{T}^N)} + \| v \|_{L^\infty(\mathcal{T}^N)} \| \nabla^s x u \|_{L^2(\mathcal{T}^N)} \right). \]
   Here and below $\nabla^s x v$ denotes the tensor of the partial derivatives of $v$ of order equal to $s$.

2. For $u \in W^{s,2}(\mathcal{T}^N)$, $\nabla^s x u \in L^\infty(\mathcal{T}^N)$, $v \in W^{s-1,2} \cap L^\infty(\mathcal{T}^N)$ and $|\alpha| \leq s$ 
   \[ \| \partial^\alpha x \left( uv - u \partial^\alpha x v \right) \|_{L^2(\mathcal{T}^N)} \leq c_s \| \nabla^s x u \|_{L^\infty(\mathcal{T}^N)} \| \nabla^{s-1} x v \|_{L^2(\mathcal{T}^N)} + c_s \| \nabla^s x u \|_{L^2(\mathcal{T}^N)} \| v \|_{L^\infty(\mathcal{T}^N)} \| \nabla^s x u \|_{L^2(\mathcal{T}^N)}. \]

3. For $u \in W^{s,2} \cap C(\mathcal{T}^N)$, and $F_s$-times continuously differentiable function on an open neighborhood of the compact set $G = \text{range} [\mathcal{T}, 1 \leq |\alpha| \leq s]$ 
   \[ \| \partial^\alpha x F(u) \|_{L^2(\mathcal{T}^N)} \leq c_s \| \partial^u F \|_{C^{s-1}(G)} \| u \|_{C^{s-1}(\mathcal{T}^N)} \| \partial^\alpha x u \|_{L^2(\mathcal{T}^N)}. \]
The existence of local-in-time smooth solutions

We establish the existence of local-in-time solutions in the Sobolev framework $W^{s,2}(\mathcal{T}^N)$ of functions with derivatives up to order $s$ square integrable in $\mathcal{T}^N$.

Let us start by recalling the following nowadays standard results (see e.g. [A. Majda (1984)]):
The existence of local-in-time smooth solutions

We establish the existence of local-in-time solutions in the Sobolev framework $W^{s,2}(\mathcal{T}^N)$ of functions with derivatives up to order $s$ square integrable in $\mathcal{T}^N$.

Let us start by recalling the following nowadays standard results (see e.g. [A. Majda (1984)]):

1. For $u, v \in W^{s,2} \cap L^\infty(\mathcal{T}^3)$ and $\alpha$ a multi-index with $|\alpha| \leq s$

   $$\|\partial_x^\alpha (uv)\|_{L^2(\mathcal{T}^3)} \leq c_s \left( \|u\|_{L^\infty(\mathcal{T}^3)} \|\nabla_x^s v\|_{L^2(\mathcal{T}^3)} + \|v\|_{L^\infty(\mathcal{T}^3)} \|\nabla_x^s u\|_{L^2(\mathcal{T}^3)} \right).$$

   Here and below $\nabla_x^s v$ denotes the tensor of the partial derivatives of $v$ of order equal to $s$. 
The existence of local-in-time smooth solutions

We establish the existence of local-in-time solutions in the Sobolev framework $W^{s,2}(\mathcal{T}^N)$ of functions with derivatives up to order $s$ square integrable in $\mathcal{T}^N$.

Let us start by recalling the following nowadays standard results (see e.g. [A. Majda (1984)]):

1. For $u, v \in W^{s,2} \cap L^\infty(\mathcal{T}^3)$ and $\alpha$ a multi-index with $|\alpha| \leq s$

   \[
   \|\partial_x^\alpha (uv)\|_{L^2(\mathcal{T}^3)} \leq c_s \left( \|u\|_{L^\infty(\mathcal{T}^3)} \|\nabla_x^s v\|_{L^2(\mathcal{T}^3)} + \|v\|_{L^\infty(\mathcal{T}^3)} \|\nabla_x^s u\|_{L^2(\mathcal{T}^3)} \right).
   \]

   Here and below $\nabla_x^s v$ denotes the tensor of the partial derivatives of $v$ of order equal to $s$.

2. For $u \in W^{s,2}(\mathcal{T}^3)$, $\nabla_x u \in L^\infty(\mathcal{T}^3)$, $v \in W^{s-1,2} \cap L^\infty(\mathcal{T}^3)$ and $|\alpha| \leq s$

   \[
   \|\partial_x^\alpha (uv) - u \partial_x^\alpha v\|_{L^2(\mathcal{T}^3)} \leq c_s \|\nabla_x u\|_{L^\infty(\mathcal{T}^3)} \|\nabla_x^{s-1} v\|_{L^2(\mathcal{T}^3)}
   + c_s \|\nabla_x^s u\|_{L^2(\mathcal{T}^3)} \|v\|_{L^\infty(\mathcal{T}^3)}.
   \]
The existence of local-in-time smooth solutions

We establish the existence of local-in-time solutions in the Sobolev framework $W^{s,2}(\mathcal{T}^N)$ of functions with derivatives up to order $s$ square integrable in $\mathcal{T}^N$.

Let us start by recalling the following nowadays standard results (see e.g. [A. Majda (1984)]):

1. For $u, v \in W^{s,2} \cap L^\infty(\mathcal{T}^3)$ and $\alpha$ a multi-index with $|\alpha| \leq s$

$$\|\partial_x^\alpha (uv)\|_{L^2(\mathcal{T}^3)} \leq c_s \left(\|u\|_{L^\infty(\mathcal{T}^3)} \|\nabla_x^s v\|_{L^2(\mathcal{T}^3)} + \|v\|_{L^\infty(\mathcal{T}^3)} \|\nabla_x^s u\|_{L^2(\mathcal{T}^3)}\right).$$

Here and below $\nabla_x^s v$ denotes the tensor of the partial derivatives of $v$ of order equal to $s$.

2. For $u \in W^{s,2}(\mathcal{T}^3)$, $\nabla_x u \in L^\infty(\mathcal{T}^3)$, $v \in W^{s-1,2} \cap L^\infty(\mathcal{T}^3)$ and $|\alpha| \leq s$

$$\|\partial_x^\alpha (uv) - u \partial_x^\alpha v\|_{L^2(\mathcal{T}^3)} \leq c_s \|\nabla_x u\|_{L^\infty(\mathcal{T}^3)} \|\nabla_x^{s-1} v\|_{L^2(\mathcal{T}^3)} + c_s \|\nabla_x^s u\|_{L^2(\mathcal{T}^3)} \|v\|_{L^\infty(\mathcal{T}^3)}.$$ 

3. For $u \in W^{s,2} \cap C(\mathcal{T}^3)$, and $F$ $s$-times continuously differentiable function on an open neighborhood of the compact set $G = \text{range}[u]$, $1 \leq |\alpha| \leq s$

$$\|\partial_x^\alpha F(u)\|_{L^2(\mathcal{T}^3)} \leq c_s \|\partial_u F\|_{C^{s-1}(G)} \|u\|^{|\alpha| - 1}_{L^\infty(\mathcal{T}^3)} \|\partial_x^\alpha u\|_{L^2(\mathcal{T}^3)}.$$