Advanced Mathematical Methods for Engineers

January 25, 2022

---

1) $\begin{cases} y'(x) = y(x)\sqrt[3]{y(x)-1} \\ y(0) = K \in \mathbb{R} \end{cases}$

a) $f(x,y) = f(y) = y\sqrt[3]{y-1} \in C^0(\mathbb{R})$ but it is <u>not</u> differentiable for $y=1$.
For the Peano theorem, we have existence of local solutions $\forall K \in \mathbb{R}$.
Uniqueness can be inferred only for $K \neq 1$ by the local Cauchy-Lipschitz theorem.
The function f is <u>not</u> globally Lipschitz and so we cannot conclude anything about global solutions.

b) $y \equiv 0$ and $y \equiv 1$ are solutions in case $K=0$ and $K=1$ respectively.
$y \equiv 0$ cannot be intersected by other solutions due to uniqueness of solutions.
Since $y' > 0$ for $y < 0$ and $y > 1$ the solutions are increasing in these sets.
Viceversa they are decreasing on the complementary sets.
The equation is a separable-variable type equation and so we have

$$g_K(y) := \int_K^y \frac{ds}{s\sqrt[3]{s-1}} = t \qquad \text{for } K \neq 0, \ K \neq 1$$

Hence, we get

b1) for $K < 0$, $\lim_{y \to 0} g_K(y) = +\infty$

and $\lim_{y \to -\infty} g_K(y) = T_1 > -\infty$

hence $\operatorname{dom} y = (T_1, +\infty)$ with

$\lim_{t \to T_1^+} y(t) = -\infty$ and

$\lim_{t \to +\infty} y(t) = 0$.

b2) for $K \in (0,1)$, $\lim_{y \to 0} g_K(y) = +\infty$ and

$\lim_{y \to 1} g_K(y) = T_2 > -\infty$ $\Rightarrow$

$\operatorname{dom}(y) = (T_2, +\infty)$ with $\lim_{t \to +\infty} y(t) = 0$

and $\lim_{t \to T_2^+} y(t) = 1$ and

$\lim_{t \to T_2^+} y'(t) = 0$

b3) for $K > 1$, $\lim_{y \to +\infty} g_K(y) = T_3 < +\infty$

and $\lim_{y \to 1} g_K(y) = T_4 > -\infty$

$\Rightarrow$ $\operatorname{dom} y = (T_4, T_3)$, moreover

$\lim_{t \to T_3^-} y(t) = +\infty$, $\lim_{t \to T_4^+} y(t) = 1$

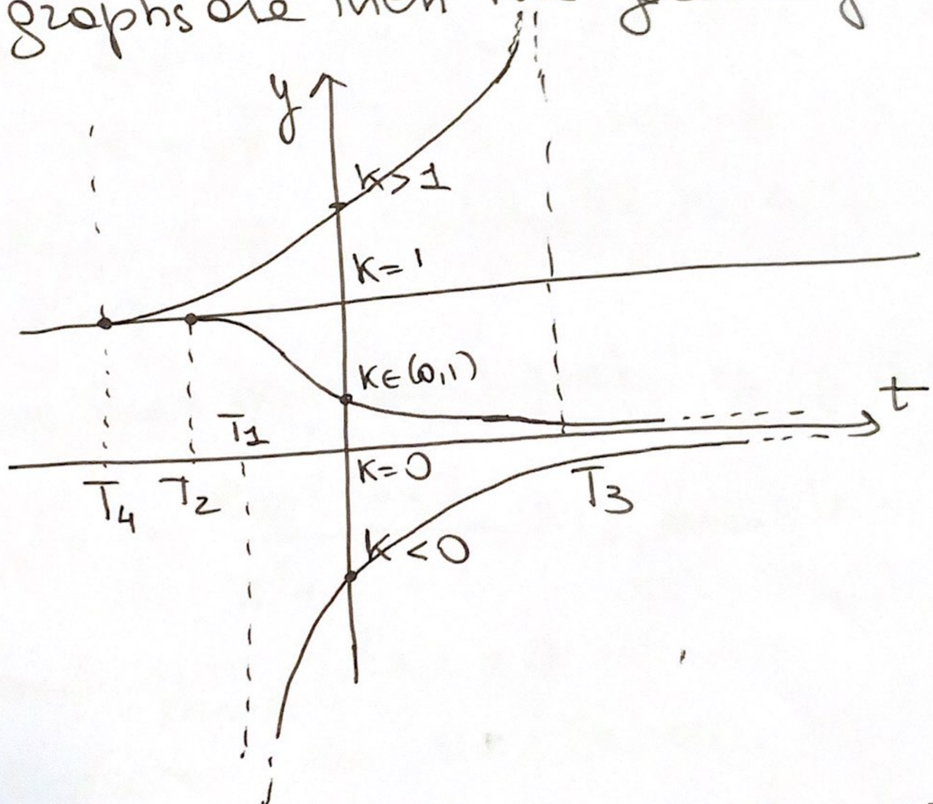
and $\lim_{t \to T_4^+} y'(t) = 0$.

Moreover, we have

$$y'' = \frac{4y-3}{3\sqrt[3]{(y-1)^2}} \cdot y'$$

and so the flex are on the line $y = \frac{3}{4}$.

The graphs are then the following:



2) Defining $y := x'$, we get the ODE system:

$$\begin{cases} x' = y \\ y' = -k(x-2)y - \tan x \end{cases}$$

The matrix of coefficients is:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2k \end{pmatrix}$$

for the linearized system in the origin.

$$\det(A-\lambda Id) = -\lambda\cdot(2k-\lambda)+1 =$$
$$= \lambda^2-2k\lambda+1 = 0$$

$$\Leftrightarrow \quad \lambda_{1,2} = \frac{k \pm \sqrt{k^2-1}}{1}$$

If $k^2-1 \geq 0$ $\quad Re(\lambda_{1,2}) = k \pm \sqrt{k^2-1} < 0$

$\Leftrightarrow \quad k+\sqrt{k^2-1} < 0 \quad \Leftrightarrow \quad k<0$

If $k^2-1<0$ $\quad Re(\lambda_{1,2}) = k < 0$

$\Leftrightarrow k<0$

$\Rightarrow$ we have asymptotic stability if $k<0$.

3) $$f_n(x) = \frac{\log(1+x)}{x^2+n^2+1}\chi_{[0,n+\sqrt{n}]}(x)$$

and $\lim_{n\to\infty} f_n(x) = 0$.

Moreover $\forall n$, $\forall x \in [0,+\infty)$

$$|f_n(x)| \leq \frac{\log(1+x)}{x^2+1} \in L^1(0,+\infty)$$

$\Rightarrow$ we can apply the Lebesgue dominated convergence theorem and so we get

$$\lim_{n\to\infty}\int_0^{+\infty} f_n(x)\,dx = \int_0^{+\infty}\lim_{n\to\infty} f_n(x) = 0.$$

4) We use the separation of variable method:
$u(x,t) = v(x)\,w(t)$ and we get

$$w'(t) = \lambda w(t) \quad \text{and}$$
$$v''(x) = \lambda v(x)$$

We first solve the second using the boundary conditions:

$$v(0) = 0, \quad v'(\pi) = 0$$

If $\lambda = \mu^2 \geq 0 \Rightarrow$ we get $v \equiv 0$ and we have only trivial solution

If $\lambda = -\mu^2 < 0 \Rightarrow \lambda_K = -\left(\frac{2K+1}{2}\right)^2, K=0,1,\ldots$

with corresponding eigenfunctions

$$v_K(x) = \operatorname{sen}\left(\frac{(2K+1)}{2}x\right)$$

$\Rightarrow$ we get

$$u_K(x,t) = c_K \operatorname{sen}\left(\frac{2K+1}{2}x\right) e^{-\left(\frac{2K+1}{2}\right)^2 t}$$

$\Rightarrow$ $$u(x,t) = \sum c_K \operatorname{sen}\left(\frac{2K+1}{2}x\right) e^{-\left(\frac{2K+1}{2}\right)^2 t}$$

formally we get

where $$c_K = \frac{2}{\pi}\int_0^{\pi} g(x) \operatorname{sen}\left(\frac{(2K+1)}{2}x\right) dx$$

are the Fourier coefficients of g.

If $g(x) = 2\operatorname{sen}\left(\frac{3}{2}x\right)$ and we take $c_1 = 2 \Rightarrow u_1$ satisfies also the initial condition $\Rightarrow$

$u(x,t) = 2e^{-\frac{9}{4}t}\operatorname{sen}\left(\frac{3}{2}x\right)$ is the solution.